Université Gustave Eiffel

MÉMOIRE

MASTER 2 Analysis and applications

Prescription of multifractal behavior of functions and measures

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Dedication

A mis padres, a quienes les debo todo.

Acknowledgments

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1 Introduction

This document concentrates on the field of multifractal analysis of functions, where given a function, measure or stochastic process X on $\mathbb{R}^d$, one mainly focuses on two aspects: first, the pointwise behavior of X at $x \in \mathbb{R}^d$ measured by an exponent $h_X(x)$, which depends on the nature of the object. Then, one computes the mapping D_X (the **multifractal spectrum** of X) defined by

$$D_X(s) = \dim_H(E_X(s)), \quad \text{where} \quad E_X(s) = \{x \in \mathbb{R}^d : h_X(x) = s\},\ s \in \mathbb{R} \cup \{+\infty\},$$

where $\dim_H$ stands for the Hausdorff dimension. This mapping provides a hierarchy between the level sets $E_X(s)$, according to their size measured by their Hausdorff dimension. The computation of D_X has been performed in diverse settings: Fourier/wavelet series, stochastic processes (Lévy, Markov, fractional Brownian motion), dynamically defined measures (Gibbs and invariant measures).

In the case where X is a real valued function $f \in L^\infty_{loc}(\mathbb{R})$, given a point $x_0 \in \mathbb{R}^d$, we may consider the *pointwise Hölder exponent* of f at x_0 as the exponent quantifying its regularity at this point, which is defined as

$$h_f(x_0) = \sup\{s > 0 : f \in C^s(x_0)\},$$

where we say that f belongs to the space $C^s(x_0)$ if there exists a constant $C > 0$, a polynomial P of degree at most $[s]$ and a neighborhood V of x_0 such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^s, \quad \text{for all } x \in V.$$

The associated multifractal spectrum (also called *singularity spectrum*) of f will be denoted as $\sigma_f(s)$. In this report, we consider the convention $\dim_H(\emptyset) = -\infty$. On the other hand, if our object is a measure μ on $[0,1]$, we consider the exponent

$$\underline{d_\mu}(x_0) = \liminf_{r \to 0} \frac{\log(\mu(B(x_0, r)))}{\log r}, \quad \text{for } x_0 \in (0,1),$$

known as the *lower local dimension* of the measure μ at the point x_0, where $B(x_0, r)$ is an euclidean ball of center x_0 and radius r.

On the other side, the multifractal analysis also aims at describing and enlightening the multiscale properties of X (i.e. statistical properties of X through scales), described by a '*scaling function*' τ_X. For instance, if again μ is a measure on $[0,1]$, we may consider the mapping

$$\tau_\mu(q) = \liminf_{j \to +\infty} \frac{1}{j} \log \left(\sum_{I \in D_j} \mu(I)^q \right), \quad \text{for } q \in \mathbb{R},$$

known as L^q-spectrum of μ, where D_j is the set of dyadic intervals of size 2^{-j}.

A key fact is that for "nice" objects (for example, Baire typical measures) D_X can be estimated on numerical data via the *multifractal formalism*, which relates D_X to the scaling function τ_X via a Legendre transform:

$$D_X(H) = \tau_X^*(H) = \inf_{q\in\mathbb{R}}(qH - \tau_X(q)).$$

Such a relation between local behaviors and global statistical properties has reflected deep and important inner structure characteristics not known before.

In this paper we propose to address the prescription problem for a univariate (i.e., for a single function) case. It is a classical problem and has been studied exhaustively by many brilliant and experienced researchers in the field of multifractal analysis, such as S. Jaffard, Buczolich, S. Seuret, J. Barral (these last two contructed function spaces in which Baire typical functions can have a prescribed multifractal behavior and satisfy a multifractal formalism, solving the so-called Frisch-Parisi conjecture, presented in a wonderful paper [1]). The first result was obtained in [2], where S. Jaffard constructed functions with prescribed multifractal spectrum using wavelet techniques and arguments from geometric number theory. Actually, our objective is to prove the main theorem showed in his paper, by using similar techniques but in a slightly different way.

1.1 Notations and definitions

Let $x \in \mathbb{R}$ be a real number, $D \subseteq \mathbb{R}$. We will consider these notations:

- $\mathcal{L}$ is the Lebesgue measure on dimension 1.
- $|D|$ is the Euclidean diameter of the set D.
- $dist(x, D)$ stands for the distance between the set D and the point x.
- $[x]$ is the greatest integer less or equal to x.
- $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2(\mathbb{R})$.

Also, we will say that a real valued function φ belongs to the class $\mathcal{C}$ if it can be written as the supremum of a countable family of functions defined as $cx\mathbb{1}_{[a,b]}(x)$, where a, b, c are constants and $\mathbb{1}_{[a,b]}$ is the indicator function of $[a, b]$.

2 Some previous results

We state some classical results on the regularity of functions and their relation to wavelets. These will be very useful in our work.

Theorem 1 (Mass distribution principle)**.** *Let A be a subset of $\mathbb{R}^d$. If there exists a non trivial finite Borel measure μ supported in A such that for some constants $C > 0$ and $s > 0$*

$$\mu(B) \leq C|B|^s, \quad \textit{for all ball } B \textit{ small enough},$$

then $\mathcal{H}^s(A) \geq \mu(A)/C$. In particular, $\dim_H(A) \geq s$.

Theorem 2. *There exists a function Ψ called wavelet function such that the family $\{\Psi_{j,k}\}_{j,k\in\mathbb{Z}}$ forms a Hilbert basis of $L^2(\mathbb{R})$, where $\Psi_{j,k} = 2^{j/2}\Psi(2^j \cdot -k)$.*

So, any function $f \in L^2(\mathbb{R})$ can be written as

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\langle f, \Psi_{j,k}\rangle\Psi_{j,k},$$

and $d_{j,k} := \langle f, \Psi_{j,k}\rangle$ are known as the *wavelet coefficients* of f, which give a time-frequency decomposition.

