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PERIMETER FUNCTIONALS ON CACCIOPPOLI PARTITIONS WITH THREE CHAMBERS

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Statutory Declaration

I hereby declare under oath that I have written the present master's thesis entitled

„Perimeter Functionals on Caccioppoli Partitions with Three Chambers"

I have written this work independently and without unauthorized outside assistance. I have not used any sources or resources other than those indicated, and have identified any literal and analogous quotations. The work has not been submitted to any examining authority in the same or a similar form. I certify that the submitted written version corresponds to the version saved on the attached medium.

City, Date	Esteban Josue Morillo Madrid

To my family, to my lifelong friends and those I was fortunate to meet during this program, to my professors who have shaped me throughout the years, and to all the people who have supported me along the way. To you I dedicate this work, with gratitude.

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1 Introduction

Imagine having a closed chamber full of water and you add drops of both oil and mercury (for lack of a better example). From the difference of densities, one can see the droplets of oil and mercury moving to the top or bottom of the chamber and slowly aggregating until they form a single concise drop. This is a broad example of an immiscible fluid system that evolves towards a stable equilibrium with time. In general, this can be done with any number of fluids (that do not mix with each other) and a keen mathematician would then be led to question themselves what type of configurations could there be, if there actually exists a minimum and if so, what regularity might its boudary have.

Mathematically, this translates to analyzing models of fluid systems where the energies are of interface types, i.e. depending on the interfaces separating the various fluids in the container. Thus, the container becomes a set Ω, the fluids become sets E_i with fixed volumes m_i that add up to the volume of the container, and the interfaces become the intersections of the boundaries between fluids $S_{ij} = \partial E_i \cap \partial E_j \cap \Omega$. A simple model for the energy would then be

$$\mathscr{F}_{\Omega,\mathbf{m}}(E_1, \ldots, E_M) = \sum_{i<j} \sigma_{ij} \cdot \text{surface area}(S_{ij}),$$

where $\sigma_{ij} > 0$ are the surface tensions between the fluids. This example with fluids was to give a general idea of how the problem can be visualized, but it is not the only instance where this functional appears. In the general realm of material science this functional is used when studying polycristaline materials, reviewed in [3, 5, 7]. In the context of imaging, the functional results useful for image segmentation, restoration, optical flow estimation and stereo reconstruction [6, 12].

The scope of this thesis will be to study the functional in a more theoretic way, within the realms of calculus of variations and geometric measure theory. Nevertheless, these problems have been studied and solved numerically in [4, 15]. We will consider only three chambers, and naturally replace the surface area for the $N-1$–dimensional Hausdorff measure, as well as replacing the topological boundaries for reduced boundaries, i.e.

$$\mathscr{F}_{\Omega,\mathbf{m}}(E_1, E_2, E_3) = \sum_{i<j} \sigma_{ij} \mathcal{H}^{N-1}(\mathcal{F}E_i \cap \mathcal{F}E_j \cap \Omega).$$

Our focus will be on minimality and regularity depending on if the weights σ_{ij} satisfy or not a triangle inequality condition that will be mentioned in its corresponding chapter.

The structure of this thesis is the following:

In Chapter 2, we introduce the analytic and geometric tools needed to understand the main results. We recall basic definitions and properties of Hausdorff measures, sets of finite perimeter and functions of bounded variations. In particular, so that we can go over the notions of reduced boundary and structure theorems such as De Giorgi's or Federer's theorems. We finish the section with standard regularity results for perimeter minimizers, such that we can guarantee smooth reduced boundaries of the minimizers up to a null set. These results will later serve as a model when studying the regularity of minimizers for the weighted perimeter functional we are considering, in Chapter 4.

In Chapter 3, we study the variational properties of the weighted perimeter functional. A central issue is whether the functional is lower semicontinuous with respect to the convergence in measure of sets. We first study the classic case where it does have this property under the triangle inequality assumption, which ensures compactness in the direct method of calculus of variations and thus existence of minimizers follows. We then contrast this with the non–lower semicontinuity case studied in [11] where the triangle inequality fails for one of the weights. Here we will impose conditions over the domain in order to assure existence of minimizers and present examples.

In Chapter 4, we turn to the structural and regularity results for minimizing configurations associated to the weighted perimeter functional. Building on the existence of minimizers, we establish strict approximation results showing that sets of finite perimeter can be approximated internally by smooth sets as done in [13], and elimination theorems that rule out the presence of negligible interfaces locally as in [9]. These results allow us to prove strong regularity properties of the minimizing and admissible configurations, specifically showing that the reduced boundaries are smooth hypersurfaces up to null sets.

The final chapter functions as a summary and final remarks on the work done in the previous ones. Included will be insight on the tools used in the proofs, importance of the results and comments on the overall work of the thesis.

Having outlined the motivation and main goal of this thesis, we move on to review preliminary material from geometric measure theory.

2 Geometric Measure-Theoretic Foundations

As previously stated, the main results of this thesis lie within the broad framework of geometric measure theory (GMT). Since this area of mathematics is quite extensive, we restrict ourselves to presenting only the necessary concepts for understanding the main results. This will serve both as an introduction and as a concise reference to the tools we employ, mainly perimeter problems. Consequently, we will not cover the underlying material of measure theory in detail, and instead refer the reader to [2] for a thorough exposition. Likewise for a deeper exploration of GMT, we recommend the references [2, 10].

Before delving into specific definitions and concepts, we fix some basic notations that will be used throughout the thesis. The symbol $N \in \mathbb{N}$ will always denote the dimension of the ambient euclidean space $\mathbb{R}^N$. The Euclidean norm is written as $\|\cdot\|$, and the open ball centered at $x \in \mathbb{R}^N$ of radius $r > 0$ is defined by

$$B_r(x) = \{y \in \mathbb{R}^N \,:\, \|y - x\| < 0\}.$$

For sets $E, F \subset \mathbb{R}^N$, we define

$$\begin{aligned} \operatorname{diam}(E) &= \sup\{\|x - y\| \,:\, x, y \in E\}, \\ \operatorname{dist}(E, F) &= \inf\{\|x - y\| \,:\, x \in E,\, y \in F\}, \end{aligned}$$

as the diameter of E and the distance between E and F, respectively. When $E = \{x\}$ is a singleton, we simply write $\operatorname{dist}(x, F)$. We also understand the Minkowski sum of two sets as

$$E + F = \{x + y \,:\, x \in E,\, y \in F\}.$$

We denote the Lebesgue measure by $|\cdot|$, the $k-$dimensional Hausdorff measure by $\mathcal{H}^k$, and the volume of the unit sphere in $\mathbb{R}^k$ by ω_k. Also, $C^m(\Omega, \mathbb{R}^N)$ denotes the space of continuously differentiable vector fields of up to derivatives of order m, $C_c^m(\Omega, \mathbb{R}^N)$ the analogue with compact support, and $L^p(\Omega)$ the standard Lebesgue spaces.

In topological terms, we use $\overset{\circ}{E}$ for the interior of E, $\overline{E}$ for its closure, and ∂E for its boundary. Finally, we write $E \Subset \Omega$ to indicate that E is compactly contained in Ω, that is, $\overline{E} \subset \Omega$.

2.1 GMT

We start by recalling the Radon measure that will be present throughout this work. As the problem in hand relies on interactions on the surfaces of fluids, it relies on surface area and thus, the most natural way to approach this is by considering the Hausdorff measure as it intrinsically generalize the notion of perimeter or surface area to k dimensions.

Definition 2.1 (Hausdorff measure)**.** Let $k \in [0, +\infty)$ and $E \subset \mathbb{R}^N$. The *$k-$dimensional Hausdorff measure* of E is defined as

$$\mathcal{H}^k(E) = \lim_{\delta \searrow 0} \mathcal{H}^k_\delta(E),$$

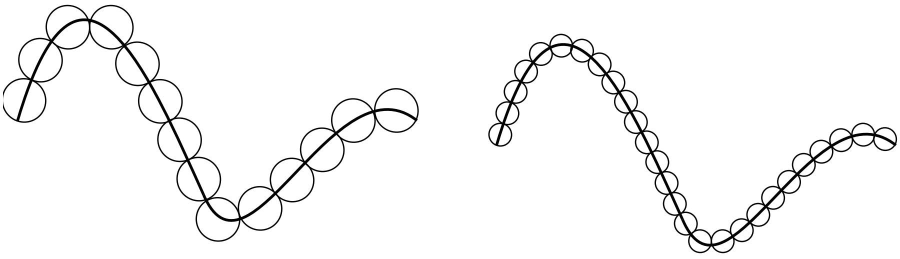

Figure 1: Illustration of the definition of the Hausdorff measure. The curve is covered with collections of small sets (here, balls), whose diameters approach to zero.

where, for $0 < \delta \leq +\infty$,

$$\mathcal{H}^k_\delta(E) = \frac{\omega_k}{2^k} \inf \left\{ \sum_{i \in I} \operatorname{diam}(E_i)^k \ : \ \operatorname{diam}(E_i) < \delta,\ E \subset \bigcup_{i \in I} E_i \right\},$$

for finite or countable covers $\{E_i\}_{i \in I}$, with $\operatorname{diam}(\emptyset) = 0$.

Remark 2.1. This definition mentions $\mathbb{R}^N$ as the ambient space but we can generally define this measure over any metric space. In fact, the definition depends only on the metric rather than its space. In the same manner, k can easily be taken as any non–negative real number [1].

Let us list some useful properties of Hausdorff measures.

Proposition 2.1 (Properties of Hausdorff measures)**.** The measure $\mathcal{H}^k$ in $\mathbb{R}^N$ has the following properties:

i. Measures $\mathcal{H}^k$ are outer measures in $\mathbb{R}^N$ and $\sigma-$additive on $\mathcal{B}(\mathbb{R}^N)$.

ii. $\mathcal{H}^k$ are invariant with respect to translations and homotheties, i.e.,

$$\mathcal{H}^k(E + x) = \mathcal{H}^k(E) \quad \text{and} \quad \mathcal{H}^k(\lambda E) = \lambda^k \mathcal{H}^k(E),$$

for any $E \subset \mathbb{R}^N$, $x \in \mathbb{R}^N$ and $\lambda > 0$. Furthermore, if $k > N$, it is identically zero.

iii. If $k > k' \geq 0$ and $\mathcal{H}^k(E) > 0$, then $\mathcal{H}^{k'}(E) = +\infty$.

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[1] ω_k can also be generalized for non–integer k as it can generally be defined as $\omega_k = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$, where Γ is the well known gamma function.

Proof. If one proves these properties for $\mathcal{H}^k_\delta$, then for $\mathcal{H}^k$ follows directly thanks to properties of the infimum. In i. $\sigma-$subadditivity follows from subadditivity of $\mathcal{H}^k_\delta$, ii. follows from invariance respect to translations and homotheties of the function $E \mapsto \operatorname{diam}(E)^k$, and iii. relies on the following inequality

$$\frac{2^k}{\omega_k}\mathcal{H}^k(E) \leq \delta^{k-k'}\frac{2^k}{\omega_k}\mathcal{H}^{k'}(E).$$

Full proof can be found in [2, Proposition 2.49]. □

However, the Hausdorff measure does not completely generalize the notion of perimeter or surface area when dealing with sets that have boundaries that do not *behave well.* This leads us to study the perimeter from a more measure–theoretic approach:

Definition 2.2 (Sets of finite perimeter)**.** Let E be an $\mathcal{L}^N-$measurable subset of $\mathbb{R}^N$. For any open set $\Omega \subset \mathbb{R}^N$ the *perimeter of E in Ω*, denoted by $P(E,\Omega)$, is defined as

$$P(E,\Omega) = \sup\left\{\int_E \operatorname{div} T\,dx \;:\; T \in C^1_c(\Omega,\mathbb{R}^N),\ \|T\|_\infty \leq 1\right\}.$$

We say that E *has finite perimeter in* Ω if $P(E,\Omega) < \infty$.

When $\Omega = \mathbb{R}^N$, we will write $P\left(E,\mathbb{R}^N\right) = P(E)$. This function coincides with the notion of perimeter in 2 dimensions or surface area in 3 dimensions when dealing with not so regular boundaries of sets, in contrast to the Hausdorff measure. For sets E with C^1 boundary, they should coincide, and indeed,

$$P(E,\Omega) = \mathcal{H}^{N-1}(\Omega \cap \partial E).$$

In the framework of semicontinuity we need to specify what type of convergence we are using. Having in mind important properties that are presented later on, we use convergence in measure.

Definition 2.3 (Convergence in measure)**.** Let $\{E_n\}_{n\in\mathbb{N}}$ and E be $\mathcal{L}^N-$measurable sets. We say that E_n *converges to E in measure in* Ω if

$$\lim_{n\to+\infty}|\Omega \cap (E \mathbin{\triangle} E_n)| = 0.$$

We say E_n *converges to E locally in measure in* Ω if it converges in measure in any open set $A \Subset \Omega$.

Remark 2.2. Convergence and local convergence in measure is equivalent to $L^1(\Omega)$ or $L^1_{\text{loc}}(\Omega)$ convergence of the characteristic functions χ_{E_n} to χ_E, respectively.

Let us now review some useful properties that will be prevalent when using the perimeter.

Proposition 2.2. Let E and F be $\mathcal{L}^N-$measurable sets and $\Omega \subset \mathbb{R}^N$ open.

i. The function $\Omega \mapsto P(E,\Omega)$ is the restriction of open sets of a Borel measure in $\mathbb{R}^N$.

ii. The function $E \mapsto P(E,\Omega)$ is lower semicontinuous with respect to local convergence in measure in Ω.

iii. The function $E \mapsto P(E,\Omega)$ is local, i.e. $P(E,\Omega) = P(F,\Omega)$ if $|\Omega \cap (E \vartriangle F)| = 0$.

iv. $P(E,\Omega) = P(E^c,\Omega)$ and

$$P(E \cup F,\Omega) + P(E \cap F,\Omega) \leq P(E,\Omega) + P(F,\Omega).$$

Proof. Lower semicontinuity of the perimeter plays an important role for us, so we will only comment on the proof for ii. The rest can be revised in [2, Proposition 3.38].

Take E and $\{E_n\}_{n\in\mathbb{N}}$ such that E_n converges to E in measure in Ω. Since this convergence is equivalent to the $L^1_{loc}(\Omega)$ convergence, one can deduce

$$\int_E \operatorname{div} T\, dx = \lim_{n\to+\infty} \int_{E_n} \operatorname{div} T\, dx \leq \liminf_{n\to+\infty} P(E_n,\Omega),$$

for any $T \in C^1_c(\Omega,\mathbb{R}^N)$ with $\|T\|_\infty \leq 1$. Since this inequality holds for any arbitrary T with favorable properties, the result follows

$$P(E,\Omega) \leq \liminf_{n\to+\infty} P(E_n,\Omega).$$

□

When dealing with minimizers, having compactness results aid greatly when working under the direct method. The following theorem functions as such.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{L}^N-$measurable sets such that*

$$\sup\{P(E_n,A) : n \in \mathbb{N}\} < +\infty \qquad \forall A \Subset \Omega \text{ open.}$$

The sequence admits a subsequence $\{E_{n_k}\}_{k\in\mathbb{N}}$ that locally converges in measure in Ω. If $|\Omega| < +\infty$ then the subsequence converges in measure in Ω.

Proof. Direct application of a compactness theorem [2, Theorem 3.23], to characteristic function. □

Other properties of interest for sets are density and regularity results. The following plays the role of an approximation theorem, telling us that sets of finite perimeter can be approximated in measure with sets with smooth boundary, as well as convergence of the perimeters.

Theorem 2.5 (Density of smooth sets)**.** *Let E be a set of finite perimeter in $\mathbb{R}^N$, $N \geq 2$. Then, there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of open sets with smooth boundary converging in measure to E and such that*

$$\lim_{n\to+\infty} P(E_n) = P(E)$$

Proof. By the isoperimetric inequality defined below, one can bound the volume of E with its perimeter which is finite, thus, $|E| < +\infty$. This way, taking a mollifying sequence $\{\rho_n\}_{n\in\mathbb{N}}$, we know that the function $u_n = \chi_E * \rho_n$ converges to χ_E in $L^1(\Omega)$ and satisfies

$$\lim_{n\to+\infty} |Du_n|(\mathbb{R}^N) = |D\chi_E|(\mathbb{R}^N) = P(E).$$

Call $F_\alpha^n = \{u_n > \alpha\}$. Using the coarea formula [2, Theorem 2.93] and Fatou's Lemma, we obtain the following estimate:

$$\begin{aligned} P(E) &= \lim_{n\to+\infty} |Du_n|(\mathbb{R}^N) \\ &= \lim_{n\to+\infty} \int_0^1 P(F_\alpha^n)\, d\alpha \\ &\geq \int_0^1 \liminf_{n\to+\infty} P(F_\alpha^n)\, d\alpha. \end{aligned}$$

By Sard's Lemma [10, Lemma 13.15], F_α^n is a smooth hypersurface for L^1-a.e. $\alpha \in (0,1)$. Choose an α with this property and such that

$$L = \liminf_{n\to+\infty} P(F_\alpha^n) \leq P(E) < +\infty.$$

Theorem 2.4 lets us take a subsequence $\{E_k\}_{k\in\mathbb{N}} = \{F_\alpha^{n_k}\}_{k\in\mathbb{N}}$ such that $P(E_k)$ converges to L. By lower semicontinuity of the perimeter $P(E) \leq L$, hence $P(E) = L$.

To prove convergence in measure, we use Chebyshev's inequality for the following bounds:

$$|E_k \backslash E| \leq \frac{1}{\alpha}\int_{\mathbb{R}^N} |u_k - \chi_E|\, dx \quad \text{and} \quad |E \backslash E_k| \leq \frac{1}{1-\alpha}\int_{\mathbb{R}^N} |u_k - \chi_E|\, dx,$$

proving convergence in measure when taking $k \to +\infty$. □

Remark 2.3. This result can be localized to sets with finite perimeter in some bounded extension domain Ω with $E \subset \Omega$. Then there exists a sequence of open sets $\{E_n \cap \Omega\}_{n\in\mathbb{N}}$ converges in measure in Ω to E and $P\left(E_n, \overline{\Omega}\right)$ converges to $P\left(E, \Omega\right)$.

Another important tool present in most results is the isoperimetric inequality. This inequality relates the volume of the set with its perimeter, which comes in handy when trying to find estimates, as one does for regularity and density results.