The following result relates the (global) smoothness of the function to how fast the wavelet coefficients decay as the scale j tends to $+\infty$.

Theorem 3. *Let $s > 0$ be a non-integer real number, $\Psi \in C^{[s]+1}(\mathbb{R})$ be a wavelet with at least $[s] + 1$ vanishing moments, and $f \in L^2(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R})$. Then $f \in C^s(\mathbb{R})$ if and only if, there exists a constant $C > 0$ such that*

$$|\langle f, \Psi_{j,k}\rangle| \leq C2^{-j(s+1/2)} \quad \textit{for all } j \geq 0, \textit{ and all } k \in \mathbb{Z}.$$

On the other hand, we also state a theorem that characterizes the point regularity of the functions with their wavelet coefficients:

Theorem 4. *Let $s > 0$ be a non-integer real number, $\Psi \in C^{[s]+1}(\mathbb{R})$ be a wavelet with at least $[s]+1$ vanishing moments, $x_0 \in \mathbb{R}$ and $f \in L^2(\mathbb{R}) \cap C^s(x_0)$. Then, there exists constants $C > 0$ and $\eta > 0$ such that for all $j \geq 0$ and all $k \in \mathbb{Z}$*

$$|x_0 - k2^{-j}| \leq \eta \Longrightarrow |\langle f, \Psi_{j,k}\rangle| \leq C2^{-j/2}(2^{-j} + |x_0 - k2^{-j}|)^s.$$

Conversely, if the last inequality holds and $f \in C^\gamma(\mathbb{R})$ for some $\gamma > 0$ then

$$f \in C^{s-\epsilon}(x_0) \quad \textit{for all } \epsilon \in (0, s).$$

Finally, we take into account the following lemma:

Lemma 1 (Billingsley Lemma)**.** *Let μ be a probability measure supported on $E \subset \mathbb{R}$, such that for $\mu-$a.e $x \in E$ it holds $\underline{d_\mu}(x) = h$. Then, $\dim_H(E) \geq h$.*

In this document, we will consider a wavelet Ψ sufficiently smooth, with a sufficient number of vanishing moments and compactly supported.

3 Main theorem

The main objective of this document is to prove the following theorem:

Theorem 5 (S. Jaffard)**.** *Let* $\varphi : (0, +\infty) \longrightarrow [0, 1] \cup \{-\infty\}$ *a function belonging to the class* $\mathcal{C}$*. Then, there exists* $f \in C(\mathbb{R})$ *such that* $\varphi = \sigma_f$*.*

First, we focus on the case when $\varphi(s) = c\, s \mathbb{1}_{[a,b]}(s)$, where $0 < a < b < +\infty$, $c > 0$ such that $cb < 1$, and we consider the following parameters, which will be used throughout this report

$$\gamma := a > 0,\ \beta := \frac{b}{a} > 1,\ \alpha := \frac{1}{cb} > 1,$$

so, we construct a function $f = f^{(\alpha,\beta,\gamma)}$. The general case will result from an application of a *superposition method*, which will be developed at the end. The proof of Theorem 5 will be presented in different sections below.

3.1 Definition of the function f

We first construct a Cantor-like set on the interval $[0, 1]$. We split it into 3 parts, considering that the ends have a length of $2^{-\alpha}$. We remove the middle one, so we obtain two intervals. We will call as I_0 to the left interval and as I_1 to the right interval. These will make up the first generation of our Cantor set.

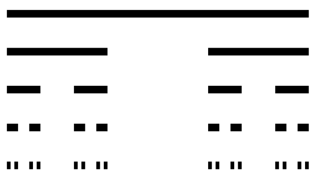

Figure 1: Cantor set K_α

We perform the same procedure, so after m steps we will get 2^m intervals $I_{\epsilon_1,\ldots,\epsilon_m}$ ($\epsilon_i \in \{0, 1\}$) of length $2^{-\alpha m}$ at the m-th generation; since $\alpha > 1$, the space between such intervals is at least $2^{-(m-1)\alpha}(1 - 2^{-\alpha+1}) > 0$. Hence, the m-th generation is made up as the union of these 2^m intervals, which we call as $K_\alpha^{(m)}$. Finally, we consider

$$K_\alpha := \bigcap_{m \geq 1} K_\alpha^{(m)}.$$

We see that K_α is a compact set. Since we need exactly 2^m intervals of length

$2^{-\alpha m}$ to cover K_α, the box dimension is

$$\begin{aligned}\dim_B(K_\alpha) &= \lim_{m\longrightarrow+\infty} \frac{\log N_{2^{-\alpha m}}(K_\alpha)}{-\log 2^{-\alpha m}} \\ &= \lim_{m\longrightarrow+\infty} \frac{\log 2^m}{\alpha \log 2^m} = \frac{1}{\alpha}.\end{aligned}$$

Moreover, we have the following result:

Lemma 2. *The Hausdorff dimension of K_α is $1/\alpha$.*

Proof. Since $\dim_H(K_\alpha) \leq \dim_B(K_\alpha)$, we get

$$\dim_H(K_\alpha) \leq 1/\alpha.$$

In order to show that $\dim_H(K_\alpha) \geq 1/\alpha$, we will build a probability measure supported on K_α verifying a scaling property. We set as μ_m to the measure

$$(2^{\alpha-1})^m \mathcal{L}|_{K_\alpha^{(m)}},$$

so, $\mu_m(I_{\varepsilon_1,\ldots,\varepsilon_m}) = 2^{-m}$ for every interval $I_{\varepsilon_1,\ldots,\varepsilon_m}$. Clearly, for every $n > m$

$$\mu_n(I_{\varepsilon_1,\ldots,\varepsilon_m}) = \mu_m(I_{\varepsilon_1,\ldots,\varepsilon_m}),$$

and get a sequence $(\mu_m)_{m\geq 1}$ of probability measures. We know that it admits a weak-converging sub-sequence to a probability measure μ, with the properties

- $supp(\mu) = K_\alpha$,
- $\mu(I_{\varepsilon_1,\ldots,\varepsilon_m}) = 2^{-m}$, for every interval $I_{\varepsilon_1,\ldots,\varepsilon_m}$, and all $m \in \mathbb{N}$,

then, $\mu(I_{\varepsilon_1,\ldots,\varepsilon_m}) = |I_{\varepsilon_1,\ldots,\varepsilon_m}|^{1/\alpha}$. Let $B := B(x,r)$ be a closed ball, $x \in K_\alpha$ and $0 < r < 2^{-(\alpha+1)}$. Therefore, there exists a unique $n \in \mathbb{N}$ such that

$$2^{-\alpha(n+1)} < r \leq 2^{-\alpha n}. \tag{1}$$

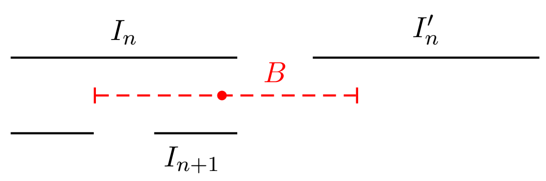

Figure 2: Ball B touching intervals of the generation n and $n+1$.

Then, B intersects at most two (even three, it depends of the parameter α) intervals I_n and I'_n of the n-th generation (see the figure 2), and at least one interval I_{n+1} for the next generation. Hence, by (1)

$$\begin{aligned}\mu(B(x,r)) &\leq \mu(I_n) + \mu(I'_n) \\ &\leq 2^2|B(x,r)|^{1/\alpha} = 2^{1/\alpha+2}r^{1/\alpha},\end{aligned}$$

also, it holds that $\mu(B(x,r)) \geq 2^{1/\alpha-1}r^{1/\alpha}$, so

$$2^{1/\alpha-1}r^{1/\alpha} \leq \mu(B(x,r)) \leq 2^{1/\alpha+2}r^{1/\alpha}. \tag{2}$$

By the Theorem 1, $\dim_H(K_\alpha) \geq 1/\alpha$. ■

We call as $\mathcal{F}_m$ to the set of the lower bounds of the intervals that make up the m-th generation $K_\alpha^{(m)}$, and we take any $x_m \in \mathcal{F}_m$. There is no reason to find $k \in \mathbb{Z}$ such that

$$x_m = k2^{-j}, \quad \text{for } j \in \mathbb{N}.$$

However, since the dyadic intervals make up a uniform partition of $[0,1]$, we can find a unique $k_{x_m} \in \{0,\dots,2^j-1\}$ verifying

$$0 \leq x_m - k_{x_m}2^{-j} \leq 2^{-j}, \tag{3}$$

which means x_m belongs to the dyadic interval $[k_{x_m}2^{-j},(k_{x_m}+1)2^{-j}[$. After this, we define the sequence $(j_m)_{m\geq1}$ by setting $j_m = [\alpha\beta m]$ and the sets $\Lambda_m^{(\alpha,\beta)}$ and $\Lambda^{(\alpha,\beta)}$ as follows

$$\begin{aligned}\Lambda_m^{(\alpha,\beta)} &= \{(j,k) : j = j_m,\ k = k_{x_m} \text{ verifying (3) and } x_m \in \mathcal{F}_m\}, \\ \Lambda^{(\alpha,\beta)} &= \bigcup_{m\geq1} \Lambda_m^{(\alpha,\beta)}.\end{aligned}$$

Finally we define the function $f^{(\alpha,\beta,\gamma)} =: f$ as follows

$$f(x) = \sum_{j\geq1}\sum_{k=0}^{2^j-1} d_{jk}\psi_{jk}(x),$$

where $d_{jk} = 2^{-(\gamma+1/2)j}$ if $(j,k) \in \Lambda^{(\alpha,\beta)}$; otherwise $d_{jk} = 0$. So, by the theorem 3 clearly $f \in C^\gamma(\mathbb{R})$.

3.2 Study of its pointwise Hölder regularity

First, we would like to study the pointwise Hölder regularity of f at the points of the Cantor set K_α. So, we take $x \in K_\alpha$, then, for all $m \geq 1$, there exists a unique $I_{\varepsilon_1,\dots,\varepsilon_m}$ which has this point. Hence, for all m there exists a pair $(j_m,k) \in \Lambda_m^{(\alpha,\beta)}$

such that

$$|x-k2^{-j_m}| \leq 2^{-\alpha m}+2^{-j_m}.$$

Now, let $\delta > 0$ be an arbitrary real number verifying $f \in C^{\delta}(x)$ and we look for any necessary condition. By Theorem (4) and the definition of the function f, for all m large enough[1]

$$\begin{aligned} 2^{-\gamma j_m} &\leq C(2^{-j_m}+|x-k2^{-j_m}|)^{\delta} \\ &\leq C(2^{-j_m}+2^{-j_m/\beta})^{\delta}, \end{aligned}$$

which implies that

$$2^{-\gamma j_m} \leq C_\delta 2^{-j_m \delta/\beta}$$

therefore $\delta \leq \gamma\beta$.

On the other hand, we call as $\widetilde{\mathcal{F}}_m$ to the set of lower bounds of the dyadic intervals corresponding to the pairs which belong to $\Lambda^{(\alpha,\beta)}$, i.e.

$$\widetilde{\mathcal{F}}_m = \{k_{x_m}2^{-j_m} : k_{x_m} \text{ is the integer verifying (3) }\}.$$

Let $\theta \geq 1$ be a parameter. We define the set

$$G_\theta = \limsup_{m \longrightarrow +\infty} \widetilde{\mathcal{F}}_m + 2[-2^{-m\alpha\theta}, 2^{-m\alpha\theta}],$$

since θ is positive, $(G_\theta)_{\theta \geq 1}$ is a non-increasing sequence. Also, if x belongs at least to G_1 then we consider the quantity

$$\theta_x := \sup\{\theta : x \in G_\theta\},$$

it is known as *approximation rate of x by* $\widetilde{\mathcal{F}}_m$.