Theorem 2.6 (Isoperimetric inequality)**.** *Let $N > 1$ be an integer and E a set of finite perimeter in $\mathbb{R}^N$. Then, either E or E^c has finite Lebesgue measure and*

$$\min\{|E|, |E^c|\} \leq \gamma_N P(E)^{\frac{N}{N-1}}, \tag{2.1}$$

where $\gamma_N = \left(N\omega_N^{\frac{1}{N}}\right)^{-\frac{N}{N-1}}$.

Proof. The main idea is to estimate the mean value of χ_E over cubes to then cover $\mathbb{R}^N$ with a disjoint family of these, in order to reconstruct $|E|$. See [2, Theorem 3.46] for details of the proof. $\square$

In fact, this estimate can be localized to balls

$$|E \cap B_r(x)| \leq \gamma_{N,t} P\left(E, B_r(x)\right)^{\frac{N}{N-1}}, \tag{2.2}$$

for some $t \in (0,1)$ such that $|E \cap B_r(x)| \leq t\,|B_r(x)|$, and $x \in \mathbb{R}^N$.

Remark 2.4. In the classic case of $N = 2$, equality is achieved by a circle. Thus being the biggest shape in terms of volume out of all the ones with a fixed perimeter.

Building upon these properties where the perimeter appears in a more prominent role, we switch focus on more structural qualities of functions of finite perimeter. For this, we introduce on a superficial level BV functions, and some useful definitions for boundaries other than the topological notion.

Definition 2.7 (Total variation of a measure)**.** Let $(X, \mathcal{M})$ be a measurable space and μ a measure. The *total variation of* μ, denoted by $|\mu|$, is defined by

$$|\mu|(E) = \sup\left\{\sum_{i=0}^{+\infty} |\mu(E_i)| \;:\; \{E_i\}_{i\in\mathbb{N}} \subset \mathcal{M} \,\text{is a partition of } E\right\}.$$

Definition 2.8. Let $\Omega \subset \mathbb{R}^N$ be open and $u \in L^1(\Omega)$. We say that u is a *function of bounded variation in* Ω if its distributional derivative is representable by a finite Radon measure in Ω, that is, if

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi \, dD_i u, \quad \forall \varphi \in C_c^\infty(\Omega), \quad i = 1, \dots, N, \tag{2.3}$$

for some $\mathbb{R}^N-$valued measure $Du = (D_1 u, \dots, D_N u)$ in Ω. The vector space of all functions of bounded variations in Ω is denoted by $BV(\Omega)$.

Remark 2.5. The integration by parts formula (2.3) also holds true for test functions in $C_c^1(\Omega)$. We can also write in vector form as follows:

$$\int_\Omega u \operatorname{div} T dx = -\sum_{i=1}^{N} \int_\Omega T \, dD_i u, \quad \forall T \in C_c^1(\Omega, \mathbb{R}^M).$$

When dealing with vector fields ($u \in BV(\Omega, \mathbb{R}^M)$) a similar equation can be written for each index.

Remark 2.6. The Sobolev space $W^{1,1}(\Omega)$ is strictly contained in $BV(\Omega)$. Indeed, the Heaviside functions is in BV while not in $W^{1,1}$ as its distributional derivative is the Dirac delta δ_0.

Definition 2.9 (Variation)**.** Let $u \in L^1_{loc}(\Omega, \mathbb{R}^M)$. The *variation of* u *in* Ω is defined as

$$V(u,\Omega) = \left\{ \sum_{j=1}^{M} \int_\Omega u^j \operatorname{div} T^j \, dx \, : \, T \in C^1_c(\Omega, \mathbb{R}^{MN}), \; \|\varphi\|_\infty \leq 1 \right\}.$$

Remark 2.7. Regarding properties of the variation:

1. If $u \in C^1(\Omega, \mathbb{R}^M)$ then integration by parts yields $V(u,\Omega) = \int_\Omega |\nabla u| \, dx$;
2. The mapping $u \mapsto V(u,\Omega)$ is lower semicontinuous in the $L^1_{loc}(\Omega, \mathbb{R}^M)$ topology.

The following property relates the variation of a function with BV functions, and the total variation of their derivatives.

Proposition 2.3 (Variation of BV functions)**.** Let $u \in L^1_{loc}(\Omega, \mathbb{R}^M)$. Then $u \in BV(\Omega, \mathbb{R}^M)$ if and only if $V(u,\Omega) < +\infty$. Furthermore, $V(u,\Omega)$ coincides with $|Du|(\Omega)$, and the map $u \mapsto |Du|(\Omega)$ is lower semicontinuous with respect to the $L^1_{loc}(\Omega, \mathbb{R}^M)$ topology.

Proof. See [2, Proposition 3.6]. ☐

Remark 2.8. From the facts presented here, it is not hard to see that $P(E,\Omega)$ coincides with the variation of the indicator function χ_E.

We can now shift to more geometric notions related to sets of finite perimeter. First, let us present the idea of tangent space that we will be using:

Definition 2.10. Let μ be an $\mathbb{R}^M-$valued Radon measure in an open set $\Omega \subset \mathbb{R}^N$, and $x \in \Omega$. We define the *Tangent space* $\operatorname{Tan}(\mu, x)$ as the set of all finite radon measures which are weak* limits of $\frac{\mu(x+\rho_n \cdot)}{|\mu|(B_{\rho_n}(x))}$, where $\{\rho_n\}_{n\in\mathbb{N}}$ is a non–negative infinitesimal sequence. The elements of $\operatorname{Tan}(\mu, x)$ are called *tangent measures.*

Now, let us see a different notion of boundary than that of the topological one. This subset of the topological boundary actually works better with the perimeter, as in this case the perimeter and Hausdorff measure coincide.

Definition 2.11 (Reduced boundary)**.** Let $E \subset \mathbb{R}^N$ be a $\mathcal{L}^N-$measurable set and Ω be the largest open set such that E is locally of finite perimeter in Ω. We call *reduced boundary of* E, denoted by $\mathcal{F}E$, the collection of all points $x \in \operatorname{supp}|D\chi_E| \cap \Omega$ such that the limit

$$v_E(x) = \lim_{r \searrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))},$$

exists and belongs to S^{N-1}.

A classic example is that of a square in $\mathbb{R}^2$. The topological boundary will include all the perimeter of the square, while the reduced boundary excludes the corners. Essentially, the reduced boundary identifies the smooth part of the topological boundary, from the point of view of calculus.

A similar idea motivates the following definition. In contrast, now we consider sets that can be covered by sufficiently regular sets up to a $\mathcal{H}^k-$null set, and applies this idea to the reduced boundary in De Giorgi's Structure Theorem.

Definition 2.12 (Rectifiable sets)**.** Let $E \subset \mathbb{R}^N$ be a $\mathcal{H}^k-$measurable set. We say that E *is countably $k-$rectifiable* if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^N$ such that

$$E \subset \bigcup_{i=0}^{+\infty} f_i(\mathbb{R}^k).$$

We say E *is countably $\mathcal{H}^k-$rectifiable* if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^N$ such that

$$\mathcal{H}^k\left(E \backslash \bigcup_{i=0}^{+\infty} f_i(\mathbb{R}^k)\right).$$

Finally, we say E *is $\mathcal{H}^k-$rectifiable* if E is countably $\mathcal{H}^k-$rectifiable and $\mathcal{H}^k(E) < +\infty$.

Theorem 2.13 (De Giorgi)**.** *Let $E \subset \mathbb{R}^N$ be a $\mathcal{L}^N-$measurable set. Then $\mathcal{F}E$ is countably $(N-1)-$rectifiable and $|D\chi_E| = \mathcal{H}^{N-1} \llcorner \mathcal{F}E$. In addition, for any $x_0 \in \mathcal{F}E$ the following properties hold:*

i. The sets $(E - x_0)/r$ locally converge in measure in $\mathbb{R}^N$ as $r \searrow 0$ to the halfspace H orthogonal to $v_E(x_0)$ and containing $v_E(x_0)$.

ii. $\mathrm{Tan}(\mathcal{H}^{N-1} \llcorner \mathcal{F}E, x_0) = \mathcal{H}^{N-1} \llcorner v_E^{\perp}(x_0)$ *and in particular*

$$\lim_{r \searrow 0} \frac{\mathcal{H}^{N-1}(\mathcal{F}E \cap B_r(x_0))}{\omega_{N-1} r^{N-1}} = 1.$$

Proof. As the proof is rather technical for what concerns the main results of this thesis. It can be found in detail in [2, Theorem 3.59]. ☐

Let us briefly go over the power of this theorem. First, it supports our claim about the reduced boundary being smooth as it is countably $(N-1)-$rectifiable. Additionally, the total variation of $D\chi_E$ is concentrated over the reduced boundary, meaning the perimeter is computed over this set. Geometrically, $i.$ can be seen as $\mathcal{F}E$ resembling the halfspace at small scales illustrated in Figure 2, and $ii.$ refers to the density of the boundary at x_0.

The following type of boundary, the so called essential boundary, allows us to think of boundary points in a different way, thus somehow completing some points excluded from the reduced boundary while still just considering the meaningful part. A comparison of the three types of boundaries can be seen in Figure 3.

Definition 2.14 (Points of density and essential boundary)**.** For Every $t \in [0,1]$ and every $\mathcal{L}^N-$measurable set $E \subset \mathbb{R}^N$ we call

$$E^t = \left\{ x \in \mathbb{R}^N \ : \ \lim_{r \searrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t \right\}$$

the set of points of E *whose density is t*. The *essential boundary of E* is denoted and defined by $\partial^* E = \mathbb{R}^N \backslash (E^0 \cup E^1)$.

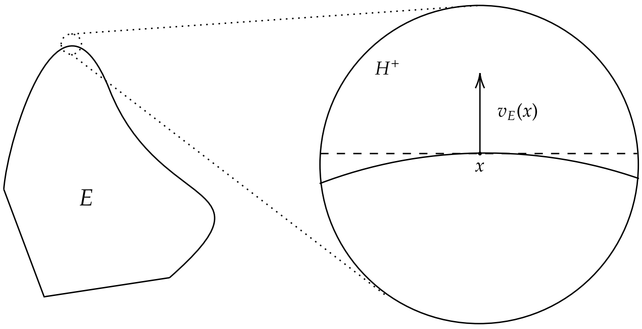

Figure 2: Visualization of De Giorgi's Theorem i. The circle on the right depicts a local zoom in on the boundary of E, approximated with a tangential line orthogonal to $v_E(x)$, which is the boundary of the halfspace denoted by H^+.

Finally, Federer's Theorem gives us insight on the density at the reduced boundary, its relation with the essential boundary and claims that both coincide outside of a $\mathcal{H}^{N-1}-$null set.

Theorem 2.15 (Federer)**.** *Let E be a set of finite perimeter in Ω. Then*

$$\mathcal{F}E \cap \Omega \subset E^{\frac{1}{2}} \subset \partial^* E \quad \textit{and} \quad \mathcal{H}^{N-1}\left((\partial^* E \cap \Omega) \backslash (\mathcal{F}E \cap \Omega)\right) = 0.$$

In particular, E has density either $0, \frac{1}{2}$ or 1 at $\mathcal{H}^{N-1}-$a.e. $x \in \Omega$ and $\mathcal{H}^{N-1}-$a.e. $x \in \partial^* E \cap \Omega$ belongs to $\mathcal{F}E$.

Proof. See [2, Theorem. 3.61]. □

This theorem proves useful when studying regularity for sets of finite perimeter in the next chapters, as with some density estimates, we will be able to prove that the boundary is contained in the essential boundary, and thus coincide with the reduced boundary $\mathcal{H}^{N-1}-$a.e.

2.2 Caccioppoli Partitions

Here we briefly introduce an idea of partitioning a set Ω into sets of finite perimeter. These partitions are called Caccioppoli partitions and have proved to be useful with its relation to piecewise constant functions over each set in the partition. In the context of this thesis, these type of partitions are going to be the admissible configurations for the main problem.

Definition 2.16 (Caccioppoli partitions)**.** Let $\Omega \subset \mathbb{R}^N$ be an open set and $I \subset \mathbb{N}$. A partition $\{E_i\}_{i \in I}$ of Ω is a Caccioppoli partition if

$$\sum_{i \in I} P\left(E_i, \Omega\right) < +\infty.$$

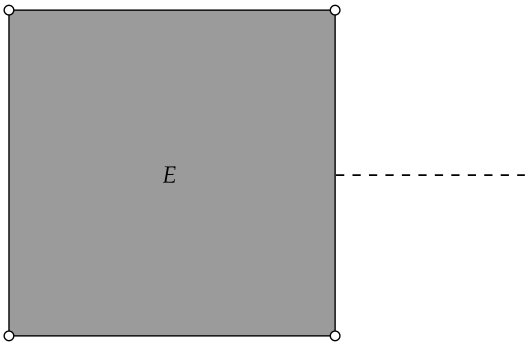

Figure 3: Consider $E = (0,1) \times (0,1) \cup (1,2) \times \left\{\frac{1}{2}\right\}$. The topological boundary will be the whole contour of the square plus the dashed line segment to the side. The essential boundary, however, considers only the perimeter of the square, as well as the reduced boundary minus the corners of the square.

A Caccioppoli partition will be said to be ordered if $|E_i| \geq |E_j|$ for any $i \leq j$.

It is in our interest to present a couple of structure and compactness properties of Caccioppoli partitions as we have already mentioned that these type of properties are of great help in our framework.

Theorem 2.17 (Local structure). *Let $\{E_i\}_{i\in I}$ be a Caccioppoli partition of Ω. Then*

$$\bigcup_{i\in I} E_i^1 \cup \bigcup_{i,j\in I\, i\neq j} (\mathcal{F}E_i \cap \mathcal{F}E_j)$$

contains $\mathcal{H}^{N-1}-$almost all of Ω.

Proof. The reader can refer to [2, Theorem 4.17] for a more formal proof. Regardless, this theorem relies heavily on Theorem 2.15. Indeed, if assuming that a point x does not belong to any reduced boundary, using the isoperimetric inequality and Federer's Theorem one attains that x must be a density point of value 1 of some set in the partition. Conversely, if it does belong to the reduced boundaries, Federer's Theorem says density must be $\frac{1}{2}$, implying it cannot be in more than two reduced boundaries. □

Essentially what we obtain from this theorem is that, for $\mathcal{H}^{N-1}-$a.e. point of Ω, it either belongs to one of the density sets E_i^1 or to the intersection of two reduced boundaries $\mathcal{F}E_i$, and no more. In less mathematical jargon, this means almost every point of Ω can be found in the interior of a set or in the intersection of the reduced boundaries of two sets, in the Caccioppoli partition of Ω.

Theorem 2.18 (Compactness). *Let $n \in \mathbb{N}$ and $\{E_{i,n}\}_{i\in I}$ be a Caccioppoli partition of $\mathbb{R}^N$ such that*

$$\sup\left\{\sum_{i\in I} P(E_{i,n}) \,:\, n \in \mathbb{N}\right\} < +\infty.$$

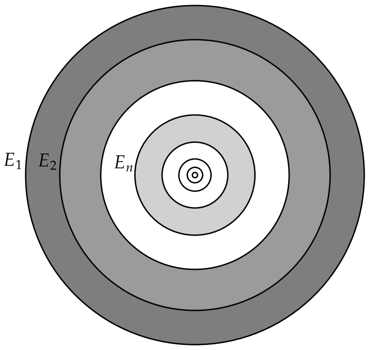

Figure 4: Example of a Caccioppoli partition of the unit ball $B_1(0)$. The partition considers rings containing just the outer perimeter. To satisfy the condition on the perimeters, their radii should decrease in such a way that $\sum_{i\in\mathbb{N}} r_i < +\infty$.

Then, if either I is finite or the partitions are ordered, there exists a Caccioppoli partition $\{E_i\}_{i\in I}$ and a subsequence $\{E_{i,n_k}\}_{k\in\mathbb{N}}$ that converges locally in measure in $\mathbb{R}^N$ to E_i for any $i \in I$.

Proof. As this is a generalization of Theorem 2.4 for a family of sets, it builds upon it for each set $E_{i,n}$ and creates a subsequence via diagonal arguments that converge to a Caccioppoli partition. See [2, Theorem 4.19]. $\square$

Remark 2.9. The same result holds if we restrict ourselves to ordered Caccioppoli partitions of a set Ω that is bounded, open and has a Lipschitz boundary. Indeed, we can add Ω^c to the Caccioppoli partitions to get them to be partitions of $\mathbb{R}^N$ and apply this theorem. Furthermore, the convergence obtained will not be local but completely in measure.

2.3 Standard Regularity and Density Results

The original type of problems that built the theory for our focal problem, were perimeter minimization problems, be it within an open set $\Omega \subset \mathbb{R}^N$ or satisfying volume constraints. These where the stepping stones for studying more complicated or general problems still regarding perimeter optimization, and some of the main results detailed in this thesis build upon the ones in this section. We start by defining what a minimizer is for a volume–constrained problem.

Definition 2.19. Let $\Omega \subset \mathbb{R}^N$ be open and $E \subset \mathbb{R}^N$ of finite perimeter. We say that E is a *volume–constrained perimeter minimizer in* Ω if

$$P(E, \Omega) \leq P(F, \Omega)$$

whenever $|\Omega \cap E| = |\Omega \cap F|$ and $E \mathbin{\triangle} F \Subset \Omega$.

Remark 2.10. The solutions to the problem

$$\min\left\{P\left(E,\Omega\right)\,:\,E\subset\Omega,\ |E|=m\right\},$$

where $0<m<|\Omega|$, are volume–constrained perimeter minimizers in Ω. Indeed, the fact that admissible sets have the same volume m and are subsets of Ω yield that the solution to the problem will satisfy

$$P\left(E,\Omega\right)\leq P\left(F,\Omega\right),\quad |E|=|F|=m\quad \textit{and}\quad E\bigtriangleup F\Subset\Omega,$$

for all admissible sets F.

These minimizers fall into a broader category of minimizers for a more general problem. These being called almost–minimizers or $(\Lambda,r_0)-$perimeter minimizers.

Definition 2.20. Let $\Omega\subset\mathbb{R}^N$ be open and $E\subset\mathbb{R}^N$ of locally finite perimeter. For $0\leq\Lambda<+\infty$ and $r_0>0$, we say that E is a $(\Lambda,r_0)-$*perimeter minimizer in* Ω if

$$P\left(E,B_r(x)\right)\leq P\left(F,B_r(x)\right)+\Lambda|E\bigtriangleup F|,\tag{2.4}$$

whenever $E\bigtriangleup F\Subset B_r(x)\cap\Omega$ and $r<r_0$.

One can see that if $\Lambda=0$ in this definition we retrieve the notion of a minimizer for the perimeter problem in Ω (in a local sense). The term $\Lambda|E\bigtriangleup F|$ acts as a higher order perturbation that generalizes the problem so it can study a broader field of examples.

Moreover, equation (2.4) is equivalent to

$$\left(1-\frac{\Lambda r}{N}\right)P\left(E,B_r(x)\right)\leq\left(1+\frac{\Lambda r}{N}\right)P\left(F,B_r(x)\right),\tag{2.5}$$

usually referred to as the weak $(\Lambda,r_0)-$minimality condition.

We mention this larger category of minimizers as they posses regularity and density results we are interested in. We will prove that volume–constrained mimimzers are as well and thus will benefit from these results. Before proving this fact, we will need a *volume fixing* lemma that allows us to change the volume of a set of finite perimeter by a small amount, at a cost proportional to that of the perimeter.

Lemma 2.21. *Consider the sets* $\Omega\subset\mathbb{R}^N$ *open and* $E\subset\mathbb{R}^N$ *of finite perimeter such that* $\mathcal{H}^{N-1}(\Omega\cap\mathcal{F}E)>0$. *Then, there exists* $\varepsilon_0>0$ *and* $C<+\infty$, *that depend on* E *and* Ω, *such that for every* $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$ *we can find a set of finite perimeter* F, *with* $E\bigtriangleup F\Subset\Omega$, *and*

$$|F|=|E|+\varepsilon,\quad |P\left(F,\Omega\right)-P\left(E,\Omega\right)|\leq C|\varepsilon|.$$

Proof. See [10, Lemma 17.21]. ☐

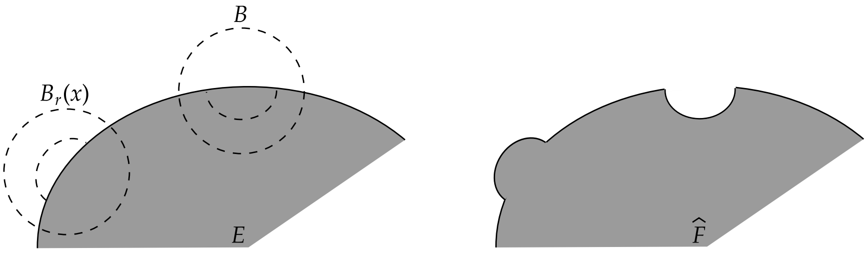

Figure 5: Visualization on how $\hat{F}$ is chosen in the proof of Proposition 2.4 by Lemma 2.21.

Ideally, choosing a convenient ε, we obtain a set similar enough to the original with a desired volume but whose perimeter does not change considerably so. This fuels the proof for the next result:

Proposition 2.4. Let $\Omega \subset \mathbb{R}^N$ be open and $E \subset \mathbb{R}^N$ a volume–constrained perimeter minimizer in Ω. Then E is a $(\Lambda, r_0)-$perimeter minimizer in Ω, for $0 \leq \Lambda < +\infty$ and $r_0 > 0$.

Proof. Fix $r_0 > 0$ and take F such that $E \triangle F \Subset B_r(x) \cap \Omega$ with $r < r_0$, and E is a volume–constrained perimeter minimizer. Consider a ball $B \subset \Omega$ such that $B_r(x)$ and B are disjoint but $F \cap B \neq \emptyset$. Applying Lemma 2.21 to F in B such that $\varepsilon = |E| - |F|$. There exists $\hat{F}$ such that

$$|\hat{F}| = |F| + \varepsilon = |E| \quad \text{and} \quad \left| P\left(\hat{F}, \Omega\right) - P\left(F, \Omega\right) \right| \leq C|\varepsilon|, \tag{2.6}$$

and $\hat{F} \triangle F \Subset B$, but $\hat{F} = F$ in $B_r(x)$. From the second condition in (2.6) and the fact that $|\varepsilon| = ||E| - |F|| \leq |E \triangle F|$ we get

$$P\left(\hat{F}, \Omega\right) \leq P\left(F, \Omega\right) + C|E \triangle F|.$$

Since $\hat{F}$ coincides with F in $B_r(x)$, then $E \triangle \hat{F} = E \triangle F \Subset B_r(x) \cap \Omega$, and from the first condition in (2.6) both $\hat{F}$ and E have the same volume, by minimality of E, we conclude

$$P\left(E, \Omega\right) \leq P\left(\hat{F}, \Omega\right) \leq P\left(F, \Omega\right) + C|E \triangle F|.$$

Taking $\Lambda = C$, E proves to be a $(\Lambda, r_0)-$perimeter minimizer. □

With this background in mind we can state some regularity and density results for almost–minimzers. The spirit of their proofs lie in their equivalent results in the next chapters, so we will omit some of them in this section but refer the reader to Sections 21 and 28 of [10], specifically Theorems 21.8 and 28.1, respectively.

Theorem 2.22. *Let $\Omega \subset \mathbb{R}^N$ be open and E be a $(\Lambda, r_0)-$perimeter minimizer in Ω with $\Lambda r_0 \leq 1$. Then, $\mathcal{F}E \cap \Omega$ is a $C^{1,\gamma}-$hypersurface for every $\gamma \in (0, \frac{1}{2})$, it is relatively open in $\partial E \cap \Omega$ and $\mathcal{F}E \cap \Omega = \partial E \cap \Omega$ up to $\mathcal{H}^{N-1}-$null set.*

The following is a density result as it gives estimates for the density quotients.

Theorem 2.23. *Let $\Omega \subset \mathbb{R}^N$ be open, and E be a $(\Lambda, r_0)-$perimeter minimizer in Ω with $\Lambda r_0 \leq 1$. Then, there exists a constant c_N such that*

$$\frac{1}{4^N} \leq \frac{|B_r(x) \cap E|}{\omega_N r^N} \leq 1 - \frac{1}{4^N} \quad \textit{and} \quad c_N \leq \frac{P\left(E, B_r(x)\right)}{r^{N-1}} \leq 3N\omega_N. \tag{2.7}$$

whenever $x \in \partial E \cap \Omega$, $B_r(x) \subset \Omega$, and $r < r_0$. In particular,

$$\mathcal{H}^{N-1}\left(\Omega \cap (\partial E \backslash \mathcal{F}E)\right) = 0. \tag{2.8}$$

Proof. Take $x \in \partial E \cap \Omega$, $d = \min\{r_0, \operatorname{dist}(x, \partial\Omega)\}$ and $\alpha(r) = |E \cap B_r(x)|$ for $0 < r < d$. Notice that $\alpha(r)$ is monotone increasing and

$$0 < \alpha(r) < \omega_N r^N,$$
$$\alpha'(r) = \mathcal{H}^{N-1}(E \cap \partial B_r(x)).$$

Clearly, for a fixed $r \in (0, d)$ with $\mathcal{H}^{N-1}(\mathcal{F}E \cap \partial B_r(x)) = 0$ and for $s \in (r, d)$, using $F = E \backslash B_r(x)$ in (2.5) we see

$$\begin{aligned}\left(1 - \frac{\Lambda s}{N}\right) P\left(E, B_r(x)\right) &\leq \left(1 + \frac{\Lambda s}{N}\right) P\left(F, B_r(x)\right) \\ &= \left(1 + \frac{\Lambda s}{N}\right) \left[\mathcal{H}^{N-1}\left(E^1 \cap \partial B_r(x)\right) + P\left(E, B_r(x) \backslash \overline{B_r(x)}\right)\right].\end{aligned} \tag{2.9}$$

Taking $s \to r^+$, and since $\mathcal{H}^{N-1}\left(E^1 \cap \partial B_r(x)\right) \leq P(B_r(x)) = N\omega_N r^{N-1}$and $\Lambda r_0 < 1$, yields

$$P\left(E, B_r(x)\right) \leq \frac{2}{1 - \frac{\Lambda r}{N}} N\omega_N r^{N-1} \leq 3N\omega_N r^{N-1},$$

which yields the right hand side of the perimeter estimates in (2.7).

If instead we add $\left(1 - \frac{\lambda r}{N}\right) \mathcal{H}^{N-1}\left(E^1 \cap \partial B_r(x)\right)$ after taking $s \to r^+$, we obtain

$$\left(1 - \frac{\lambda r}{N}\right) P\left(E \cap B_r(x)\right) \leq 2\mathcal{H}^{N-1}\left(E^1 \cap \partial B_r(x)\right).$$

By isoperimetric inequality and again since $\Lambda r_0 < 1$

$$\frac{N\omega_N^{\frac{1}{N}}}{2} \alpha(r)^{\frac{N-1}{N}} \leq 2\alpha'(r).$$

dividing by $\alpha(r)^{\frac{N-1}{N}}$, noticing that $\alpha(r)^{\frac{1-N}{N}}\alpha'(r) = \left(\alpha(r)^{\frac{1}{N}}\right)'$ and integrating over $(0, r)$ results in

$$\frac{N\omega_N^{\frac{1}{N}}}{4}r \leq \alpha(r)^{\frac{1}{N}}.$$

Thus, the left hand side of the volume estimates in (2.7) follows directly from this inequality. The upper bound can be deduced the same way using a symmetry argument, as $\mathbb{R}^N\backslash E$ is also a $(\Lambda, r_0)-$perimeter minimizer. Indeed, since

$$|(\mathbb{R}^N\backslash E) \cap B_r(x)| = |B_r(x)\backslash(E \cap B_r(x))|,$$

and the already computed lower bound, we can write

$$\frac{1}{4^N} \leq \frac{|B_r(x)\backslash(E \cap B_r(x))|}{\omega_N r^N} = 1 - \frac{|E \cap B_r(x)|}{\omega_N r^N},$$

where the lower bound follows. With this bound and the relative isoperimetric inequality (2.2), we can deduce the left side of the perimeter bound in (2.7).

Finally, From the volume estimates we have that $\partial E \cap \Omega \subset \partial^* E \cap \Omega$. Thus, restricting the result in Federer's Theorem yields (2.8). $\square$

The set $\Omega \cap (\partial E\backslash\mathcal{F}E)$ has been studied and characterized further in view of this theorem, in the following result:

Theorem 2.24. *Let $\Omega \subset \mathbb{R}^N$ be open and E be a $(\Lambda, r_0)-$perimeter minimizer in Ω with $\Lambda r_0 \leq 1$. The following statements hold true:*

i. If $2 \leq N \leq 7$, then $\Omega \cap (\partial E\backslash\mathcal{F}E)$ is empty;

ii. If $N = 8$, then $\Omega \cap (\partial E\backslash\mathcal{F}E)$ has no accumulation points in Ω;

iii. If $N \geq 9$, then $\mathcal{H}^k(\Omega \cap (\partial E\backslash\mathcal{F}E)) = 0$ for every $k > N - 8$.

There exists also results regarding the curvature of the perimeter minimizers, for which we need to go over briefly about these concepts. Nevertheless, this will be a surface level review as one can dive deep into proper definitions and applications in Sections 11.3 and 17.3 from [10]. We will only focus on the definitions that we will use in Chapter 3.

Definition 2.25 (Mean curvature vector). Let $\Gamma \subset \mathbb{R}^N$ be a C^2-hypersurface. We define the *mean curvature vector* $\mathbf{H}_\Gamma : \Gamma \to \mathbb{R}^N$ as a continuous vector field normal to Γ, such that

$$\int_\Gamma \operatorname{div}^\Gamma T \, d\mathcal{H}^{N-1} = \int_\Gamma T \cdot \mathbf{H}_\Gamma \, d\mathcal{H}^{N-1} + \int_{\partial\Gamma} (T \cdot \nu_{\partial\Gamma}^\Gamma) \, d\mathcal{H}^{N-2}, \tag{2.10}$$

for every $T \in C_c^1(\mathbb{R}^N, \mathbb{R}^N)$, $\nu_{\partial\Gamma}^\Gamma : \partial\Gamma \to S^{N-1}$ is a continuously differentiable vector field normal to $\partial\Gamma$, and $\operatorname{div}^\Gamma T$ is the tangential divergence of T on Γ given by the formula

$$\operatorname{div}^\Gamma T = \operatorname{div} T - (\nabla T \, \nu_\Gamma) \cdot \nu_\Gamma,$$

where $\nu_\Gamma : \Gamma \to S^{N-1}$ is any unit normal vector field to Γ.

Here, the term *normal to a set* is in the classic sense when talking about vector fields, that is, a vector field $\nu(x)$ is normal to a set Γ if it is orthogonal to the tangent space of Γ at x.

Remark 2.11. The scalar mean curvature $H_\Gamma : \Gamma \to \mathbb{R}^N$ of Γ depends on the mean curvature vector and a unit vector field ν_Γ to Γ, and it's defined by the relation

$$\mathbf{H}_\Gamma = H_\Gamma \nu_\Gamma.$$

In the planar case, when $N = 2$, $|H_\Gamma|$ coincides with the classic notion of curvature of a flat curve.

If we now consider $\Gamma = \partial E$, where $E \subset \mathbb{R}^N$ is open and has a C^2-boundary, equation (2.10) is simplified to

$$\int_{\partial E} \operatorname{div}^{\partial E} T \, d\mathcal{H}^{N-1} = \int_{\partial E} T \cdot \mathbf{H}_{\partial E} \, d\mathcal{H}^{N-1}, \tag{2.11}$$

for all $T \in C_c^1(\mathbb{R}^N, \mathbb{R}^N)$. This equation can be even further generalized and localized over an open set Ω, replacing $\mathcal{F}E$ instead of ∂E via distributional arguments [2], having in this sense also $T \in C_c^\infty(\Omega, \mathbb{R}^N)$.

Falling back to volume–constrained perimeter problems, the following theorem gives us insight to the curvature of the minimizers:

Theorem 2.26. *Let $\Omega \subset \mathbb{R}^N$ be an open set. If E is a volume–constrained perimeter minimizer, then there exists $\lambda \in \mathbb{R}$ such that*

$$\int_{\mathcal{F}E} \operatorname{div}^{\mathcal{F}E} T \, d\mathcal{H}^{N-1} = \lambda \int_{\mathcal{F}E} (T \cdot \nu_E) \, d\mathcal{H}^{N-1}, \tag{2.12}$$

for all $T \in C_c^\infty(\Omega, \mathbb{R}^N)$. In particular, E has constant mean curvature in Ω equal to λ.

Proof. The proof relies on Lemma 2.21 to construct competitor sets in order to produce zero first order perimeter variation. The proof is rather technical for the focus of this thesis so only general ideas will be discussed but can be found in [10, Theorem 17.20].

The first step proves that for $T \in C_c^\infty(\Omega, \mathbb{R}^N)$, with $\operatorname{supp} T \Subset B_r(x)$ for some $x \in \Omega$, if T produces a zero order volume variation of E, then T produces zero first order perimeter variation. That is,

$$\int_{\mathcal{F}E} (T \cdot \nu_E) \, d\mathcal{H}^{N-1} = 0$$

implies

$$\int_{\mathcal{F}E} \operatorname{div}^{\mathcal{F}E} T \, d\mathcal{H}^{N-1} = 0.$$

---

[2] This distributional argument is not so important for the following chapters so if the reader is further interested we refer them to [10, Remark 17.7]

To do this, one can follow a similar line of thought as in the proof of Proposition 2.4. It relies first in taking a local variation $\{f_t\}_{|t|<\varepsilon}$ in $B_r(x)$, which adds volume to E while approximating the perimeter locally in a way that

$$P\left(f_t(E),\Omega\right) = P\left(E,\Omega\right) + t\int_{\mathcal{F}E} \operatorname{div}^{\mathcal{F}E} T \, d\mathcal{H}^{N-1} + O(t^2).$$

Then, use Lemma 2.21 to create a competitor set E_t which on top of this, takes away this change of volume in another ball $B_s(y)$ disjoint from the first one, and coincides with E everywhere else, in order to again approximate the perimeter but now have the same volume volume $|E_t| = |E|$. By minimality of E and the approximating bounds, one achieves

$$0 \leq P\left(E_t,\Omega\right) - P\left(E,\Omega\right) \leq t\int_{\mathcal{F}E} \operatorname{div}^{\mathcal{F}E} T \, d\mathcal{H}^{N-1} + O(t^2),$$

which yields the desired result as one can take t and $-t$ converging to zero.

The second step takes $T_1, T_2 \in C_c^\infty(\Omega, \mathbb{R}^N)$, such that $\operatorname{supp} T_i \Subset B_{r_0}(x_i)$ and

$$\int_{\mathcal{F}E} (T_i \cdot \nu_E) \, d\mathcal{H}^{N-1} \neq 0,$$

for $i \in \{1,2\}$. We use these to define

$$T = T_1 - \frac{\int_{\mathcal{F}E} (T_1 \cdot \nu_E) \, d\mathcal{H}^{N-1}}{\int_{\mathcal{F}E} (T_2 \cdot \nu_E) \, d\mathcal{H}^{N-1}} T_2,$$

in order to achieve $\int_{\mathcal{F}E} (T \cdot \nu_E) \, d\mathcal{H}^{N-1} = 0$. Thus, from the first step it will have zero first order perimeter variation which entails

$$\frac{\int_{\mathcal{F}E} \operatorname{div}^{\mathcal{F}E} T_1 \, d\mathcal{H}^{N-1}}{\int_{\mathcal{F}E} (T_1 \cdot \nu_E) \, d\mathcal{H}^{N-1}} = \frac{\int_{\mathcal{F}E} \operatorname{div}^{\mathcal{F}E} T_2 \, d\mathcal{H}^{N-1}}{\int_{\mathcal{F}E} (T_2 \cdot \nu_E) \, d\mathcal{H}^{N-1}},$$

thus constant. This can be extended to a general $T \in C_c^\infty(\Omega.\mathbb{R}^N)$ using partitions of unity.

Comparing equation (2.12) with (2.11) and definition of $\mathbf{H}_{\mathcal{F}E}$, one sees that the mean curvature will be equal to the constant λ.