Now, we are interested in studying the pointwise Hölder regularity in the points x belonging to any G_θ. So, let us take $x \in G_\theta$, then

$$|x-k_{x_m}2^{-j_m}| \leq 2^{-m\alpha\theta+1}$$

for infinitely many $m \geq 1$. So, similarly to the case of the points of K_α, if $f \in C^{\delta}(x)$ then $\delta \leq \beta\gamma/\theta$, which implies that $h_f(x) \leq \beta\gamma/\theta$. Since θ is arbitrary, then

$$h_f(x) \leq \frac{\beta\gamma}{\theta_x}. \tag{4}$$

[1] In Theorem (4) we need that $|x-k2^{-j_m}| \leq \eta$, where η is a constant, so we take m large enough verifying $2^{-\alpha m}+2^{-j_m} \leq \eta$.

Moreover, we will prove the following lemma

Lemma 3. *For all $x \in K_\alpha$, $h_f(x) = \dfrac{\beta\gamma}{\theta_x}$. Otherwise, $h_f(x) = +\infty$.*

Proof. Let us take $x \in K_\alpha$ and $\theta \geq 1$ such that $x \in G_\theta$ (θ exists because $K_\alpha \subseteq G_1$). Thanks to the inequality (4), it is enough to prove that

$$f \in \bigcap_{\varepsilon>0} C^{\beta\gamma/\theta_x-\varepsilon}(x).$$

Let $\widetilde{\varepsilon} > 0$ be a positive real number. Since $f \in C^\gamma(\mathbb{R})$, by Theorem 4 it is enough to find a constant $C > 0$ such that for all $m \geq 1$

$$2^{(-\gamma+1/2)j_m} \leq C 2^{-j_m/2}(2^{-j_m} + |x - k_{x_m} 2^{-j_m}|)^{\beta\gamma/\theta_x - \widetilde{\varepsilon}},$$

in other words,

$$2^{j_m}(2^{-j_m} + |x - k_{x_m} 2^{-j_m}|)^{\beta/\theta_x - \widetilde{\varepsilon}/\gamma} \geq C.$$

Also, we can choose $\varepsilon := \widetilde{\varepsilon}/\eta$, where $\eta > 0$ is large enough verifying

$$\eta > \frac{1}{\theta_x}\left(\frac{\beta\gamma}{\theta_x} - \widetilde{\varepsilon}\right), \tag{5}$$

and such that $x \notin G_{\theta_x+\varepsilon}$. Therefore, there exists an integer $M := M_\varepsilon > 0$ large enough such that

$$|x - k_{x_m} 2^{-j_m}| > 2^{-m\alpha(\theta_x+\varepsilon)+1} \quad \text{for all } m \geq M.$$

Hence[2]

$$2^{j_m}(2^{-j_m} + |x - k_{x_m} 2^{-j_m}|)^{\beta/\theta_x - \widetilde{\varepsilon}/\gamma} \geq 2^{j_m} \cdot \frac{1}{2}\left(2^{-j_m(\beta/\theta_x - \widetilde{\varepsilon}/\gamma)} + C_{\widetilde{\varepsilon}} 2^{-m\alpha(\beta/\theta_x - \widetilde{\varepsilon}/\gamma)(\theta_x+\varepsilon)}\right),$$

and by the condition (5) it holds that $(\beta/\theta_x - \widetilde{\varepsilon}/\gamma)(\theta_x + \varepsilon) < \beta$; so for all $m \geq M$

$$\begin{aligned} 2^{j_m}(2^{-j_m} + |x - k_{x_m} 2^{-j_m}|)^{\beta/\theta_x - \widetilde{\varepsilon}/\gamma} &\geq 2^{j_m} \cdot \frac{1}{2}\left(2^{-j_m(\beta/\theta_x - \widetilde{\varepsilon}/\gamma)} + C_{\widetilde{\varepsilon}} 2^{-m\alpha\beta}\right) \\ &\geq \frac{1}{2}\left(2^{-j_m(\beta/\theta_x - \widetilde{\varepsilon}/\gamma - 1)} + C_{\widetilde{\varepsilon}} 2^{-\{m\alpha\beta\}}\right). \end{aligned}$$

Now, we note that $2^{-\{m\alpha\beta\}} \geq 1/2$ for all m, therefore we obtain

$$2^{j_m}(2^{-j_m} + |x - k_{x_m} 2^{-j_m}|)^{\beta/\theta_x - \widetilde{\varepsilon}/\gamma} \geq C_{\widetilde{\varepsilon}}, \quad \text{for all } m \geq M.$$

[2] By using the inequality $(a+b)^s \geq \frac{1}{2}(a^s + b^s)$, where $a, b, s > 0$.

For the case $1 \leq m < M$, we observe that

$$2^{j_m}(2^{-j_m} + |x - k_{x_m} 2^{-j_m}|)^{\beta/\theta_x - \widetilde{\varepsilon}/\gamma} \geq 2^{j_1}\left(2^{-j_M(\beta/\theta_x - \widetilde{\varepsilon}/\gamma)}\right),$$

so, by choosing $C = \min\{2^{j_1}\left(2^{-j_M(\beta/\theta_x - \widetilde{\varepsilon}/\gamma)}\right), C_{\widetilde{\varepsilon}}\}$ as the required constant, we are done.

Now, let $x \notin K_\alpha$. Since K_α is compact, there exists $\delta > 0$ such that $dist(x, K_\alpha) = \delta$. So, we can find $M > 0$ large enough such that $x \notin K_\alpha^{(m)}$ and $2^{-j_m} \leq \delta/2$, for all $m \geq M$. Therefore

$$\begin{aligned} |x - k_{x_m} 2^{-j_m}| &\geq |x - x_m| - |x_m - k_{x_m} 2^{-j_m}| \\ &\geq \frac{\delta}{2} \quad \text{for all } x_m \in \mathcal{F}_m, \end{aligned}$$

so, for all $m \geq M$ and any $s > 0$

$$2^{j_m}(2^{-j_m} + |x - k_{x_m} 2^{-j_m}|)^s \geq C_\delta,$$

for the case $1 \leq m < M$, we proceed in the same way as before. So, as s is arbitrary, $h_f(x) = +\infty$. ■

3.3 Hausdorff dimension of the set $\widetilde{G}_\theta$

Now, the objective is to characterize the level set $E_f(s)$ by applying the Lemma 3.