□

Remark 2.12. Lemma 2.21 proves useful not only to prove Theorem 2.26, but also when making competitor sets in the later chapters. Certainly, when defining sets we need to be wary that the volume is maintained in order for them to be admissible, thus in cases where we cannot, we use this lemma to somewhat change the shape and make it admissible at a cost of a small quantity. We will mention it again once needed.

3 Lower Semicontinuity and the Existence of Minimizers

In Chapter 1 we vaguely introduced the perimeter functional that we will be working with. Time has come to present it in full detail. Consider the functional

$$\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}) = \sum_{i<j\in\{1,2,3\}} \sigma_{i,j}\mathcal{H}^{N-1}(S_{ij}), \tag{3.1}$$

where $\mathcal{E} = (E_1, E_2, E_3)$ is a Caccioppoli partition of Ω, $\mathbf{m} = (m_1, m_2, m_3)$ with $m_i > 0$, $S_{ij} = \mathcal{F}E_i \cap \mathcal{F}E_j \cap \Omega$, and $\sigma_{ij} \geq 0$, for $i < j \in \{1, 2, 3\}$. The set $\mathcal{A}_{\Omega,\mathbf{m}}$ denotes the set of admissible partitions that satisfy:

i. $|E_i| = m_i$, for $i \in \{1, 2, 3\}$;

ii. $|E_i \cap E_j| = 0$ for all $i < j \in \{1, 2, 3\}$;

iii. $|\Omega\backslash(E_1 \cup E_2 \cup E_3)| = 0$.

A central question is under which assumptions on the weights σ_{ij} the functional admits minimizers. A central condition is reminiscent of a triangle inequality, and opens up two paths of study: when the inequality holds true or not. The more natural condition (mathematically at least) is the following:

$$\begin{array}{ll} \sigma_{ij} \leq \sigma_{ik} + \sigma_{kj}, & \text{for all } i, j, k \in \{1, 2, 3\},\ \text{with } i \neq j \neq k,\ i \neq k \\ \sigma_{ij} = \sigma_{ji}, & \text{for all } i, j \in \{1, 2, 3\},\ \text{with } i \neq j \end{array} \tag{3.2}$$

One may then wonder what happens if these conditions are not satisfied. This question was answered in [11] but will be studied here in detail. this is not just an academic problem. For certain fluids the interface between them does not satisfy these conditions entirely. To have an idea of what happens in this case, we consider the extra condition for a pair of indices:

$$\sigma_{13} > \sigma_{12} + \sigma_{23}, \tag{3.3}$$

while σ_{12} and σ_{23} still satisfy (3.2).

Most results in this section will be proved in the general case where $\sigma_{ij} > 0$ but we are interested in the type of partitions where E_1 and E_3 do not interact, that is, $\sigma_{13} = 0$ (see Figure 6). We begin by proving lower semicontinuity of $\mathscr{F}_{\Omega,\mathbf{m}}$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Under conditions (3.2), the functional defined in (3.1) is lower semicontinuous with respect to convergence in measure.*

Proof. Consider $\mathcal{E}_n = (E_{n,1}, E_{n,2}, E_{n,3})$ be a sequence of partitions of Ω such that $E_{n,i}$ converge to E_i in measure in Ω, for $i = 1, 2, 3$.

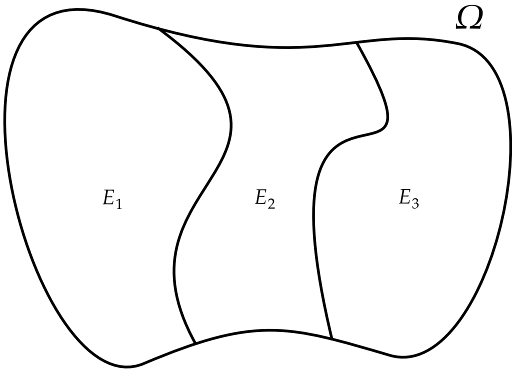

Figure 6: Representation of admissible configurations when $\sigma_{13} = 0$.

Step 1: Let us assume that $\mathcal{H}^{N-1}(S_{12}) = \mathcal{H}^{N-1}(S_{23}) = 0$. The functional will simplify down to

$$\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}) = \sigma_{13}\mathcal{H}^{N-1}(S_{13}).$$

In general, applying Theorem 2.17, we can write the perimeter of E_i in terms of the Hausdorff measure of S_{ij}

$$\begin{aligned} P\left(E_1,\Omega\right) &= \mathcal{H}^{N-1}(S_{12}) + \mathcal{H}^{N-1}(S_{13}),\\ P\left(E_2,\Omega\right) &= \mathcal{H}^{N-1}(S_{12}) + \mathcal{H}^{N-1}(S_{23}),\\ P\left(E_3,\Omega\right) &= \mathcal{H}^{N-1}(S_{13}) + \mathcal{H}^{N-1}(S_{23}). \end{aligned} \tag{3.4}$$

Under the assumptions for this step we have $\mathcal{H}^{N-1}(S_{13}) = P\left(E_1,\Omega\right) = P\left(E_3,\Omega\right)$, and thus $\mathscr{F}_{\Omega,\mathbf{m}}$ can be rewritten as

$$\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}) = \sigma_{12}P\left(E_1,\Omega\right) = \sigma_{12}P\left(E_3,\Omega\right),$$

or, in a more convenient way,

$$\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}) = \frac{\sigma_{12}}{2}\left(P\left(E_1,\Omega\right) + P\left(E_3,\Omega\right)\right). \tag{3.5}$$

Now, for the sequence of partitions $\mathcal{E}_n$, we can manipulate the first two equations in (3.4) to obtain

$$\sigma_{13}\mathcal{H}^{N-1}(S_{n,13}) = \frac{\sigma_{13}}{2}\left(P\left(E_{n,1},\Omega\right) + P\left(E_{n,3},\Omega\right)\right) - \frac{\sigma_{13}}{2}\left(\mathcal{H}^{N-1}(S_{n,12}) + \mathcal{H}^{N-1}(S_{n,23})\right),$$

allowing us to bound the functional on the sequence partition as follows

$$\begin{aligned}\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}_n) &= \frac{\sigma_{13}}{2}\left(P\left(E_{n,1},\Omega\right)+P\left(E_{n,3},\Omega\right)\right)-\frac{\sigma_{13}}{2}\left(\mathcal{H}^{N-1}(S_{n,12})+\mathcal{H}^{N-1}(S_{n,23})\right)\\ &+\sigma_{12}\mathcal{H}^{N-1}(S_{n,12})+\sigma_{23}\mathcal{H}^{N-1}(S_{n,23})\\ &= \frac{\sigma_{13}}{2}\left(P\left(E_{n,1},\Omega\right)+P\left(E_{n,3},\Omega\right)\right)\\ &+\left(\sigma_{12}-\frac{\sigma_{13}}{2}\right)\mathcal{H}^{N-1}(S_{n,13})+\left(\sigma_{23}-\frac{\sigma_{13}}{2}\right)\mathcal{H}^{N-1}(S_{n,23})\\ &\geq \frac{\sigma_{13}}{2}\left(P\left(E_{n,1},\Omega\right)+P\left(E_{n,2},\Omega\right)\right)\\ &+\left(\sigma_{12}+\sigma_{23}-\sigma_{13}\right)\min\left\{\mathcal{H}^{N-1}(S_{n,12}),\mathcal{H}^{N-1}(S_{n,23})\right\}.\end{aligned}$$

Due to Hausdorff measures being positive and conditions (3.2), the second term on the right side of the previous inequality is positive and thus

$$\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}_n) \geq \frac{\sigma_{13}}{2}\left(P\left(E_{n,1},\Omega\right)+P\left(E_{n,3},\Omega\right)\right). \tag{3.6}$$

This inequality, together with (3.5) and lower semicontonuity of the perimeter, results in

$$\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}) \leq \liminf_{n\to+\infty}\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}_n).$$

Step 2: Suppose now that $\mathcal{H}^{N-1}(S_{12}) \leq \varepsilon$ and $\mathcal{H}^{N-1}(S_{23}) \leq \varepsilon$, for an arbitrary $\varepsilon > 0$. In this case we have

$$\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}) \leq \sigma_{13}\mathcal{H}^{N-1}(S_{13}) + C\varepsilon, \tag{3.7}$$

where C is some constant depending on the σ_{ij} [3].

From (3.4) we have that $\mathcal{H}^{N-1}(S_{13}) \leq P\left(E_1,\Omega\right)$ and $\mathcal{H}^{N-1}(S_{13}) \leq P\left(E_3,\Omega\right)$, thus

$$\sigma_{13}\mathcal{H}^{N-2}(S_{13}) \leq \frac{\sigma_{13}}{2}\left(P\left(E_1,\Omega\right)+P\left(E_3,\Omega\right)\right).$$

This fact together with (3.7) yields

$$\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}) \leq \frac{\sigma_{13}}{2}\left(P\left(E_1,\Omega\right)+P\left(E_3,\Omega\right)\right)+C\varepsilon.$$

Since estimates done to obtain (3.6) still hold for this case, in addition to the lower semicontinutiy of the perimeter and the last inequality, we obtain

$$\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}) \leq \liminf_{n\to+\infty}\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}_n) + C\varepsilon.$$

Step 3: Now we proceed to prove the general statement of the theorem. Let us take an arbitrary $\varepsilon > 0$. Since $\mathcal{H}^{N-1} \llcorner S_{ij}$ is a Radon measure, by inner regularity of Radon measures [2, Proposition 1.43], we know that there is a compact set $K_{ij} \subset S_{ij}$ such that

$$\mathcal{H}^{N-1}(S_{ij}\backslash K_{ij}) < \varepsilon,$$

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[3] As the constant is not of importance for us we will use the same notation C regardless if C changes

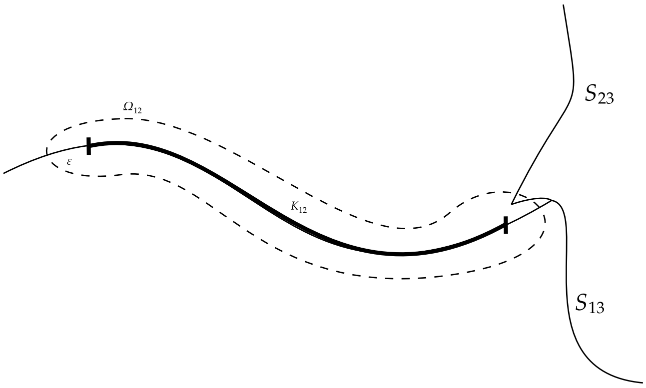

Figure 7: Up close rendition of a three set configuration with the choices of K_{12} and Ω_{12} in the proof of Theorem 3.1. Condition $\mathcal{H}^{N-1}(\Omega_{12} \cap S_{23}) \leq \varepsilon$ can be seen on the right as a small portion of E_3 intersects Ω_{12}.

for each $i < j \in \{1, 2, 3\}$. Notice that these K_{ij} are pairwise disjoint since S_{ij} are also pairwise disjoint due to the Caccioppoli Structure Theorem. Take an $\varepsilon-$neighborhood Ω_{ij} of K_{ij} such that they are also disjoint and that

$$\mathcal{H}^{N-1}(S_{ij} \backslash \Omega_{ij}) < \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}(\Omega_{ij} \cap S_{kl}) < \varepsilon, \tag{3.8}$$

where $k, l \in \{1, 2, 3\}$ such that $(i, j) \neq (k, l)$. The second condition allows us to apply *step* 2 over each Ω_{ij} so that

$$\mathscr{F}_{\Omega_{ij},\mathbf{m}}(\mathcal{E}) \leq \liminf_{n \to +\infty} \mathscr{F}_{\Omega_{ij},\mathbf{m}}(\mathcal{E}_n) + C\varepsilon$$

For simplicity, we write $\Omega_0 = \bigcup_{ij} \Omega_{ij}$ and $\Omega_1 = \Omega \backslash \Omega_0$. From the first condition in (3.8) we can say that

$$\mathscr{F}_{\Omega_1,\mathbf{m}}(\mathcal{E}) < C\varepsilon,$$

and since $\{\Omega_{12}, \Omega_{23}, \Omega_{13}, \Omega_1\}$ is a partition of Ω, previous estimates allow us to attain the following bound

$$\begin{aligned}
\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}) &= \sum_{ij} \mathscr{F}_{\Omega_{ij},\mathbf{m}}(\mathcal{E}) + \mathscr{F}_{\Omega_1,\mathbf{m}}(\mathcal{E}) \\
&\leq \liminf_{n \to +\infty} \sum_{ij} \mathscr{F}_{\Omega_{ij},\mathbf{m}}(\mathcal{E}_n) + C\varepsilon \\
&\leq \liminf_{n \to +\infty} \mathscr{F}_{\Omega_0,\mathbf{m}}(\mathcal{E}_n) + C\varepsilon \\
&\leq \liminf_{n \to +\infty} \mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}_n) + C\varepsilon,
\end{aligned}$$

where the last estimate is obtained from $\Omega_0 \subset \Omega$ and classic measure properties. Taking $\varepsilon \to 0$ in the previous inequality completes the proof. □

Remark 3.1. As the functional is bounded from below, lower semicontinuous, and possesses compactness properties given by Theorem (2.18), we can conclude existence of minimizers for the problem in view of the direct method. In fact, Ambrosio and Braides proved that condition (3.2) is not only necesary but sufficient to ensure lower semicontinuity of $\mathscr{F}_{\Omega,\mathbf{m}}$ in [1].

Sadly, when considering condition (3.3), existence of minimizers cannot be guaranteed, but can be recovered in specific types of domains. To this end, we begin by defining isoperimetrically foliated domains.

Definition 3.2 (Isoperimetrically foliated domains)**.** Let $\Omega \subset \mathbb{R}^N$ be an open set and suppose that for any $0 < m < |\Omega|$ [4], there exists a minimizer for the problem

$$\min \{P(E, \Omega) : E \subset \Omega, |E| = m\}. \tag{3.9}$$

We will say that Ω *is an isoperimetrically foliated domain* if there exists a selection of minimizers $E^{(m)}$ of this problem such that if $m_1 < m_2$ then

$$E^{(m_1)} \subset E^{(m_2)} \quad \text{and} \quad \mathcal{H}^{N-1}\left(\partial E^{(m_1)} \cap \partial E^{(m_2)} \cap \Omega\right) = 0.$$

Remark 3.2. The existence of minimizers for problem (3.9) is guaranteed when Ω is an open and bounded set of finite perimeter, and the fixed volume is $0 < m \leq |\Omega|$, as in view of [10, Proposition 12.30]. If $\Omega = \mathbb{R}^N$ then we get the classic isoperimetric problem, where we know the solution is a ball with volume m. If Ω is the unit square and $m \leq \frac{1}{\pi}$, the minimizers are quarter discs centered at any of the vertices, while if $\frac{1}{\pi} < m < 1 - \frac{1}{\pi}$ the minimizers are vertical or horizontal stripes.

Remark 3.3. Sets $\partial E^{(m)}$ are C^∞ up to null sets with Hausdorff dimension $N-8$ thanks to Theorems 2.22, 2.23 and 2.24, for $\Lambda = 0$. This fact appears important and recurrently in the later sections when talking about regularity.

If we restrict our domain to be isoperimetrically foliated, then we can guarantee existance of minimizers under (3.3), as stated in the following theorem:

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^N$ be an isoperimetrically foliated domain and $\boldsymbol{m} = (m_1, m_2, m_3)$ with $0 < m_i < +\infty$, for $i \in \{1, 2, 3\}$, and $m_1 + m_2 + m_3 = |\Omega|$. Then $\mathscr{F}_{\Omega,\boldsymbol{m}}$ attains its minimum in $\mathcal{A}_{\Omega,\boldsymbol{m}}$.*

Proof. Take $\mathcal{E} = (E_1, E_2, E_3) \in \mathcal{A}_{\Omega,\mathbf{m}}$. Thanks to (3.3) we can estimate the functional as follows:

$$\begin{aligned}\mathscr{F}_{\Omega,\mathbf{m}} &\geq \sigma_{12}\mathcal{H}^{N-1}(S_{12}) + \sigma_{23}\mathcal{H}^{N-1}(S_{23}) + (\sigma_{12} + \sigma_{23})\mathcal{H}^{N-1}(S_{13}) \\ &= \sigma_{12}\left(\mathcal{H}^{N-1}(S_{12}) + \mathcal{H}^{N-1}(S_{13})\right) + \sigma_{23}\left(\mathcal{H}^{N-1}(S_{23}) + \mathcal{H}^{N-1}(S_{13})\right) \\ &= \sigma_{12}P(E_1, \Omega) + \sigma_{23}P(E_3, \Omega) \\ &\geq \sigma_{12}P\left(E^{(m_1)}, \Omega\right) + \sigma_{23}P\left(E^{(m_1+m_2)}, \Omega\right),\end{aligned} \tag{3.10}$$

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[4] It is worth mentioning that $|\Omega|$ does not necessarily need to be finite.

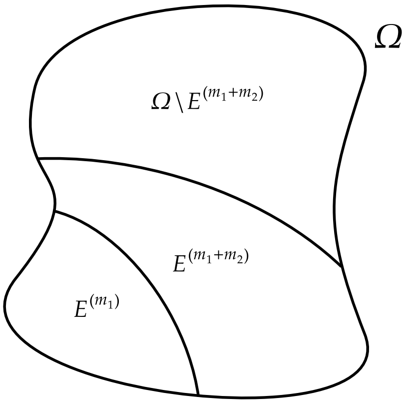

Figure 8: Optimal configuration for $\mathscr{F}_{\Omega,\mathbf{m}}$ in an isoperimetrically foliated domain. Here, $E_1 = E^{(m_1)}$, $E_2 = E^{(m_1+m_2)}\setminus E^{(m_1)}$ since $E^{(m_1)}$ is contained in $E^{(m_1+m_2)}$, and E_3 would be the rest.

where the last inequality holds thanks to the sets $E^{(m_1)}$ and $E^{(m_1+m_2)}$ minimizing each perimeter, and noticing that the perimeter of E_3 in Ω will be the same as the perimeter of its complement E_3^c in Ω, which has volume $m_1 + m_2$.

Now, we define $\mathcal{E}^* = (E_1^*, E_2^*, E_3^*) = \left(E^{(m_1)}, E^{(m_1+m_2)}\setminus E^{(m_1)}, \Omega\setminus E^{(m_1+m_2)}\right)$ and we prove that it belongs to $\mathcal{A}_{\Omega,\mathbf{m}}$ and the functional in this partition has the value of the lower bound in (3.10). By definition, we have that

$$|E_1^*| = |E^{(m_1)}| = m_1 \quad \text{and} \quad |E_2^*| = |E^{(m_1+m_2)}| - |E^{(m_1)}| = m_2,$$

as well as $|\Omega\setminus(E_1^* \cup E_2^* \cup E_3^*)| = |\Omega\setminus\Omega| = 0$. Furthermore, all three sets are mutually disjoint by definition and the fact that $E^{(m_1)} \subset E^{(m_1+m_2)}$ since Ω is isopermetrically foliated. Thus $|E_i^* \cap E_j^*| = 0$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. This proves $\mathcal{E}^* \in \mathcal{A}_{\Omega,\mathbf{m}}$ and, since E_1^* does not share a boundary with E_3^*, we have

$$\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}^*) = \sigma_{12}\mathcal{H}^{N-1}(S_{12}^*) + \sigma_{23}\mathcal{H}^{N-1}(S_{23}^*) = \sigma_{12}P\left(E^{(m_1)}, \Omega\right) + \sigma_{23}P\left(E^{(m_1+m_2)}, \Omega\right),$$

completing the proof. □

Remark 3.4. Once again, thanks to the regularity of sets $\partial E^{(m)}$ in view of Remark 3.3, the sets constructed in the proof of Theorem 3.3 have smooth boundaries up to a null set of Hausdorff dimension of at most $N - 8$.

Remark 3.5. The existence of an isoperimetric foliation of Ω is sufficient but not necessary for the existence of minimizers. In fact, the existence is ensured if there exists $E^{(m_1)}$ and $E^{(m_1+m_2)}$ minimizers of the perimeter problem with fixed volume constraint, such that $E^{(m_1)} \subset E^{(m_1+m_2)}$ and $\mathcal{H}^{N-1}(\mathcal{F}E^{(m_1)} \cap \mathcal{F}E^{(m_1+m_2)}) = 0$.

These extra assumptions over the sets or domain are crucial, as if we do not take them we can construct examples where $\mathscr{F}_{\Omega,\mathbf{m}}$ does not attain a minimum, but first we can mention some examples. Classic ones are $\mathbb{R}^N$ or $B_1(0)$, since balls with the given volumes can foliate both. The unit square in $\mathbb{R}^2$ is also isoperimetrically foliated when $m_1, m_3 < \frac{\pi}{16}$ and $m_2 = 1 - m_1 - m_2$. Moreover, the minimizer is comprised of two quarter circles centered in opposite vertices of the square, and their complement in Ω. Mainly:

$$\begin{aligned} E_1^* &= \Omega \cap \left\{ (x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 < \frac{4m_1}{\pi} \right\}, \\ E_3^* &= \Omega \cap \left\{ (x,y) \in \mathbb{R}^2 \,:\, (x-1)^2 + (y-1)^2 < \frac{4m_2}{\pi} \right\}, \\ E_2^* &= \Omega \backslash \overline{(E_1^* \cup E_2^*)}. \end{aligned}$$

In contrast, when $m_1 < \frac{1}{\pi}$ and $\frac{1}{\pi} < m_2 < 1 - \frac{1}{\pi}$, then the unit square is not isoperimetrically foliated. Indeed, $E^{(m_1)}$ will be a quarter circle and $E^{(m_2)}$ a stripe that cannot contain it, i.e. $E^{(m_1)} \not\subset E^{(m_2)}$.

In the same spirit as this last example, the following proposition shows that there is no minimizer under certain conditions.

Proposition 3.1. Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, $\sigma_{12} = \sigma_{23} = 1$, $\sigma_{13} > 2$, $m_1 = \frac{\pi(1+\varepsilon^2)}{16}$, $m_2 = \frac{1}{2} - m_1$, $m_3 = \frac{1}{2}$, for some $0 < \varepsilon << 1$. Then $\mathscr{F}_{\Omega,\mathbf{m}}$ has no minimum on $\mathcal{A}_{\Omega,\mathbf{m}}$.

Proof. We will reason by contradiction, i.e. suppose there is and call it $\mathcal{E}^* = (E_1^*, E_2^*, E_3^*)$. By Proposition 4.3 which will be covered in detail in the next chapter, we have that $\mathcal{H}^{N-1}(S_{13}) = 0$, and by Theorem 2.26, S_{12} and S_{12} would have constant curvature in $\mathbb{R}^2$, which we know are either straight lines or circular arcs meeting $\partial\Omega$ orthogonally.

Suppose both are straight lines. This means $\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}^*) = 2$. But, if we take $\mathcal{E}$ such that

$$\begin{aligned} E_1 &= \Omega \cap \left(\left\{ (x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 < \frac{1}{2} \right\} \cup \left\{ (x,y) \in \mathbb{R}^2 \,:\, x^2 + (y-1)^2 = \frac{\varepsilon}{2} \right\} \right), \\ E_3 &= \Omega \cap \left\{ (x,y) \in \mathbb{R}^2 \,:\, x > \frac{1}{2} \right\}, \\ E_2 &= \Omega \backslash \overline{(E_1 \cup E_3)}, \end{aligned}$$

we obtain

$$\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}) = 1 + \frac{\sqrt{\pi(1+\varepsilon)}}{4} < 2 = \mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}^*).$$

It is easy to see that $\mathcal{E} \in \mathcal{A}_{\Omega,\mathbf{m}}$ since each set has the prescribed volumes, the x component in E_1 are strictly less than $\frac{1}{2}$ and thus not intersecting with E_3, and by defintion E_2 is does not intersect with the other sets, and they make up a partition of Ω. This contradicts minimality of $\mathcal{E}^*$.

Now, suppose both are circular arcs with centers in corners of $\partial\Omega$. Notice that their centers cannot be in consecutive corners due to the volume constraints, as these would then overlap and thus not be in $\mathcal{A}_{\Omega,\mathbf{m}}$, so they must be in opposite corners. In this case, they would have the shape

$$\begin{aligned} E_1^* &= \Omega \cap \left\{ (x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 < \frac{(1+\varepsilon)^2}{4} \right\}, \\ E_3^* &= \Omega \backslash \left\{ (x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 < \frac{2}{\pi} \right\}, \\ E_2^* &= \Omega \backslash \overline{(E_1^* \cup E_3^*)}. \end{aligned}$$

This yields

$$\mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}^*) = \sqrt{\frac{\pi}{2}} + \frac{\pi(1+\varepsilon)}{4} > 2,$$

so still greater than the straight line configuration, and by considering $\mathcal{E}$ as in the previous case, this contradicts once again the minimality of $\mathcal{E}^*$.

Failing in both cases, $\mathscr{F}_{\Omega,\mathbf{m}}$ does not attain its minimum in $\mathcal{A}_{\Omega,\mathbf{m}}$. □

Remark 3.6. The construction of the minimizing configuration in Theorem 3.3 resembles that of the configurations of interest. Graphically, one can compare Figures 8 and 6 to notice that in both cases, E_1 and E_3 do not touch. In the first case, it reduces to $\mathcal{H}^{N-1}(S_{13}) = 0$, while our interests lie when $\sigma_{13} = 0$. Regardless, any of the cases would result in sets behaving in such manner.

Naturally, the next step is to study regularty and come up with density estimates for these minimizers. We mentioned little about the regularity of the boundary in this chapter for minimizers of the perimeter problem with volume constraints, but we will delve deeper for our functional in the next sections.

4 Regularity and Structural Properties of Partitions

We are now aware that minimizers for $\mathscr{F}_{\Omega,\mathbf{m}}$ can exist under certain conditions. A natural next step is to wonder about their shape, or in a more passive but still significant way, wonder about regularity and approximation. Indeed the central focus of this chapter is to shine a light on these results and study in detail two important theorems: Strict Interior Approximation Theorem and Elimination Theorem. The former, as the name implies, gives us a way to approximate the sets from within the domain, which is useful when having partitions as the idea is to stay within Ω when considering competitor configurations. The latter, will give us insight in the geometry of the boundaries, as it will exclude unnecessary interactions in the interfaces. These two theorems provide a foundation for admissible and minimizing configurations, inherent to these types of problems.

4.1 Strict Interior Approximation Theorem

In Chapter 2, we saw that sets of finite perimeter can be approximated by smooth sets that also approximate the perimeter, in view of Theorem 2.5. Nevertheless, we cannot generally hope to approximate an open set Ω strictly from within.

For bounded Lipschitz domains this problem has been handled, but when considering a general set, no strict approximation from within has been achieved. In [13] it was shown that with an extra assumption for the boundary of the open set Ω, asking for the boundary to be well behaved, we can obtain strict approximation with smooth sets that are strictly contained in Ω. The following theorem is taken from the aforementioned paper which we will study in detail here.

Theorem 4.1 (Strict interior approximation of the perimeter)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded and open set such that*

$$P(\Omega) = \mathcal{H}^{N-1}(\partial\Omega). \tag{4.1}$$

Then, for every $\varepsilon > 0$ there exists an open set Ω_ε with smooth boundary such that

$$\Omega_\varepsilon \Subset \Omega, \quad \Omega \backslash \Omega_\varepsilon \subset \mathcal{N}_\varepsilon(\partial\Omega) \cap \mathcal{N}_\varepsilon(\partial\Omega_\varepsilon), \quad P(\Omega_\varepsilon) \leq P(\Omega) + \varepsilon. \tag{4.2}$$

Here we used $\mathcal{N}_\varepsilon$ for ε-neighborhoods of sets in $\mathbb{R}^N$.

Before the proof of this theorem, some remarks regarding the statement of this result are warranted.

Remark 4.1.

1. As a consequence of Theorem 2.13, whenever the perimeter of Ω is finite, we have the relation
$$P(\Omega) = \mathcal{H}^{N-1}(\mathcal{F}\Omega).$$
This allows us to rephrase hypothesis (4.1) into
$$\mathcal{H}^{N-1}(\partial\Omega \backslash \mathcal{F}\Omega) = 0. \tag{4.3}$$

2. From the conclusion of the theorem we can infer a couple of results. Mainly,
$$\bigcup_{\varepsilon>0}\Omega_\varepsilon=\Omega \quad \text{and} \quad \lim_{\varepsilon\searrow 0}|\Omega\backslash\Omega_\varepsilon|=0.$$
By the lower semicontinuity of the perimeter and the smoothness of $\partial\Omega_\varepsilon$, we have that the perimeter of Ω_ε approximates the perimeter of Ω, as was intended.

Continuing on with a useful lemma necessary for the proof of the theorem, let us establish some notation. Let T be a vector subspace of $\mathbb{R}^N$. For any $y\in\mathbb{R}^N$, we denote by $T(y)$ as the orthogonal projection of y onto T, and by $T^\perp(y)=y-T(y)$ the orthogonal complement of $T(y)$. We also define the cone with axis T and opening parameter ε as
$$C_\varepsilon(T)=\left\{y\in\mathbb{R}^N\,:\,\|T^\perp(y)\|\leq\varepsilon\|T(y)\|\right\},$$
and the sets
$$B_r^\pm(x)=\{y\in B_r(x)\,:\,\pm y\cdot v_\Omega(x)\geq 0\},$$
which represents the intersection of the ball centered at x and of radius $r>0$, and the halfspace determined by the outward or inward unit normal of Ω at x, respectively.

Lemma 4.2. *Let $\Omega\subset\mathbb{R}^N$ be a $\mathcal{L}^N$−measurable set. For every $\varepsilon>0$, $\mathcal{F}\Omega$ can be decomposed into countably many disjoint Borel subsets $\{R_i\}$ such that if $x\in R_i$, then*
$$\left|\Omega\cap B_r^+(x)\right|\leq\varepsilon^2r^N$$
for all $r\in\left[0,\frac{1}{i}\right]$, and
$$R_i\cap B_{\frac{1}{i}}(x)\subset x+C_\varepsilon\left(\operatorname{Tan}(\mathcal{F}\Omega,x)\right).$$

Proof. We start off by defining the following sets for $j\in\mathbb{N}$:
$$A_j=\left\{x\in\mathcal{F}\Omega\,:\,\begin{array}{l}\mathcal{H}^{N-1}\left(\mathcal{F}\Omega\cap B_\rho(x)\right)\geq\frac{1}{2}\omega_{N-1}\rho^{N-1}\\ \left|\Omega\cap B_\rho^+(x)\right|\leq\varepsilon^2\rho^N\end{array}\quad\forall\rho\leq\frac{1}{j}\right\}\setminus\bigcup_{t=1}^{j-1}A_t \tag{4.4}$$
Clearly, these sets are disjoint, as each is defined in the complement of the previous ones. They are Borel sets since the functions
$$x\mapsto\int_\Omega\chi_{B_\rho^+(x)}\,dy=|\Omega\cap B_\rho^+(x)| \quad\text{and}\quad x\mapsto\int_{\mathcal{F}\Omega}\chi_{B_\rho(x)}\,d\mathcal{H}^{N-1}=\mathcal{H}^{N-1}(\mathcal{F}\Omega\cap B_\rho(x))$$
are measurable by Fubini's Theorem. Moreover, by Theorem 2.13,
$$\bigcup_{j\in\mathbb{N}}A_j=\mathcal{F}\Omega.$$

Indeed, from property *ii*,
$$\lim_{\rho\searrow 0}\frac{\mathcal{H}^{N-1}(\mathcal{F}\Omega\cap B_\rho(x))}{\omega_{N-1}\rho^{N-1}}=1,$$

so if we fix $\varepsilon < \frac{1}{2}$, there exists $\rho > 0$ small enough such that

$$1 - \frac{\mathcal{H}^{N-1}(\mathcal{F}\Omega \cap B_\rho(x))}{\omega_{N-1}\rho^{N-1}} \leq \frac{1}{2}$$

$$\frac{1}{2}\omega_{N-1}\rho^{N-1} \leq \mathcal{H}^{N-1}(\mathcal{F}\Omega \cap B_\rho(x)).$$

Similarly, using property i, let H^- and H^+ be the inner and outer halfspaces at x, respectively. Then,

$$\left| \frac{\Omega - x}{\rho} \triangle H^- \right| \to 0,$$

and intersecting with $B_1(0)$ and then with H^+ gives

$$\left| \frac{\Omega - x}{\rho} \cap B_1^+(0) \triangle \emptyset \right| = \frac{|\Omega \cap B_\rho^+(x)|}{\rho^N} \to 0,$$

which implies

$$|\Omega \cap B_\rho^+(x)| \leq \varepsilon^2 \rho^N.$$

So, for a small enough ρ so that both inequalities hold, and $\rho \leq \frac{1}{j}$ for some $j \in \mathbb{N}$, we conclude that $x \in A_j$.

Let $T = \operatorname{Tan}(\mathcal{H}^{N-1} \llcorner \mathcal{F}\Omega, x) = \operatorname{Tan}(\mathcal{F}\Omega, x)$. We claim that for every $x \in A_j$, there exists $r_x > 0$ such that

$$A_j \cap B_{r_x}(x) \subset x + C_\varepsilon(T). \tag{4.5}$$

We prove this by contradiction. Suppose not, i.e., we can fix a radius $r_1 > 0$ and find $x_1 \in A_j \cap B_{r_1}(x) \setminus (x + C_\varepsilon(T))$. Iteratively, we can construct a sequence

$$\{x_n\}_{n \in \mathbb{N}} \subset A_j \setminus (x + C_\varepsilon(T))$$

such that $x_n \to x$ as $n \to \infty$.

Set $\rho_n = \|x - x_n\|$. We also claim that

$$B_{c\rho_n}(x_n) \cap (x + C_{\frac{\varepsilon}{2}}(T)) = \emptyset, \tag{4.6}$$

for some constant $c(\varepsilon) > 0$. Assume, for simplicity, $x = 0$. Let $y \in B_{c\rho_n}(x_n)$. Then,

$$\|y\| \geq \|x_n\| - \|y - x_n\| > \rho_n - c\rho_n = (1-c)\rho_n,$$

$$\|T^\perp(y)\| \geq \|y\| - \|T(y)\| > (1-c)\rho_n - \|T(y)\|,$$

$$\|T(y)\| \leq \|T(x_n)\| + \|T(x_n) - T(y)\| < (1+c)\rho_n.$$

Therefore,

$$\frac{\|T^\perp(y)\|}{\|T(y)\|} \geq \frac{(1-c)\rho_n}{(1+c)\rho_n} - 1.$$

If we choose $c < \frac{2}{\varepsilon+4}$, then the above inequality implies $y \notin C_{\frac{\varepsilon}{2}}(T)$, proving (4.6).

Now, since $B_{c\rho_n}(x_n) \subset B_{(c+1)\rho_n}(x)$, we have

$$\mathcal{H}^{N-1} \llcorner \mathcal{F}\Omega \left(B_{(c+1)\rho_n}(x) \setminus (x + C_{\frac{\varepsilon}{2}}(T))\right) \geq \mathcal{H}^{N-1}(\mathcal{F}\Omega \cap B_{c\rho_n}(x_n)).$$

From the definition of A_j, for n large enough,

$$\mathcal{H}^{N-1} \llcorner \mathcal{F}\Omega \left(B_{(c+1)\rho_n}(x) \setminus (x + C_{\frac{\varepsilon}{2}}(T))\right) \geq \frac{1}{2}\omega_{N-1}(c\rho_n)^{N-1}.$$

Rescaling Ω to $\Omega_n = \frac{\Omega - x}{\rho_n}$, this becomes

$$\mathcal{H}^{N-1} \llcorner \mathcal{F}\Omega_n \left(B_{c+1}(0) \setminus C_{\frac{\varepsilon}{2}}(T)\right) \geq \frac{1}{2}\omega_{N-1}c^{N-1} > 0.$$

But any weak* limit of the measures $\mathcal{H}^{N-1} \llcorner \mathcal{F}\Omega_n$ must be supported on T by Theorem 2.13, which contradicts this inequality. Therefore, (4.5) holds.