Let us define the following set

$$\widetilde{G}_\theta = \bigcap_{\theta' < \theta} G_{\theta'} \setminus \bigcup_{\theta' > \theta} G_{\theta'},$$

so, $\widetilde{G}_\theta \subset G_{\theta'}$ for all $\theta' < \theta$, then

$$\dim_H(\widetilde{G}_\theta) \leq \dim_H(G_{\theta'}) \quad \text{for all } \theta' < \theta. \tag{6}$$

Let $\theta' < \theta$. We want to prove that

$$\dim_H(G_{\theta'}) \leq \frac{1}{\alpha\theta'},$$

for this purpose, we take any $d > \frac{1}{\alpha\theta'}$ and we will show that $dim_H(G_{\theta'}) \leq d$. By definition, it is clear that for each $k \geq 1$, $\{B(x_m, 2^{-m\alpha\theta'+1})\}_{x_m \in \widetilde{\mathcal{F}}_m, m \geq k}$ is a covering of $G_{\theta'}$. So, by setting $a_m := |B(x_m, 2^{-m\alpha\theta'+1})|$ we get

$$\begin{aligned}\mathcal{H}^d_{a_m}(G_{\theta'}) &\le \sum_{m\ge k}\sum_{x_m\in\widetilde{\mathcal{F}}_m} a_m^d \\ &= 4^d \sum_{m\ge k} 2^{-(\alpha\theta' d-1)m},\end{aligned}$$

so, we can take k large enough in such a way $\mathcal{H}^d_{a_m}(G_{\theta'}) < 1$, therefore $\mathcal{H}^d(G_{\theta'}) < 1$ and we are done. So

$$\dim_H(G_{\theta'}) \le \frac{1}{\alpha\theta'} \quad \text{for all } \theta' < \theta,$$

which, together with inequality (6), implies that

$$\dim_H(\widetilde{G}_\theta) \le \frac{1}{\alpha\theta}.$$

In order to prove that $\dim_H(\widetilde{G}_\theta) \ge \frac{1}{\alpha\theta}$, we proceed as in the proof of the Lemma 2; we will construct a probability measure supported on G_θ, but also with a scaling property. So, we consider the following result whose demonstration is inspired by [3]:

Lemma 4. *There exists a probability measure μ_θ supported on G_θ verifying*

$$\mu_\theta(D) \le |D|^{\frac{1}{\alpha\theta}}(\log|D|)^2 \quad \textit{for all ball } D\subset\mathbb{R} \textit{ small enough.}$$

Proof. We will divide the proof in two stages.
Stage 1. *Construction of the measure μ_θ supported on G_θ.* First, given a closed ball $B = B(x,r)$, we set $cB := B(x, cr)$ and $B^\rho := B(x, r^\rho)$, where $x\in\mathbb{R}$ and $r, c, \rho > 0$. Also, for simplicity, we will denote as C_p to any constant depending on the parameter p.

Since $K_\alpha \subset G_1$ then

$$[0,1] = G_1 \quad \mu - a.e. \tag{7}$$

where μ is the measure supported on K_α, which we built in the Lemma 2.

We now construct a Cantor set $K_{\alpha,\theta}\subset G_\theta$ and simultaneously a probability measure μ_θ supported by this set, with the scaling property

$$\mu_\theta(D) \le |D|^{\frac{1}{\alpha\theta}}(\log|D|)^2 \quad \text{for all ball } D\subset\mathbb{R} \text{ small enough.} \tag{8}$$

We begin with the construction of the first generation of the Cantor set as follows: we select a finite set $\{B_{\phi(m)}\}$ from the family $\{B_j\}_{j\ge1}$, which is an ordering of the family $\{B(\lambda_m, 2^{-m\alpha+1})\}_{\lambda_m \in \widetilde{\mathcal{F}}_m, m\geq 1}$. We choose $\phi(1)$ in such a way

$$\left|B^{\theta}_{\phi(1)}\right| < \frac{1}{e}, \tag{9}$$

this condition will be useful in the next stage. Now, $\phi(2)$ is the first index such that $B_{\phi(2)}$ is not included in $5B_{\phi(1)}$, $\phi(3)$ is the first index such that $B_{\phi(3)}$ is not included in $5B_{\phi(1)} \cup 5B_{\phi(2)}$; recursively, $\phi(l)$ is the first index such that $B_{\phi(l)}$ is not included in

$$\bigcup_{i=1}^{l-1} 5B_{\phi(i)}. \tag{10}$$

We stop this extraction at the first index N such that

$$\mu\left(\bigcup_{i=1}^{N} 5B_{\phi(i)}\right) \geq \frac{1}{2}; \tag{11}$$

this index exists because each ball B_m which has not been selected among the family $\{B_{\phi(i)}\}_{1\leq i\leq N}$ is included in the union of the balls previously selected, then

$$\bigcup_{j=\phi(1)}^{\phi(N)} B_j \subset \bigcup_{i=1}^{N} 5B_{\phi(i)}.$$

Since $\mu - a.e.\, x \in [0,1]$ belongs to G_1,

$$\mu\left(\bigcup_{j=\phi(1)}^{n} B_j\right) \longrightarrow 1, \quad \text{as } n \longrightarrow +\infty.$$

Hence, the inequality (11) follows if N is large enough. Also, since the radius of the balls make up a non-increasing sequence, the balls $B_{\phi(i)}$ are disjoint. In addition, by the subadditivity and scaling properties (2) of μ