To complete the proof, fix an enumeration $(j(i), k(i)) \in \mathbb{N} \times \mathbb{N}$ such that $j(i), k(i) \leq i$, and define

$$R_i = \left\{ x \in A_{j(i)} \,:\, r_x \geq \frac{1}{k(i)} \right\} \setminus \bigcup_{l=1}^{i-1} R_l.$$

These sets are pairwise disjoint, just like the A_j, and

$$\bigcup_{i \in \mathbb{N}} R_i = \mathcal{F}\Omega.$$

Indeed, for any $x \in \mathcal{F}\Omega$, there exists $j \in \mathbb{N}$ such that $x \in A_j$ and $r_x > 0$, so we can find $k \in \mathbb{N}$ with $r_x \geq \frac{1}{k}$. Then, choosing i such that $j(i) = j$ and $k(i) = k$, and $i \geq \max\{j, k\}$, gives $x \in R_i$. Moreover, for $x \in R_i$, we have

$$|\Omega \cap B_r^+(x)| \leq \varepsilon^2 r^N \quad \text{for } r \in \left[0, \frac{1}{j}\right] \subset \left[0, \frac{1}{i}\right],$$

and since $r_x \geq \frac{1}{i}$,

$$R_i \cap B_{\frac{1}{i}}(x) \subset A_j \cap B_{r_x}(x) \subset x + C_\varepsilon(T).$$

Hence, the sets R_i satisfy the desired properties. □

Remark 4.2. As a fun fact, whenever $P(\Omega)$ is finite, it coincides with the total variation $|D\chi_\Omega|(\mathbb{R}^N)$. So the approximating sequence $\left\{\chi_{\Omega_{\frac{1}{n}}}\right\}_{n \in \mathbb{N}}$, which will appear in the proof of the theorem further down the line, converges strictly to χ_Ω in $BV(\mathbb{R}^N)$.

Having this decomposition for $\mathcal{F}\Omega$ allows us to prove the Strict interior approximation Theorem.

Proof of Theorem 4.1. We begin by relaxing some conditions that the approximating sets Ω_ε must satisfy. Notably, it is not necessary for them to have smooth boundaries as we can later smooth them via mollification, as in the proof of Theorem 2.5. Specifically, we take $u_n = \chi_{\Omega_\varepsilon} * \rho_n$, where ρ_n is a mollifier, which converges to $\chi_{\Omega_\varepsilon}$ in $L^1(\Omega)$. Then, by Sard's Lemma (see [10, Lemma 13.15]), the level sets $F^n_\alpha = \{u_n > \alpha\}$ have smooth boundaries for L^1-a.e. $\alpha \in (0,1)$. Choosing such an α, we obtain a sequence converging in measure to Ω_ε with converging perimeters.

It also suffices to prove the inclusion

$$\Omega \backslash \Omega_\varepsilon \subset \mathcal{N}_\varepsilon(\partial\Omega), \tag{4.7}$$

in place of (4.2). To see this, suppose we construct a sequence $\tilde{\Omega}_{\frac{1}{n}}$ satisfying (4.7) for each $n \in \mathbb{N}$, and assume by contradiction that no such $\tilde{\Omega}_{\frac{1}{n}}$ satisfies (4.2) for $\varepsilon \geq \frac{1}{n}$. That is,

$$\Omega \backslash \tilde{\Omega}_{\frac{1}{n}} \not\subset \mathcal{N}_{\frac{1}{n}}\left(\partial\tilde{\Omega}_{\frac{1}{n}}\right).$$

Then we can choose $x_n \in \left(\Omega\backslash\tilde{\Omega}_{\frac{1}{n}}\right) \setminus \mathcal{N}_{\frac{1}{n}}\left(\partial\tilde{\Omega}_{\frac{1}{n}}\right)$, which converges to some $x \in \overline{\Omega}$ as $\operatorname{dist}\left(x_n, \overline{\Omega}\right) \to 0$. For large n, we have

$$\left|\operatorname{dist}\left(x, \tilde{\Omega}_{\frac{1}{n}}\right) - \operatorname{dist}\left(x_n, \tilde{\Omega}_{\frac{1}{n}}\right)\right| \leq |x_n - x| < \frac{\varepsilon}{2},$$

and hence

$$\begin{aligned} \operatorname{dist}\left(x, \tilde{\Omega}_{\frac{1}{n}}\right) &> \operatorname{dist}\left(x_n, \tilde{\Omega}_{\frac{1}{n}}\right) - \frac{\varepsilon}{2} \\ &= \operatorname{dist}\left(x_n, \partial\tilde{\Omega}_{\frac{1}{n}}\right) - \frac{\varepsilon}{2}. \end{aligned}$$

Since $x_n \notin \mathcal{N}_\varepsilon(\partial\tilde{\Omega}_{\frac{1}{n}})$, then $\operatorname{dist}\left(x, \tilde{\Omega}_{\frac{1}{n}}\right) > \frac{\varepsilon}{2}$. Taking $y \in \Omega$ with $\operatorname{dist}\left(y, \tilde{\Omega}_{\frac{1}{n}}\right) \geq \frac{\varepsilon}{2}$ so $y \in \Omega\backslash\tilde{\Omega}_{\frac{1}{n}} \subset \mathcal{N}_{\frac{1}{n}}(\partial\Omega)$ which is a contradiction since $\operatorname{dist}(y, \partial\Omega) > \frac{1}{n}$.

We proceed to construct the approximating sets. If $P(\Omega) = +\infty$, we simply set

$$\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{\varepsilon}{2}\},$$

which trivially satisfies (4.2).

Focus now in the case $P(\Omega) < +\infty$. Fix $\varepsilon \leq \frac{1}{2}$ and consider the sets R_i from Lemma 4.2. By the definition of Hausdorff measure, for $\delta = \frac{2}{3}\min\left\{\frac{1}{i}, \varepsilon\right\}$, we can cover R_i by balls $B_{r_{ij}}(x_{ij})$ with $3r_{ij} \leq \min\left\{\frac{1}{i}, \varepsilon\right\}$ such that

$$R_i \subset \bigcup_j B_{r_{ij}}(x_{ij}) \quad \text{and} \quad \sum_j \omega_{N-1} r_{ij}^{N-1} \leq \mathcal{H}^{N-1}(R_i) + \frac{\varepsilon}{2^i}. \tag{4.8}$$

For each ball, choose $\tilde{x}_{ij} \in R_i \cap B_{r_{ij}}(x_{ij})$ and define $T_{ij} = \mathrm{Tan}(\mathcal{F}\Omega, \tilde{x}_{ij})$. The smallness of r_{ij} ensures the estimate

$$|\Omega \cap B^+_{3r_{ij}}(\tilde{x}_{ij})| \leq \varepsilon^2 3^N r_{ij}^N. \tag{4.9}$$

Moreover, we have

$$R_i \cap B_{r_{ij}}(x_{ij}) \subset \tilde{x}_{ij} + C_\varepsilon(T_{ij}),$$

which yields the inclusion

$$R_i \cap B_{r_{ij}}(x_{ij}) \subset \left\{y : \|T_{ij}(y - x_{ij})\| < r_{ij},\ \|T_{ij}^\perp(y - \tilde{x}_{ij})\| < 2\varepsilon r_{ij}\right\}.$$

To better control the boundary, define

$$\mathcal{C}(t) = \left\{y \in \Omega : \|T_{ij}(y - x_{ij})\| < r_{ij},\ T_{ij}^\perp(y - \tilde{x}_{ij}) = t\nu_\Omega(\tilde{x}_{ij})\right\}.$$

Then using (4.9),

$$\int_{2\varepsilon r_{ij}}^{3\varepsilon r_{ij}} \mathcal{H}^{N-1}(\mathcal{C}(t))\,dt \leq |\Omega \cap B^+_{3r_{ij}}(\tilde{x}_{ij})| \leq \varepsilon^2 3^N r_{ij}^N,$$

hence, by integral properties [5], we can choose $h_{ij} \in [2\varepsilon r_{ij}, 3\varepsilon r_{ij}]$ such that

$$\mathcal{H}^{N-1}(\mathcal{C}(h_{ij})) \leq \frac{\varepsilon^2 3^N r_{ij}^N}{\varepsilon r_{ij}} = \varepsilon 3^N r_{ij}^{N-1}. \tag{4.10}$$

Let

$$C_{ij} = \left\{y : \|T_{ij}(y - x_{ij})\| < r_{ij},\ \|T_{ij}^\perp(y - \tilde{x}_{ij})\| < h_{ij}\right\} \Subset B_\varepsilon(\tilde{x}_{ij}).$$

The boundary ∂C_{ij} consists of a top face, a bottom face, and side walls. Their measures are bounded by (4.10), $\omega_{N-1} r_{ij}^{N-1}$, and $2h_{ij}(N-1)\omega_{N-1} r_{ij}^{N-2} \leq 6\varepsilon(N-1)\omega_{N-1} r_{ij}^{N-1}$, respectively. Thus,

$$\mathcal{H}^{N-1}(\Omega \cap \partial C_{ij}) \leq \left(\omega_{N-1} + 6\varepsilon(N-1)\omega_{N-1} + \varepsilon 3^N\right) r_{ij}^{N-1}. \tag{4.11}$$

To cover $\partial\Omega \setminus \mathcal{F}\Omega$, which has zero $\mathcal{H}^{N-1}-$measure by (4.1), choose balls $B_{\rho_k}(y_k)$ with $\rho_k \leq \frac{\varepsilon}{2}$ and $B_{\rho_k}(y_k) \cap \partial\Omega \neq \emptyset$ such that

$$\sum_k \rho_k^{N-1} \leq \varepsilon.$$

Then $\{C_{ij}, B_{\rho_k}(y_k)\}$ covers $\partial\Omega$, which is compact, so we can extract a finite subcover, which we will index in the same way for sake of simplicity, and define

$$S = \bigcup C_{ij} \cup \bigcup B_{\rho_k}(y_k), \quad \Omega_\varepsilon = \Omega \setminus \overline{S}.$$

The set Ω_ε is open, relatively compact in Ω, and satisfies $\Omega \setminus \Omega_\varepsilon \subset \mathcal{N}_\varepsilon(\partial\Omega)$.

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[5] If $\int_a^b f(x)\,dx \leq M$, then there exists $x \in [a,b]$ such that $f(x) \leq \frac{M}{b-a}$.

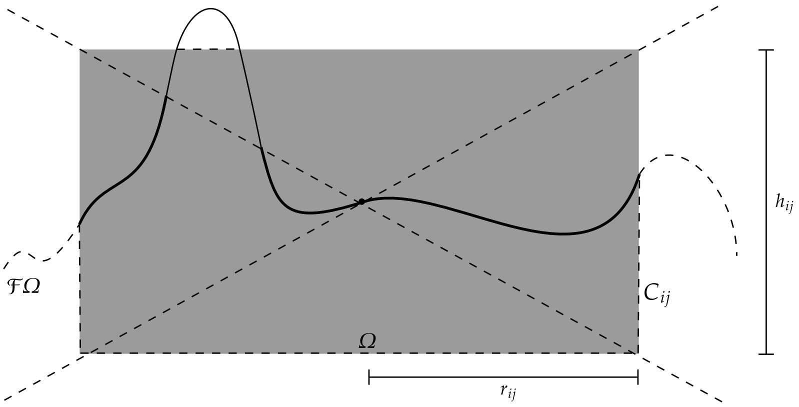

Figure 9: Covering cylinders C_{ij} from Theorem 4.1. The thick curve portion are contained inside the cone and controlled by $\omega_{N-1}r_{ij}^{N-1}$, the less thick curve is the small section controlled by $\varepsilon 3^N r_{ij}^{N-1}$, and the dotted portions in the boundary can by $6\varepsilon(N-1)\omega_{N-1}r_{ij}^{N-1}$.

To estimate the perimeter,

$$\begin{aligned}
P(\Omega_\varepsilon) &\leq \mathcal{H}^{N-1}(\partial\Omega_\varepsilon) \\
&\leq \sum_{i,j}\mathcal{H}^{N-1}(\Omega\cap\partial C_{ij}) + \sum_k \mathcal{H}^{N-1}(B_{\rho_k}(y_k)) \\
&\leq (1+A_N\varepsilon)\sum_{i,j}\omega_{N-1}r_{ij}^{N-1} + A_N\sum_k \rho_k^{N-1} \\
&\leq (1+A_N\varepsilon)\left(\sum_i \mathcal{H}^{N-1}(R_i) + \frac{\varepsilon}{2^i}\right) + A_N\varepsilon \\
&= (1+A_N\varepsilon)(P(\Omega)+\varepsilon) + A_N\varepsilon \\
&\leq P(\Omega) + (1 + A_N P(\Omega) + 2A_N)\,\varepsilon,
\end{aligned}$$

where $A_N = \max\left\{6(N-1) + \frac{3^N}{\omega_{N-1}}, N\omega_{N-1}\right\}$. Rescaling ε yields the desired approximation, completing the proof. $\square$

The reader can notice that the results in this section do not apply only to minimizers of our functional $\mathscr{F}_{\Omega,\mathbf{m}}$. It is a more general result, but will appear with some importance in regularity from the next section of this chapter.

4.2 Elimination Theorem and Regularity Results

A backbone to some of these results is the Elimination Theorem or sometimes called Infiltration Theorem, as it describes how, for locally minimizing configurations, if the sets are separated by a surface inside a ball, then in a smaller ball we will find only two sets being separated. Essentially, we will always find only two sets interacting with each other locally, allowing us to construct competitor sets with less trouble.

This result has been proven initially if the interfaces were flat in [14]. In contrast, the Elimination Theorem here proves that regardless of the shape of the surface, just based on the minimality of the configuration, we are in a framework where we have the elimination property. In [9] the theorem is proved in this general case, for $M \in \mathbb{N}$ number of chambers partitioning Ω. Considering only three chambers simplifies the process completely but is worth studying in this thesis. In a first approach, in order to work under lower semicontinuity, we will need a strict version of condition (3.2), this is,

$$\begin{array}{ll} \sigma_{ij} < \sigma_{ik} + \sigma_{kj}, & \text{for all } i,j,k \in \{1,2,3\}, \text{ with } i \neq j \neq k, \, i \neq k \\ \sigma_{ij} = \sigma_{ji}, & \text{for all } i,j \in \{1,2,3\}, \text{ with } i \neq j, \end{array} \tag{4.12}$$

or equivalently

$$\sigma_{ij} \leq \sigma_{ik} + \sigma_{kj} - \delta, \quad \text{for all } i,j,k \in \{1,2,3\}, \text{ with } i \neq j \neq k, \, i \neq k, \tag{4.13}$$

for some $\delta > 0$.

Let us get acquainted with the elimination property:

Property (EP)**.** Let $\mathcal{E} = (E_1, E_2, E_3)$ be a locally minimizing configuration in $B_R \subset \Omega$. We say $\mathcal{E}$ has the *elimination property*, denoted by (EP), if there exists constants $\eta, r_0 > 0$ with $r_0 < R$, such that, if $0 < \rho < r_0$, then

$$|E_k \cap B_\rho| \leq \eta \rho^N \quad \Rightarrow \quad |E_k \cap B_{\frac{\rho}{2}}| = 0, \tag{4.14}$$

for $k \in \{1,2,3\}$.

Remark 4.3. In [9], expression (4.14) would have $k = 3$ as Leonardi fixes the first two sets of $\mathcal{E}$. Nevertheless, one can fix any two sets of the minimizing configuration to obtain the remaining set as the one being excluded locally, which is why we consider k any number in $\{1,2,3\}$. Regardless, without loss of generality, the proofs will be done for $k = 3$.

Now, we present the aforementioned Elimination Theorem.

Theorem 4.3 (Elimination)**.** *Let $\mathcal{E}^* = (E_1^*, E_2^*, E_3^*)$ be a locally minimizing Caccioppoli partition for $\mathscr{F}_{\Omega,m}$ in $B_R \subset \Omega$, and that coefficients σ_{ij} satisfy* (4.12)*. Then $\mathcal{E}^*$ has (EP).*

The proof of this theorem is based on two smaller results that are interesting to study on their own. Due to the fact that we will do compact variations of the partition to suggest competitor partitions for the minimizer, we introduce the notion of a cut.

Definition 4.4 (Cut)**.** For a Caccioppoli partition $\mathcal{E} = (E_1, E_2, \ldots, E_M)$ of Ω, a *cut $\mathcal{K}$ relative to E_1, E_2* is a bipartition of the set of indexes $\{1, 2, ..., M\}$ into two sets K_1 and K_2 such that $i \in K_i$ for $i = 1, 2$.

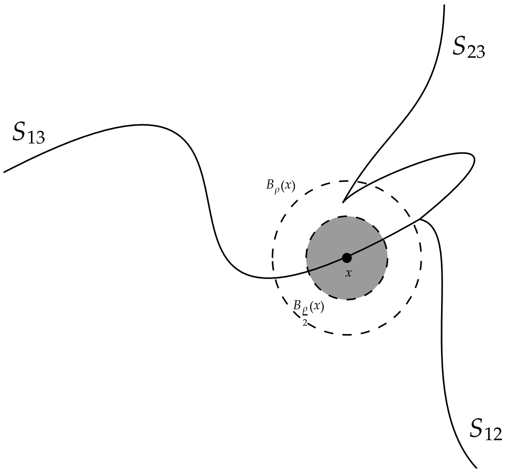

Figure 10: Elimination property applied to three chambers.

With this we can define competitor partitions in a smaller ball of radius $r < R$. We call this new partition $\mathcal{E}^r = \mathcal{E}^{\mathcal{K},r} = \left(\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_M\right)$, where

$$\hat{E}_i = \begin{cases} E_i \backslash B_r & i > 2 \\ E_i \cup \bigcup_{j \in K_i} (E_j \cap B_r) & \text{otherwise,} \end{cases}$$

for $i \in \{1, 2, \ldots, M\}$. We will also denote the variation of our functional over these partitions by

$$\Delta \mathscr{F} = \mathscr{F}_{B_R,\mathbf{m}}(\mathcal{E}^r) - \mathscr{F}_{B_R,\mathbf{m}}(\mathcal{E}) \quad \text{and} \quad \Delta_r \mathscr{F} = \mathscr{F}_{B_r,\mathbf{m}}(\mathcal{E}^r) - \mathscr{F}_{B_r,\mathbf{m}}(\mathcal{E}),$$

and the improperly called area of $\mathcal{K}$ inside B_r as

$$A_r^{\mathcal{K}} = \sum_{\substack{h \in K_1, \\ t \in K_2, \\ (h,t) \neq (1,2)}} \mathcal{H}^{N-1}(S_{ij}^r),$$

where $S_{ij}^r = S_{ij} \cap B_r$. Additionally, we will write $\hat{S}_{ij}^r$ when referring to the sets S_{ij}^r of the competitor partition. Under the framework of this thesis, we consider just three chambers so the only cuts considered here can be $K_1 = \{1\}$ and $K_2 = \{2, 3\}$, or $K_1 = \{1, 3\}$ and $K_2 = \{2\}$, which we will denote by $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. We also choose indexes 1 and 2 without loss of generality as the proofs would be done in the same way if we were to take any other combination of indexes. For visual aid one can imagine the competitor partition $\mathcal{E}^r$ as illustrated in Figure 11.

We now have the tools to prove the Decay and Balancing lemmas.

Lemma 4.5 (Decay)**.** *Suppose $\mathcal{E}^* = (E_1^*, E_2^*, E_3^*)$ is a locally minimizing Caccioppoli partition inside $B_R \subset \Omega$ and that there exists a constant $C < 0$ such that for almost all $0 < r < R$ there is at least one cut $\mathcal{K}$ such that*

$$\Delta_r \mathscr{F} \leq -CP(E_3, B_r)\,. \tag{4.15}$$

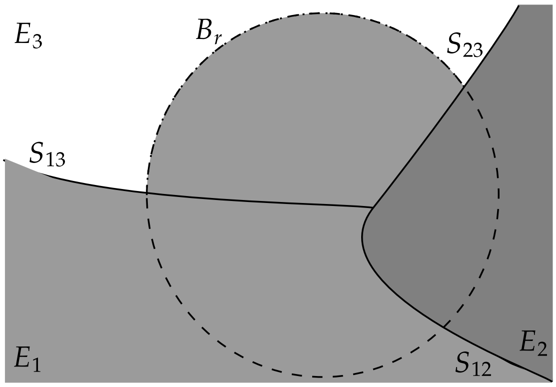

Figure 11: Partition $\mathcal{E}^r$ respect to the cut $\mathcal{K}_2$. The portion of E_3 inside the ball B_r is added to E_1 (light gray), while E_2 (dark gray) remains the same.

Then $\mathcal{E}^*$ has (EP). Moreover, we can take

$$\eta = \left(\frac{C}{2(C+C_1)}\right)^N \omega_N \quad \text{and} \quad C_1 = \max_{i<j} \sigma_{ij}.$$

Proof. Start by calling $\alpha(r) = |E_3 \cap B_r|$. It is easy to see that this function is monotone increasing and that $\alpha'(r) = \mathcal{H}^{N-1}(E_3 \cap \partial B_r)$ for a.e. $r \in (0, R)$. Notice also that, since we have $\mathcal{H}^{N-1}(S_{ij} \cap \partial B_r) = 0$ for a.e. r and all i, j, we can write

$$\mathscr{F}_{B_R,\mathbf{m}}(\mathcal{E}) = \mathscr{F}_{B_R\setminus\overline{B_r},\mathbf{m}}(\mathcal{E}) + \mathscr{F}_{B_r,\mathbf{m}}(\mathcal{E}), \tag{4.16}$$

for any Caccioppoli partition $\mathcal{E}$.

Continuing the proof by contradiction, fix $r_0 = R$ and take $\rho \in (0, r_0)$ such that

$$\alpha(\rho) \leq \eta\rho^N \quad \text{and} \quad \alpha\left(\frac{\rho}{2}\right) > 0. \tag{4.17}$$

Take $r \in \left(\frac{\rho}{2}, \rho\right)$. From minimality of $\mathcal{E}^*$ and (4.16) we can write

$$\begin{aligned} 0 &\leq \mathscr{F}_{B_R,\mathbf{m}}(\mathcal{E}^r) - \mathscr{F}_{B_R,\mathbf{m}}(\mathcal{E}^*) \\ &\leq \mathscr{F}_{B_R\setminus\overline{B_r},\mathbf{m}}(\mathcal{E}^r) + \mathscr{F}_{B_r,\mathbf{m}}(\mathcal{E}^r) - \mathscr{F}_{B_R\setminus\overline{B_r},\mathbf{m}}(\mathcal{E}^*) - \mathscr{F}_{B_r,\mathbf{m}}(\mathcal{E}^*) \\ &\leq \Delta_r\mathscr{F} + \left(\mathscr{F}_{B_R\setminus\overline{B_r},\mathbf{m}}(\mathcal{E}^r) - \mathscr{F}_{B_R\setminus\overline{B_r},\mathbf{m}}(\mathcal{E}^*)\right). \end{aligned}$$

Notice that in $B_R\setminus\overline{B_r}$, $\hat{S}_{12}$ and S_{12} are the same, while either $\hat{S}_{13}$ or $\hat{S}_{23}$ are the same to their respective original sets except by a term given in $E_3^* \cap \partial B_r$ depending on the cut. This means that

$$\mathscr{F}_{B_R\setminus\overline{B_r},\mathbf{m}}(\mathcal{E}^r) - \mathscr{F}_{B_R\setminus\overline{B_r},\mathbf{m}}(\mathcal{E}^*) = \sigma_{h3}\mathcal{H}^{N-1}(E_3^* \cap \partial B_r),$$

where h is either 1 or 2 depending on the cut. This way, we can bound the previous inequality as

$$0 \leq \Delta_r \mathscr{F} + C_1 \alpha'(r). \tag{4.18}$$

On the other hand, by the isoperimetric inequality (Theorem 2.6), we have that

$$\begin{aligned} \alpha(r)^{\frac{N-1}{N}} &\leq \frac{1}{N\omega_N^{\frac{1}{N}}} P\left(E_3 \cap B_r\right) \\ N\omega_N^{\frac{1}{N}} \alpha(r) - \alpha'(r) &\leq P(E_3 \cap B_r) - \alpha'(r) \\ N\omega_N^{\frac{1}{N}} \alpha(r) - \alpha'(r) &\leq P\left(E_3, B_r\right). \end{aligned} \tag{4.19}$$

This allows us to bound (4.18) with (4.15) and (4.19) in such a way that

$$\begin{aligned} 0 &\leq \Delta_r \mathscr{F} + C_1 \alpha'(r) \\ 0 &\leq -CP\left(E_3, B_r\right) + C_1 \alpha'(r) \\ 0 &\leq C\left(\alpha'(r) - N\omega_N^{\frac{1}{N}} \alpha(r)^{\frac{N-1}{N}}\right) + C_1 \alpha'(r) \\ CN\omega_N^{\frac{1}{N}} \alpha(r)^{\frac{N-1}{N}} &\leq (C + C_1)\alpha'(r) \\ \frac{C\omega_N^{\frac{1}{N}}}{C + C_1} &\leq \frac{1}{N} \alpha(r)^{\frac{1-N}{N}} \alpha'(r) = \left(\alpha(r)^{\frac{1}{N}}\right)'. \end{aligned}$$

Integrating both sides of this inequality over $\left(\frac{\rho}{2}, \rho\right)$, and by (4.17),

$$\begin{aligned} \frac{C\omega_N^{\frac{1}{N}}}{2(C + C_1)} \rho &\leq \alpha(\rho)^{\frac{1}{N}} - \alpha\left(\frac{\rho}{2}\right)^{\frac{1}{N}} \\ \alpha\left(\frac{\rho}{2}\right)^{\frac{1}{N}} &\leq \alpha(\rho)^{\frac{1}{N}} - \frac{C\omega_N^{\frac{1}{N}}}{2(C + C_1)} \rho \\ \alpha\left(\frac{\rho}{2}\right)^{\frac{1}{N}} &\leq \eta^{\frac{1}{N}} \rho - \frac{C\omega_N^{\frac{1}{N}}}{2(C + C_1)} \rho. \end{aligned}$$

If $\eta = \left(\frac{C}{2(C+C_1)}\right)^N \omega_N$ in the last inequality, then $\alpha\left(\frac{\rho}{2}\right) \leq 0$ which contradicts our initial assumption, as we wanted. □

The following allows us to obtain (4.15) from a weaker, yet more achievable formulation.

Lemma 4.6 (Balancing)**.** *Suppose there exists a constant $\delta > 0$ and a cut $\mathcal{K}$ such that*

$$\Delta_r \mathscr{F} \leq -\delta A_r^{\mathcal{K}}. \tag{4.20}$$

Then there exists $C = C(\delta, C_0, C_1) > 0$, where $C_0 = \min_{i<j} \sigma_{ij}$ and $C_1 = \max_{i<j} \sigma_{ij}$, such that

$$\Delta_r \mathscr{F} \leq -CP\left(E_3, B_r\right). \tag{4.21}$$

Proof. Without loss of generality, the proof will be done for the cut $\mathcal{K}_1$. In this case $A_r^{\mathcal{K}} = \mathcal{H}^{N-1}(S_{13}^r)$, and

$$\mathscr{F}_{B_r,\mathbf{m}}(\mathcal{E}^r) = \sigma_{12}\mathcal{H}^{N-1}(\hat{S}_{12}^r),$$

since $\mathcal{H}^{N-1}(\hat{S}_{13}) = \mathcal{H}^{N-1}(\hat{S}_{23}) = 0$, as inside B_r there is no $\hat{E}_3$. Furthermore, we can write $\hat{S}_{12}^r = S_{13}^r \cup S_{12}^r$, so that

$$\begin{aligned}
\Delta_r\mathscr{F} &= \mathscr{F}_{B_r,\mathbf{m}}(\mathcal{E}^r) - \mathscr{F}_{B_r,\mathbf{m}}(\mathcal{E}) \\
&= \sigma_{12}\mathcal{H}^{N-1}(\hat{S}_{12}^r) - \sigma_{12}\mathcal{H}^{N-1}(S_{12}^r) - \sigma_{13}\mathcal{H}^{N-1}(S_{13}^r) - \sigma_{23}\mathcal{H}^{N-1}(S_{23}^r) \\
&= \sigma_{12}\left(\mathcal{H}^{N-1}(S_{13}^r) + \mathcal{H}^{N-1}(S_{12}^r)\right) - \sigma_{12}\mathcal{H}^{N-1}(S_{12}^r) - \sigma_{13}\mathcal{H}^{N-1}(S_{13}^r) - \sigma_{23}\mathcal{H}^{N-1}(S_{23}^r) \\
&= (\sigma_{12} - \sigma_{13})A_r^{\mathcal{K}} - \sigma_{23}\mathcal{H}^{N-1}(S_{23}^r) \\
&\leq (C_1 - C_0)A_r^{\mathcal{K}} - C_0\mathcal{H}^{N-1}(S_{23}^r). \qquad (4.22)
\end{aligned}$$

On the other hand, from Theorem 2.17, we can decompose the perimeter as

$$P\left(E_3, B_r\right) = \mathcal{H}^{N-1}(S_{13}^r) + \mathcal{H}^{N-1}(S_{23}^r),$$

which we can rewrite as

$$-\mathcal{H}^{N-1}(S_{23}^r) = A_r^{\mathcal{K}} - P\left(E_3, B_r\right).$$

Substituting this into (4.22) yields

$$\begin{aligned}
\Delta_r\mathscr{F} &\leq (C_1 - C_0)A_r^{\mathcal{K}} + C_0(A_r^{\mathcal{K}} - P\left(E_3, B_r\right)) \\
&= C_1 A_r^{\mathcal{K}} - C_0 P\left(E_3, B_r\right).
\end{aligned}$$

This bound, together with (4.20), results in the bound

$$\Delta_r\mathscr{F} \leq \min\left\{-\delta A_r^{\mathcal{K}}, C_1 A_r^{\mathcal{K}} - C_0 P\left(E_3, B_r\right)\right\}, \qquad (4.23)$$

and recalling the real number inequality [6]

$$\min\left\{-\delta_1 a, \delta_2 a - \delta_3 b\right\} \leq -\frac{\delta_1\delta_3}{\delta_2 + \delta 1}b, \qquad (4.24)$$

for $\delta_i > 0$ and $a, b \in \mathbb{R}$, and applying it to (4.23) yields

$$\Delta_r\mathscr{F} \leq -\frac{\delta C_0}{C_1 + \delta}P\left(E_3, B_r\right).$$

Writing $C = -\frac{\delta C_0}{C_1+\delta}$, we obtain the relation that we were looking for. $\square$

Remark 4.4. Key changes in the proof when considering the only other cut $\mathcal{K}_2$ are that, now, $A_r^{\mathcal{K}} = \mathcal{H}^{N-1}(S_{23}^r)$ and one would now have to replace $\mathcal{H}^{N-1}(S_{23}^r)$ for $\mathcal{H}^{N-1}(S_{13}^r)$ in (4.22).

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[6] This inequality can be proven easily by direct computation choosing each caso of the minimum.

In view of these lemmas, the path to prove the Elimination Theorem should be straightforward. If hypothesis of Lemma 4.5 are met, we obtain (EP), and for this, we can verify if the weaker formulation in Lemma 4.6 holds true. With the layout set, we can tackle the proof of Theorem 4.3.

Proof Theorem 4.3. Start by taking a minimal cut in the sense that it minimizes the quantity

$$\sigma(\mathcal{K}) = A_r^{\mathcal{K}} + \mathcal{H}^{N-1}(S_{12}^r),$$

Suppose then that the minimal cut is $\mathcal{K}_1$. Since $A_r^{\mathcal{K}} = \mathcal{H}^{N-1}(S_{13}^r)$ in this case, by minimality we have

$$A_r^{\mathcal{K}} \leq \mathcal{H}^{N-1}(S_{23}^r). \tag{4.25}$$

By estimations done in the proof of Lemma 4.6 we know that

$$\Delta_r \mathscr{F} = (\sigma_{12} - \sigma_{13}) A_r^{\mathcal{K}} - \sigma_{23} \mathcal{H}^{N-1}(S_{23}^r).$$

From strict triangle inequality (4.13) specifically $\sigma_{12} \leq \sigma_{13} + \sigma_{23} - \delta$, and thus

$$\begin{aligned} \Delta_r \mathscr{F} &\leq (\sigma_{23} - \delta) A_r^{\mathcal{K}} - \sigma_{23} \mathcal{H}^{N-1}(S_{23}^r) \\ &= \sigma_{23}(A_r^{\mathcal{K}} - \mathcal{H}^{N-1}(S_{23}^r)) - \delta A_r^{\mathcal{K}} \\ &\leq -\delta A_r^{\mathcal{K}}, \end{aligned}$$

which is enough to apply the Balancing Lemma.

If the minimal cut is $\mathcal{K}_2$, the process is identical aside from substituting $\mathcal{H}^{N-1}(S_{23}^r)$ for $\mathcal{H}^{N-1}(S_{13}^r)$. □

Remark 4.5. Regarding the proofs of these results:

1. The problem becomes significantly more difficult when dealing with M number of chambers in the results. Leonardi tackled this using specific results in graph theory, seeing each chamber as a node in a connected graph, with the weights going through the edges being the Hausdorf measure of S_{ij} (see [9]) and a special decomposition of the quantity $\sigma(\mathcal{K})$ into flow functions.

2. A cautious reader could notice that if we modify partitions the way we did in the proofs, it may result in a non admissible competitor sets due to the volume constraints maybe not holding. This is in fact the case, but can be amended under the view of Lemma 2.21. Indeed, proven by Almgren in [8], one can compensate the missing volume by adding this variation into another set while also approximating the perimeter by a factor of this same volume change. Since this change is infinitesimal, it turns out to be negligible when taken into account in the proofs.

3. These results also apply to the case where E_1 and E_3 are separated entirely by E_2. Some terms simplify even further in the lower semicontnuity results as well as the elimination results, since $\mathcal{H}^{N-1}(S_{13}) = 0$ in this case. Regardless, this quantity did not interfere in most of the computations.

Consider now the second set of conditions (3.3) and let us explore an analogous theorem and resulting regularity and density properties for the minimzers. To this end, call $\mathcal{M}_{\Omega,\mathbf{m}}$ the set of all minimizers of $\mathscr{F}_{\Omega,\mathbf{m}}$.

Theorem 4.7. *Let* $\mathcal{E}^* = (E_1^*, E_2^*, E_3^*)$ *be a locally minimizing Caccioppoli partition for* $F_{\Omega,m}$ *in* $B_R \subset \Omega$*, and that coefficients* σ_{ij} *satisfy* (3.3)*. Then* $\mathcal{E}^*$ *has (EP) for* $k \in \{1,3\}$*.*

Proof. From (3.3) we can deduce that

$$\sigma_{12} < \sigma_{13} + \sigma_{23} \quad \text{and} \quad \sigma_{23} < \sigma_{12} + \sigma_{13}.$$

Then, in the proof of Theorem 4.3 we can still use the variant of the triangle inequality (4.13) which proofs it for $k = 3$. Following the same train of thought but now fixing sets E_2^* and E_3^* for their respective cuts, one proofs that $\mathcal{E}^*$ has (EP) for $k = 1$. □

We can now follow–up this version of the Elimination Theorem with some properties.

Lemma 4.8. *Let* $\Omega \subset \mathbb{R}^N$ *be open. If* $\mathcal{E} \in \mathcal{M}_{\Omega,\mathbf{m}}$*, then for every* $x \in \mathcal{F}E_1 \cap \mathcal{F}E_2 \cap \Omega$ *or* $x \in \mathcal{F}E_3 \cap \mathcal{F}E_2 \cap \Omega$*, there exists* $r > 0$ *such that*

$$B_r(x) \cap E_3 = \emptyset \quad \text{or} \quad B_r(x) \cap E_1 = \emptyset,$$

respectively.

Proof. It suffices to prove for $x \in \mathcal{F}E_1 \cap \mathcal{F}E_2 \cap \Omega$ as the other case is analogous. By Theorem 2.15, E_1 and E_2 have density $\frac{1}{2}$ at x, so that E_3 would have density 0 at x. Thus, for any $\varepsilon > 0$, there exists $r_0 > 0$ such that

$$\frac{|B_{r_0}(x) \cap E_3|}{|B_{r_0}(x)|} \leq \varepsilon,$$

so that

$$|B_{r_0}(x) \cap E_3| \leq \varepsilon |B_{r_0}(x)| = \varepsilon \omega_N r_0^N.$$

By Theorem 4.7, we obtain

$$\left| B_{\frac{r_0}{2}}(x) \cap E_3 \right| = 0.$$

Taking $r < \frac{r_0}{2}$ yields $B_r(x) \cap E_3 = \emptyset$. □

This lemma gives us an idea of how the minimizing configuration is going to be within Ω. Essentially it tells us that, at most, only two sets will be interacting at a time. The next following results hands us bounds for regulartiy results.