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{N} 5B_{\phi(i)}\right) &\leq \sum_{i=1}^{N} \mu\left(5B_{\phi(i)}\right)\\ &\leq C_\alpha \sum_{i=1}^{N} \mu\left(B_{\phi(i)}\right),\end{aligned}$$

in other words

$$\sum_{i=1}^{N} \mu\left(B_{\phi(i)}\right) \geq \frac{1}{C_\alpha}. \tag{12}$$

The N balls $B^{\theta}_{\phi(i)}$ make up the first generation balls of our generalized Cantor set. The measure μ_θ will be supported by the union of these balls and we take

$$\mu_\theta\left(B^\theta_{\phi(i)}\right) := \frac{\mu(B_{\phi(i)})}{\sum_{j=1}^N \mu\left(B_{\phi(j)}\right)}.$$

The scaling property of μ implies that $\mu\left(B_{\phi(i)}\right) \le C_{\alpha,\theta}|B^\theta_{\phi(i)}|^{\frac{1}{\alpha\theta}}$, and by the inequality (12) we get

$$\mu_\theta\left(B^\theta_{\phi(i)}\right) \le C_{\alpha,\theta}\left|B^\theta_{\phi(i)}\right|^{\frac{1}{\alpha\theta}}. \tag{13}$$

We will not construct the second generation of the Cantor set by 'subdividing' each $B^\theta_{\phi(i)}$ (as we did for the second generation of K_α) but rather we will consider some other balls contained in it. Let $J \ge 1$ be a large enough integer such that

$$\frac{1}{2\varepsilon_J} \ge e^{1/2\varepsilon_{\phi(N)}}, \tag{14}$$

where ε_j is the radius of the ball B_j. Also, let us take one the balls $B^\theta_{\phi(i)}$; since $\cup_{j\ge J}B_j$ covers almost every point of $B^\theta_{\phi(i)}$ (because of property (7)), we can as above select a finite number of balls $B_{\phi(i,1)}, \dots, B_{\phi(i,N(i))}$ from the sequence $\{B_j\}_{j\ge J}$ such that

$$\mu\left(\bigcup_{j=1}^{N(i)} 5B_{\phi(i,j)}\right) \ge \frac{1}{2}\mu\left(B^\theta_{\phi(i)}\right).$$

Again, the balls $B_{\phi(i,j)}$ are disjoint, so that

$$\sum_{j=1}^{N(i)} \mu\left(B_{\phi(i,j)}\right) \ge \frac{1}{2C_\alpha}\mu\left(B^\theta_{\phi(i)}\right). \tag{15}$$

The balls $\{B^\theta_{\phi(i,j)}\}_{1\le j\le N(i),1\le i\le N}$ are the second generation balls and we naturally take

$$\mu_\theta\left(B^\theta_{\phi(i,j)}\right) := \mu_\theta\left(B^\theta_{\phi(i)}\right)\frac{\mu(B_{\phi(i,j)})}{\sum_{j=1}^{N(i)} \mu\left(B_{\phi(i,j)}\right)} \quad \text{for all } i, j. \tag{16}$$

The scaling property of μ again implies that $\mu(B_{\phi(i,j)}) \le C_{\alpha,\theta}|B^\theta_{\phi(i,j)}|^{\frac{1}{\alpha\theta}}$, from which by the inequality (15) the following bound holds

$$\mu_\theta\left(B^\theta_{\phi(i,j)}\right) \le C_{\alpha,\theta}\left|B^\theta_{\phi(i,j)}\right|^{\frac{1}{\alpha\theta}} \frac{\mu_\theta\left(B^\theta_{\phi(i)}\right)}{\mu\left(B^\theta_{\phi(i)}\right)} \quad \text{for all } i, j. \tag{17}$$

This construction is iterated and we thus obtain a generalized Cantor set $K_{\alpha,\theta}$ and a probability measure μ_θ supported by it.

Stage 2. *Scaling property of μ_θ.* We will call as fundamental balls to the balls constructed at each generation. Also, if B is a fundamental ball, we denote as $\widehat{B}$ the fundamental ball from which B was obtained (*the 'father' of B*).

On the other hand, let us remember that by (14) the diameters of the fundamental balls have been chosen such that if B is a fundamental ball of the $n-th$ generation

$$\frac{1}{|B|} \geq e^{\sup 1/|J|},$$

where the supremum is taken over all fundamental balls J of the previous generation, hence

$$\frac{1}{|J|} \leq \log \frac{1}{|B|} \quad \text{for all } J. \tag{18}$$

Now, our objective is to prove that every interval D satisfies the property (8). We will first check it for the fundamental balls, by induction on the generation of the ball. So, by the inequalities (9) and (13)

$$-\log \left| B^\theta_{\phi(i)} \right| > 1 \quad \text{for all } i, \tag{19}$$

in other words

$$\left(\log \left| B^\theta_{\phi(i)} \right|\right)^2 > 1 \quad \text{for all } i. \tag{20}$$

Now, we assume that (8) holds for the balls of the generation $n-1$. The analogue of (17) at the n-th generation states that any fundamental ball B satisfies

$$\mu_\theta(B) \leq C_{\alpha,\theta} \, |B|^{\frac{1}{\alpha\theta}} \, \frac{\mu_\theta\left(\widehat{B}\right)}{\mu\left(\widehat{B}\right)} \leq C_{\alpha,\theta} \, |B|^{\frac{1}{\alpha\theta}} \, \frac{\mu_\theta\left(\widehat{B}\right)}{|\widehat{B}|^{1/\alpha}},$$

which, using the induction hypothesis, implies that

$$\begin{aligned} \mu_\theta(B) &\leq C_{\alpha,\theta} \, |B|^{\frac{1}{\alpha\theta}} \, |\widehat{B}|^{\frac{1}{\alpha\theta}} \frac{(\log |\widehat{B}|)^2}{|\widehat{B}|^{1/\alpha}} \\ &\leq C_{\alpha,\theta} \, |B|^{\frac{1}{\alpha\theta}} \, |\widehat{B}|^{\frac{1}{\alpha}(\frac{1}{\theta}-1)} (\log |\widehat{B}|)^2. \end{aligned}$$