Proposition 4.1. Let $\Omega \subset \mathbb{R}^N$ be open, $\mathcal{E} \in \mathcal{M}_{\Omega,\mathbf{m}}$, and $R > 0$. Then, there exists $\gamma \in (0,1)$ and $r > 0$ such that, for every $x \in \partial E_i$ with $B_R(x) \subset \Omega$ and $0 < \rho < r$, it holds

$$\gamma \leq \frac{|E_i \cap B_\rho(x)|}{\omega_N \rho^N} \leq 1 - \gamma, \tag{4.26}$$

for $i = 1, 3$.

Proof. Once again, we will do the proof only for $i = 1$, i.e. take $x \in \partial E_1$. By definition of topological boundary we can say that $\left|B_{\frac{\rho}{2}}(x) \cap E_1\right| \neq 0$, and thus by the contrapositive of Theorem 4.3

$$|B_\rho(x) \cap E_1| > \omega_N \eta \rho^N.$$

We can take a small enough $\gamma > \eta$ to obtain the left inequality in (4.26)

$$\frac{|B_\rho(x) \cap E_1|}{\omega_N \rho^N} > \eta > \gamma.$$

For the right hand side of the inequality we will consider $\gamma > \eta$, and suppose the estimate does not hold, i.e.

$$\frac{|B_\rho(x) \cap E_1|}{\omega_N \rho^N} > 1 - \eta.$$

By Federer's Theorem, we can deduce that

$$\frac{|B_\rho(x) \cap E_3|}{\omega_N \rho^N} \leq \sum_{i=2}^{3} \frac{|B_\rho(x) \cap E_i|}{\omega_N \rho^N} \leq \eta,$$

since the densities need to add up to 1. So, we get $|B_\rho(x) \cap E_3| \leq \eta \omega_N \rho^N$ which by Theorem 4.3, yields $\left|B_{\frac{\rho}{2}}(x) \cap E_3\right| = 0$, meaning $x \in \partial E_2$. Summarizing, we have that $x \in \partial E_1 \cap \partial E_2$ and by Theorem 2.22 we get a contradiction of the form $\gamma < 1 - \frac{1}{4^N}$ for a large enough γ. □

Proposition 4.2. Let $\Omega \subset \mathbb{R}^N$ be open, $\mathcal{E} \in \mathcal{M}_{\Omega,\mathbf{m}}$, and $R > 0$. Then, there exists $\theta, r > 0$ such that for all $x \in \partial E_i$ with $B_R(x) \subset \Omega$ and $0 < \rho < r$, the following inequality holds

$$P\left(E_i, B_\rho(x)\right) \geq \theta \rho^{N-1},$$

for $i = 1, 3$.

Proof. Without loss of generality, we will only prove for $i = 1$. From Proposition 4.1 we have

$$\gamma \leq \frac{|E_1 \cap B_\rho(x)|}{\omega_N \rho^N}.$$

Applying the relative isoperimetric inequality to the set $E_1 \cap B_\rho(x)$ we obtain

$$\gamma \leq \frac{\gamma_N}{\omega_N \rho^N} P\left(E_1, B\rho(x)\right)^{\frac{N}{N-1}}$$
$$(\gamma \omega_N)^{\frac{N-1}{N}} \rho^{N-1} \leq P\left(E_1, B\rho(x)\right).$$

Calling $\theta = (\gamma_N \omega_N)^{\frac{N-1}{N}}$ we get the desired lower bound for the perimeter. □

With these results one obtains the following powerful corollary, in my opinion. It entails that within Ω, the topological and reduced boundaries of minimizing sets E_i coincide up to a $\mathcal{H}^{N-1}-$null set. In other words, the boundaries are in some sense well behaved and in the framework of applying Theorem 4.1.

Corollary 4.9. *If $\mathcal{E} \in \mathcal{M}_{\Omega,\boldsymbol{m}}$, then*

$$\mathcal{H}^{N-1}\left((\partial E_i \cap \Omega)\backslash(\mathcal{F}E_i \cap \Omega)\right) = 0,$$

for $i = 1, 3$.

Proof. Again we will only prove for $i = 1$ as the other case follows similarly. By Proposition 4.2 we have

$$\begin{aligned} |D\chi_{E_1}|(B_\rho(x)) = P\left(E_1, B_\rho(x)\right) &\geq \theta\rho^{N-1} \\ &\geq \frac{\theta}{\omega_{N-1}}\omega_{N-1}\rho^{N-1} \\ &= \frac{\theta}{\omega_{N-1}}\mathcal{H}^{N-1}(B_\rho(x)). \end{aligned} \tag{4.27}$$

We can extend this result to any Borel set $B \subset \partial E_1$ using a covering argument. Mainly, by the Vitali–Besicovitch Covering Theorem [2, Theorem 2.19], there exists a disjoint family of balls $\{B_{\rho_i}(x_i)\}$, with $x_i \in B$, such that

$$\mathcal{H}^{N-1}\left(B\backslash\bigcup_i B_{\rho_i}(x_i)\right) = 0.$$

Since the $\mathcal{H}^{N-1}$ dominates the perimeter, this implies

$$|D\chi_{E_1}|\left(B\backslash\bigcup_i B_{\rho_i}(x_i)\right) = 0,$$

This way, $|D\chi_{E_1}|(B) = |D\chi_{E_1}|\left(\bigcup_i B_{\rho_i}(x_i)\right)$ and $\mathcal{H}^{N-1}(B) = \mathcal{H}^{N-1}\left(\bigcup_i B_{\rho_i}(x_i)\right)$, as the balls are disjoint. So to generalize (4.27) it suffices to obtain the same lower bound for the union of the balls as follows

$$\begin{aligned} |D\chi_{E_1}|\left(\bigcup_i B_{\rho_i}(x_i)\right) &= \sum_i |D\chi_{E_1}|\left(B_{\rho_i}(x_i)\right) \\ &\geq \frac{\theta}{\omega_{N-1}}\sum_i \mathcal{H}^{N-1}\left(B_{\rho_i}(x_i)\right) \\ &= \frac{\theta}{\omega_{N-1}}\mathcal{H}^{N-1}\left(\bigcup_i B_{\rho_i}(x_i)\right). \end{aligned}$$

Taking $B = \partial E_1\backslash(\mathcal{F}E_1 \cap \Omega)$ and recalling $D\chi_{E_1}$ is concentrated on $\mathcal{F}E_1$, then

$$0 = |D\chi_{E_1}|(B) \geq \frac{\theta}{\omega_{N-1}}\mathcal{H}^{N-1}(B) \geq 0,$$

thus $\mathcal{H}^{N-1}(B) = 0$. □

Remark 4.6. This corollary can be proven directly using Federer's Theorem, Theorem 2.15. Indeed, taking $r \searrow 0$ in Proposition 4.1, we get that every $x \in \partial^* E_1$, implying $\partial E_1 \cap \Omega \subset \partial^* E_1 \cap \Omega$. This allows us to restrict the result in Federer's Theorem to $\partial E_1 \cap \Omega$.

Proposition 4.3. Let $\Omega \subset \mathbb{R}^N$ be open. For any $\mathcal{E} \in \mathcal{M}_{\Omega,\mathbf{m}}$, the following equality holds:

$$\mathcal{H}^{N-1}\left(\mathcal{F}E_1 \cap \mathcal{F}E_3 \cap \Omega\right) = 0.$$

Proof. Suppose by contradiction that

$$\mathcal{H}^{N-1}\left(\mathcal{F}E_1 \cap \mathcal{F}F_3 \cap \Omega\right) > 0.$$

By Corollary 4.9, $\mathcal{H}^{N-1}\left((\partial E_1 \cap \Omega)\backslash(\mathcal{F}E_1 \cap \Omega)\right) = 0$, so by Theorem 4.1 in view of Remark 4.1, for $\varepsilon > 0$, there exists $E_1^\varepsilon \Subset E_1$ such that

$$\left|P\left(E_1, \Omega\right) - P\left(E_1^\varepsilon, \Omega\right)\right| \leq \varepsilon \quad \text{and} \quad |E_1| - |E_1^\varepsilon| < \varepsilon. \tag{4.28}$$

Consequently one can see also that $\mathcal{H}^{N-1}\left(\mathcal{F}E_1 \cap \mathcal{F}E_1^\varepsilon \cap \Omega\right) = 0$.

Define $E_3^\varepsilon = E_3$ and $E_2^\varepsilon = \Omega\backslash\left(\overline{E}_1^\varepsilon \cup \overline{E}_3^\varepsilon\right)$. Sadly these new sets do not have the same volume so they cannot be considered as competitors for the original configuration. To recover the volumes, consider $x \in \mathbb{R}^N$ such that its density in E_2 is 1. This entails that density in E_1 and E_3 are negligible, i.e. for a small enough $r > 0$

$$\left|E_1 \cap B_r(x)\right| \leq \eta \left|B_r(x)\right| \quad \text{and} \quad \left|E_3 \cap B_r(x)\right| \leq \eta \left|B_r(x)\right|,$$

then by Theorem 4.3 we get that $\left|E_1 \cap B_{\frac{r}{2}}(x)\right| = \left|E_3 \cap B_{\frac{r}{2}}(x)\right| = 0$. Thus $B_{\frac{r}{2}}(x) \subset E_2$, and rewriting $\frac{r}{2}$ as just $r \in (0,1)$ for simplicity, we can see that

$$\left|E_2 \cap B_r(x)\right| = \left|B_r(x)\right|.$$

Now, take $\varepsilon < |B_r(x)|$ small enough and $r' > 0$ such that $|B_{r'}(x)| = |E_1| - |E_1^\varepsilon|$, and define

$$\tilde{E}_1^\varepsilon = E_1^\varepsilon \cap B_{r'}(x), \quad \tilde{E}_2^\varepsilon = E_2^\varepsilon \backslash B_{r'}(x), \quad \tilde{E}_3^\varepsilon = E_3^\varepsilon.$$

Volume for these sets is the same as $\mathbf{m}$, so they become a competitor for minimizer that we will call $\tilde{\mathcal{E}}$, and we proceed to estimate its functional value. Since E_1^ε is contained in E_1 by definition, then there is no overlap with E_3, implying

$$\mathcal{H}^{N-1}\left(\mathcal{F}\tilde{E}_1^\varepsilon \cap \mathcal{F}\tilde{E}_3^\varepsilon \cap \Omega\right) = 0.$$

Having $\mathcal{F}\tilde{E}_2^\varepsilon \cap \mathcal{F}\tilde{E}_3^\varepsilon \cap \Omega \subset \mathcal{F}E_3 \cap \Omega$ and by Theorem 2.17, we can write

$$\begin{aligned}\mathcal{H}^{N-1}\left(\mathcal{F}\tilde{E}_2^\varepsilon \cap \mathcal{F}\tilde{E}_3^\varepsilon \cap \Omega\right) &\leq \mathcal{H}^{N-1}\left(\mathcal{F}E_3 \cap \Omega\right) \\ &= \mathcal{H}^{N-1}\left(S_{13}\right) + \mathcal{H}^{N-1}\left(S_{23}\right).\end{aligned}$$

Since $|B_{r'}(x)| < \varepsilon$, then $r' < \left(\frac{\varepsilon}{\omega_N}\right)^{\frac{1}{N}}$. This coupled with the perimeter inequality in (4.28) yields

$$\begin{aligned}\mathcal{H}^{N-1}\left(\mathcal{F}\tilde{E}_1^{\varepsilon} \cap \mathcal{F}\tilde{E}_2^{\varepsilon} \cap \Omega\right) &= \mathcal{H}^{N-1}\left(\mathcal{F}E_1^{\varepsilon} \cap \Omega\right) + P(B_{r'}(x)) \\ &\leq \mathcal{H}^{N-1}\left(\mathcal{F}E_1 \cap \Omega\right) + \varepsilon + N\omega_N \left(\frac{\varepsilon}{\omega_N}\right)^{\frac{N-1}{N}} \\ &\leq \mathcal{H}^{N-1}\left(S_{12}\right) + \mathcal{H}^{N-1}\left(S_{13}\right) + \varepsilon + N\omega_N^{\frac{1}{N}}\varepsilon^{\frac{N-1}{N}}.\end{aligned}$$

These estimates allow us to bound $\mathscr{F}_{\Omega,\mathbf{m}}$, together with condition (3.3), in such a way that

$$\begin{aligned}\mathscr{F}_{\Omega,\mathbf{m}}(\tilde{\mathcal{E}}) &\leq \sigma_{12}\left(\mathcal{H}^{N-1}\left(S_{12}\right) + \mathcal{H}^{N-1}\left(S_{13}\right) + \varepsilon + N\omega_N^{\frac{1}{N}}\varepsilon^{\frac{N-1}{N}}\right) + \sigma_{23}\left(\mathcal{H}^{N-1}\left(S_{13}\right) + \mathcal{H}^{N-1}\left(S_{23}\right)\right) \\ &= \mathscr{F}_{\Omega,\mathbf{m}}(\mathcal{E}) + \tilde{\varepsilon},\end{aligned}$$

where $\tilde{\varepsilon} = \sigma_{12}\left(\varepsilon + N\omega_N^{\frac{1}{N}}\varepsilon^{\frac{N-1}{N}}\right)$. Taking $\varepsilon \to 0$ yields a contradiction with the minimality condition of $\mathcal{E}$. □

This proposition gives us a geometrical understanding for the minimizing configurations. Ideally, sets E_1 and E_3 will not touch unless it is in a $\mathcal{H}^{N-1}-$null set, meaning that E_1 and E_3 will be separated by E_2, as was the general idea for configurations in this work. Considering further more a foliated domain, then E_1 will be contained in E_2 which will separate it from E_3, or vice versa.

The next and final theorem generalizes Theorem 2.22 for our case of Caccioppoli partitions, achieving regularity in the boundaries.

Theorem 4.10. *Let $\Omega \subset \mathbb{R}^N$ be open. If $\mathcal{E} \in \mathcal{M}_{\Omega,\mathbf{m}}$, then for any $i \in \{1,2,3\}$, the sets $\partial E_i \cap \Omega$ are of class C^∞ up to $\mathcal{H}^{N-1}-$null sets.*

Proof. From Corollary 4.9 we know that $\mathcal{F}E_i = \partial E_i$ up to $\mathcal{H}^{N-1}-$null sets in Ω, for $i = 1, 3$. By Proposition 4.3, we can conclude that if $x \in \partial E_1 \cap \Omega$ then necessarily $x \in S_{12}$, and together with Lemma 4.8, there exists $r > 0$ such that

$$E_3 \cap B_r(x) = \emptyset.$$

This implies that $\partial E_1 \cap B_r(x) = \partial E_2 \cap B_r(x)$, and by Theorem 2.22 we can conclude that these sets are C^∞ outside a $\mathcal{H}^{N-1}-$null set. Since $B_r(x) \Subset \Omega$ and these claims hold true for almost any $x \in \partial E_1 \cap \Omega$, then $\partial E_1 \cap \Omega$ is C^∞ as well. We can conclude that $\partial E_3 \cap \Omega$ is also C^∞, following the same line of reasoning.

Furthermore, sets $\partial E_1 \cap \partial E_2 \cap \Omega$ and $\partial E_2 \cap \partial E_3 \cap \Omega$ are C^∞ based on $x \in S_{12}$ or $x \in S_{13}$, a.e. and in their respective case. Hence, $\partial E_2 \cap \Omega$ is also C^∞ up to $\mathcal{H}^{N-1}-$null sets. □

To summarize the results, aside from some very useful bounds for the volume or perimeter of the minimizing partition, we were able to conclude that the boundary of the sets is very well behaved. This in a sense that The reduced boundary of the sets coincide with the essential boundary, thus integrating the measure theoretic thematic with the topological one, and great regularity for them. The backbone for these results, Theorems 4.3 and 4.7, were crucial as one can think that they allowed us to reduce the boundaries locally to only the essentials, in turn erasing any *strange* interaction between the three sets, into just two. Once again, this can be generalized for any amount of sets in the partition but we have cemented the idea on how this result will work in the general scope.

5 Closing Remarks: From Variational Principles to Structure

In this thesis we studied the weighted perimeter functional on Caccioppoli partitions with three chambers, motivated by models of immiscible fluids and, more generally, by interface problems in GMT. The main questions concerned the existence of minimizers, as well as their structure and regularity under different assumptions on the weights. Although these issues have already been addressed in [9, 10, 13], here we present them in detail, aiming to provide a clear starting point for readers interested in sets of finite perimeter and related variational problems, while also showing classical tools that give a systematic framework.

Chapter 3, focused on the variational properties of the functional. We proved lower semicontinuity under triangle inequality conditions on the weights, proving first for small interactions in the interfaces and build upon that on smaller sets of the interface with the inner regularity of Radon measures to achieve lower semicontinuity. This ensured existence of minimizers via the direct method. In contrast, when the inequality was violated we showed that lower semicontinuity fails, leading to non–existence of minimizers in general. To address the issue, introduced isoperimetrically foliated domains, which provided sufficient compactness to recover existence.

Chapter 4 was devoted to structural and regularity results for the minimizers. First establishing a strict approximation theorem, showing sets of finite perimeter can be approximated by smooth sets strictly contained within the domain, while also approximating the perimeter. This was done by a covering argument on the boundary, using Lemma 4.2 to cover the reduced boundary and balls for the rest such that the distance from the set was small enough and the perimeters approximated well. This provided a flexible approximation tool that could be applied to admissible partitions. The Elimination Theorem, was proven with the help of the Decay and Balancing lemmas. After creating a competitor set given by a cut of the partition, the Balancing Lemma proved that the difference in energy between this and the minimizer is bounded by the perimeter of the set we are cutting out locally, while the Decay Lemma uses this fact to ensure the elimination property. The importance of this result lies in its ability to simplify the admissible class to structurally minimal partitions, thereby excluding degenerate geometries. Together, these two ingredients allowed us to derive strong regularity results akin to the ones revised in Section 2.3.

Overall, the methods employed emphasize the interplay between compactness, approximation, and structural simplification in variational analysis. The results recover the classical picture when the weights satisfy a triangle inequality, but also extend to more delicate cases where this assumption fails. In this way, the thesis contributes to the understanding of minimizing partitions and to the broader development of tools for weighted interface problems in GMT.

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