By the inequality (18)

$$\frac{1}{|\widehat{B}|} \leq \log \frac{1}{|B|} = |\log |B||,$$

also, (19) implies that any fundamental ball B satisfies $\log(1/|B|) > 0$. Hence

$$(\log|\widehat{B}|)^2 \leq (\log|\log|B||)^2,$$

on the other hand

$$\begin{aligned}|\widehat{B}|^{\frac{1}{\alpha}(\frac{1}{\theta}-1)} &\leq \left(\log\frac{1}{|B|}\right)^{\frac{1}{\alpha}(1-\frac{1}{\theta})} \\ &\leq |\log|B||,\end{aligned}$$

because $\frac{1}{\alpha}(1-\frac{1}{\theta}) \leq 1$. Therefore

$$\begin{aligned}\mu_\theta(B) &\leq C_{\alpha,\theta}\,|B|^{\frac{1}{\alpha\theta}}\,|\log|B||(\log|\log|B||)^2 \\ &\leq C_{\alpha,\theta}\,|B|^{\frac{1}{\alpha\theta}}\,(\log|B|)^2.\end{aligned}$$

Let now D be an arbitrary ball. Later, we will see how small the ball must be in order to satisfy the property (8). If D does not intersect the Cantor set $K_{\alpha,\theta}$ then $\mu_\theta(D) = 0$. Otherwise, D intersects at least one ball from each generation of the Cantor set. If D intersects a single ball in each generation, then $D \cap K_\alpha$ is a single point x_0, which again implies that $\mu_\theta(D) = \mu_\theta(\{x_0\}) = 0$. On the other hand, let B be the fundamental ball of the smallest generation which intersects D and such that at least two children of B intersects D; there exists exactly one such ball. Indeed, if there was a ball $\underline{B}$ from the same generation as B, which touches D, then we will have two cases:

- B and $\underline{B}$ have the same father. It immediately contradicts that B belongs to the smallest generation satisfying the condition.
- B and $\underline{B}$ do not have the same father. Since the balls are contained in their fathers, D intersects them. So, we recursively apply the same idea, and at the end D will touch two balls with the same father (we can assume that $[0,1]$ is the father of the balls of the first generation), returning to the previous case.

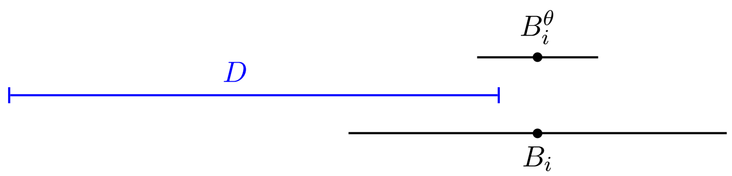

Figure 3: Estimation for the measure of $B_i \cap D$.

If $|D| \geq |B|$, since D touches only the fundamental ball B from its generation

$$\mu_\theta(D) = \mu_\theta(D \cap B) \leq \mu_\theta(B) \\ \leq C_{\alpha,\theta}\, |B|^{\frac{1}{\alpha\theta}}\, (\log |B|)^2.$$

The function $\rho \longmapsto \rho^{\frac{1}{\alpha\theta}}(\log \rho)^2$ is increasing in a neighborhood of the origin, which will be called as $\mathcal{V}_0$. We take the ball D small enough so that it satisfies $|D| < |\mathcal{V}_0|$; hence

$$\mu_\theta(D) \leq |D|^{\frac{1}{\alpha\theta}}\, (\log |D|)^2.$$

Otherwise, $|D| < |B|$ and we denote as $B_1^\theta, \ldots, B_p^\theta$ the children of B intersecting D. Without lost of generality, B_i is small enough, so $|D| > |B_i|$[3], and since $|B_i| >> |B_i^\theta|$, then there exists a constant $C > 0$, which does not depend on the balls, such that

$$\mu(B_i \cap D) \geq C\mu(B_i) \quad \text{for all } 1 \leq i \leq p. \tag{21}$$

This estimate holds because D contains points of B_i^θ which are very close to the center of B_i, so we could say that $\mu(B_i \cap D) \sim \mu(B_i)/2$ (see the figure 3).

On the other hand, by using (15) and (16) applied at the corresponding generation, it holds that

$$\mu_\theta(D) \leq \sum_{i=1}^{p} \mu_\theta(B_i^\theta) \\ \leq C_\alpha\, \mu_\theta(B) \frac{\sum_{i=1}^{p} \mu(B_i)}{\mu(B)}.$$

Hence, by the estimation (21) and $B_i \cap B_j$ are disjoint, one has

$$\mu_\theta(D) \leq C_\alpha\, \mu_\theta(B) \frac{\mu(D)}{\mu(B)} \\ \leq C_{\alpha,\theta}\, |B|^{\frac{1}{\alpha}(\frac{1}{\theta}-1)}\, (\log |B|)^2\, |D|^{\frac{1}{\alpha}}.$$

The function $\rho \longmapsto \rho^{\frac{1}{\alpha}(\frac{1}{\theta}-1)}(\log \rho)^2$ is decreasing near the origin and without lost of generality, $|B|$ is small enough [4], so D is also small. Therefore, one finally gets

$$\mu_\theta(D) \leq C_{\alpha,\theta}\, |D|^{\frac{1}{\alpha\theta}}\, (\log |D|)^2.$$

■

[3] Otherwise, D could not touch more than one ball B_i, because thanks to the way we choose them, as we can see in (10), they are 'sufficiently' separated. Actually, the distance is at least two times the diameter of some other ball.

[4] We can always choose balls small enough in the construction of the Cantor set $K_{\alpha,\theta}$. Actually, we can choose $\phi(1)$ in such a way that $|B^\theta_{\phi(1)}|$ is small enough.

On the other hand, since $(G_\theta)_{\theta\geq 1}$ is a non-increasing sequence

$$G_\theta \subset \bigcap_{\theta'<\theta} G_{\theta'},$$

so

$$\widetilde{G}_\theta \supset G_\theta \setminus \bigcup_{\theta'>\theta} G_{\theta'}.$$

Also, it holds that

$$\mu_\theta(G_{\theta'}) = 0 \quad \text{for all } \theta' > \theta. \tag{22}$$

Indeed, if $\theta' > \theta$ by the *Borel Cantelli Lemma*, it is enough to prove

$$\sum_{m\geq 1} \mu_\theta\left(G_{\theta'}^m\right) < +\infty,$$

where $G_{\theta'}^m := \widetilde{\mathcal{F}}_m + 2[-2^{-m\alpha\theta'}, 2^{-m\alpha\theta'}]$. By applying the property (8) to the intervals shown in $G_{\theta'}^m$, for all m large enough

$$\begin{aligned}\mu_\theta\left(G_{\theta'}^m\right) &\leq \sum_{x_m\in\widetilde{\mathcal{F}}_m} \mu_\theta\left(B(x_m, 2^{-m\alpha\theta'+1})\right)\\ &\leq C_{\alpha,\theta}\, 2^m \left(2^{-m\alpha\theta'+2}\right)^{\frac{1}{\alpha\theta}} (\log(2^{-m\alpha\theta'+2}))^2,\end{aligned}$$

hence, for m large enough

$$\mu_\theta\left(G_{\theta'}^m\right) \leq C_{\alpha,\theta}\, 2^{-(\theta'/\theta-1)m} (\log(2^{-m\alpha\theta'+2}))^2,$$

since $\theta' > \theta$, we are done.

In addition, by the monotonicity of $(G_\theta)_{\theta\geq 1}$

$$G_\theta \supset \bigcup_{\theta'>\theta} G_{\theta'},$$

thus, by (22) and since $supp(\mu_\theta) \subset G_\theta$

$$\begin{aligned}\mu_\theta\left(G_\theta \setminus \bigcup_{\theta'>\theta} G_{\theta'}\right) &= \mu_\theta(G_\theta) - \mu_\theta\left(\bigcup_{\theta'>\theta} G_{\theta'}\right)\\ &= 1,\end{aligned}$$

so that $\mu_\theta(\widetilde{G}_\theta) = 1$ as well. Now, thanks to the property (8) we obtain

$$\underline{d_{\mu_\theta}}(x) \geq \frac{1}{\alpha\theta}, \quad \text{for } \mu_\theta - a.e.\, x \in G_\theta. \tag{23}$$

Indeed, for $\mu_\theta - a.e. x \in G_\theta$ and $r > 0$ small enough, by (8)

$$\frac{\log \mu_\theta(B(x,r))}{\log r} \geq \frac{1}{\alpha\theta}\left(\frac{\log |(B(x,r)|}{\log r}\right) + 2\frac{\log |\log |B(x,r)||}{\log r} + \frac{C_{\alpha,\theta}}{\log r}$$

and by taking $r \longrightarrow 0$, we are done. Finally, the *Billingsley Lemma* say us that (23) implies

$$\dim_H(\widetilde{G}_\theta) \geq \frac{1}{\alpha\theta}.$$

3.4 Multifractal spectrum of f

Thanks to the Lemma 3, for $\gamma \leq s \leq \beta\gamma$

$$\begin{aligned} E_f(s) &= \{x \in K_\alpha : \theta_x = \frac{\beta\gamma}{s}\} \\ &= \bigcap_{\theta' < \beta\gamma/s} G_{\theta'} \setminus \bigcup_{\theta' > \beta\gamma/s} G_{\theta'}, \end{aligned}$$

i.e. $E_f(s) = \widetilde{G}_{\beta\gamma/s}$, otherwise $E_f(s) = \emptyset$. So, by the previous chapter

$$\sigma_f(s) = \frac{s}{\alpha\beta\gamma} = \varphi(s).$$

3.5 General case of the main theorem

First, without lost of generality, we can assume that f is compactly supported, for example, by multiplying f with a function $\xi \in C_c^\infty(\mathbb{R})$, equal to 1 in a neighborhood of K_α. Moreover, even if we must consider a translation and dilation of the domain (i.e. to take $f(px+q)$), we can assume that the support is included in a given interval I because it does not change its multifractal spectrum. Since $\varphi \in \mathcal{C}$, there exists a family of positive constants $\{(a_n, b_n, c_n)\}_{n\in\mathbb{N}}$ such that

$$\varphi = \sup_{n\in\mathbb{N}} \varphi_n,$$

where $\varphi_n(s) = c_n\, s\mathbb{1}_{[a_n,b_n]}$. By the previous sections, for each n we can construct a family of functions $f_n := f^{(\alpha_n,\beta_n,\gamma_n)}$ compactly supported in I_n, where $\{I_n\}_n$ is a countable partition of $\mathbb{R}$, and such that $\sigma_{f_n} = \varphi_n$. Hence, it is enough to take $f = \sum_{n\geq 1} f_n \in C(\mathbb{R})$; so $E_f(s) = \bigcup_n E_{f_n}(s)$ and therefore

$$\sigma_f = \sup_n \sigma_{f_n} = \varphi.$$

4 Conclusions

In this paper we have addressed the problem of prescribing multifractal behavior for a univariate and one-dimensional case. We have successfully demonstrated a classical result in multifractal analysis for functions using mainly wavelet techniques.

The ideas that have been used in this paper should be useful in tackling more difficult problems such as the multivariate case, i.e, for n objects $(X_i)_{i=1,...n}$, we would like to study the intersection of level sets

$$E_{x_1,...,X_n}(s_1, s_2, ...s_n) = \bigcap_{i=1}^{n} E_{X_i}(s_i),$$

so that we must construct deterministic and random objects, functions and probability measures whose *multivariate multifractal properties* is prescribed in advance. It is strongly recommended to study these problems because there are many phenomena where the data are intrinsically correlated.

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