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Master thesis

Partial regularity of minimizers of free discontinuity problems

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To

Daysi and David.

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Contents

Notations

$a \wedge b$	minimun between a, b
$a \vee b$	maximun between a, b.
$\mathcal{L}^n$	Lebesgue measure of dimension n.
$\mathcal{H}^d$	Hausdorff measure of dimension d.
$\overline{E}$	Topological closure of E.
∂E	Topological boundary of E.
$B_r(x)$	Open ball with center in x and radius r ($x = 0$ can be omited).
$\omega_n r^n$	volume of the ball of radius r
$ap \lim_{y \to x} f(y)$	approximate limit
$apDf(x)$	approximate differential
Du	Distributional derivative of u
$D^a u$	absolutely continuous part of derivative
$D^s u$	singular part of derivative
$D^j u$	jump part of derivative
$D^c u$	Cantor part of derivative
$\|Du\|$	total variation of Du
$\|\partial E\|$	perimeter of E
$\partial^* E$	reduced boundary of E
$\partial_* E$	measure theoretical boundary of E

Introduction

The Mumford-Shah conjecture says, in dimension 2, that the singular set of a reduced minimizer of the Mumford-Shah functional is locally a C^1 curve, except at a finite number of points and, the physic point of view of the Mumford-Shah problem is related to image segmentation. The conjecture was proposed in [23].

This famous problem is about the minimization of the functional

$$J(K,u) := \int_{\Omega\setminus K} |\nabla u|^2 + \alpha (u-g)^2 \, dx + \beta \mathcal{H}^{n-1}(K \cap \Omega),$$

where $\Omega \subset \mathbb{R}^n$, $g \in L^\infty(\Omega)$ and the positive parameters α, β are given; the problem is to minimize J in the set of the admissible pairs

$$\mathcal{A} := \left\{ (K,u); K \subset \overline{\Omega}, u \in W^{1,2}_{loc}(\Omega\setminus K) \right\}.$$

To approach to the possible solution of the conjecture, the method is to minimize the weak formulation of J, i.e., minimize the functional

$$F(u) := \int_\Omega |\nabla u|^2 + \alpha(u-g)^2 dx + \beta\mathcal{H}^{n-1}(S_u),$$

where S_u is the jump set of $u \in SBV(\Omega)$.

We do not have yet the solution of this conjecture, but some results have been obtained, simultaneously and independently, by A. Bonnet ([9, 8, 10]), by L. Ambrosio, N. Fusco and D. Pallara ([6, 4]) and by G. David ([13]). We will follow the work done by L. Ambrosio, N. Fusco and D. Pallara, for that reason, is important to study first the theory of special bounded variation functions.

In the first chapter, we give some basic (more or less) results which come from Measure Theory and Geometric Measure Theory.

The second chapter is devoted to the basic theory of BV functions, for instance a very important tool the coarea formula. In this chapter, a lot of proofs of the theorems in this chapter could be found in [5, 17].

In chapter three, we study the construction of special bounded variation functions ($SBV(\Omega)$). Almost all of the results of this chapter have their proofs which are based in the work of L. Ambrosio in [3]. Here, we can find two important results, compactness of $SBV(\Omega)$ and the chain rule in $SBV(\Omega)$, which are fundamental to show the existence of minimzers of Mumford-Shah functional,

In the last chapter, we follow the work of [16] to find a minimizers of the next problem

$$\min_{u=g, u\in SBV(\Omega)} \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u), \tag{1}$$

where $g \in L^\infty(\Omega)$, and Ω is a domain with Lipschitz boundary. Furthermore, we follow the work of Babadjian and Giacomini (see [7]) with the difference that we are supposing that we only have Dirichlet condition. Our aim is obtain the existence of minimizers, quasi-minimizers and some lower bounds properties (i.e., Alfhors-David regularity) and the essential closely of the jump set in the problem (1).

The last section is devoted to prove the existence of minimizers for the strong formulation of the Mumford-Shah functional using the existence of minimizers and quasiminimizers for the weak formulation and a lower density bound (Alfhors regularity) of the jump set.

At the end, there is one annex, which is a basic result of Geometric Measure Theory.

Chapter 1

Preliminaries

In this chapter, we present some important tools (theorems and definitions) from Geometric Measure Theory and Measure Theory.

Definition 1.1 (Approximate limit)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$. We say that $l \in \mathbb{R}^m$ is the approximate limit of f as $y \to x$, written*

$$ap \lim_{y \to x} f(y) = l,$$

if for each $\epsilon > 0$,

$$\lim_{r \to 0} \frac{\mathcal{L}^n \left(B_r(x) \cap \{ |f - l| \geq \epsilon \} \right)}{\mathcal{L}^n \left(B_r(x) \right)} = 0.$$

Definition 1.2 (Approximate differentiability)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$. We say that f is approximately differentiable at $x \in \mathbb{R}^n$ if there exists a linear mapping*

$$L : \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$ap \lim_{y \to x} \frac{|f(y) - f(x) - L(x - y)|}{|x - y|} = 0.$$

Remark 1.1. *In the last definition, if L exists, it is unique and we write it as $apDf(x)$*

Definition 1.3 (Densities). *Let $E \subset \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is a point of density* 1 *for E if*

$$\lim_{r\to 0} \frac{\mathcal{L}^n\left(B_r\left(x\right)\cap E\right)}{\omega_n r^n} = 1$$

and a point of density 0 *for E if*

$$\lim_{r\to 0} \frac{\mathcal{L}^n\left(B_r\left(x\right)\cap E\right)}{\omega_n r^n} = 0.$$

Definition 1.4. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^k$ be a Borel map, and let $\mu = (\mu_1, \ldots, \mu_p)$ be a vector measure with finite total variation in $\mathbb{R}^n$. We canonically define a vector measure $\varphi_\#\left(u\right)$ in $\mathbb{R}^k$ by setting*

$$\varphi_\#\left(\mu\right)\left(F\right) = \mu\left(\varphi^{-1}\left(F\right)\right)$$

for any Borel set $F \subset \mathbb{R}^k$.

Lemma 1.1. *Under the assumptions above, the image of φ contains a Borel set in which $\varphi_\#\left(|\mu|\right)$ is supported. In addition, if φ is one to one the following equality holds*

$$|\varphi\left(\mu\right)| = \varphi_\#\left(|\mu|\right). \tag{1.1}$$

Proof. According to Lusin's Theorem, there exists (K_h) an increasing sequence of compact subsets of Ω such that the restriction of φ to K_h is continuous and $|\mu|\left(\Omega\backslash K_h\right) \to 0$. Then, $\varphi\left(K_h\right)$ are compact sets whose union covers $\varphi_\#\left(|\mu|\right)$-almost all of $\mathbb{R}^k$. Indeed,

$$\mu\left(\phi^{-1}\left(K_h\right)\right) \le \mu\left(K_h\right) \to \mu\left(\Omega\right).$$

If φ is one to one, we only need to prove that

$$|\varphi_\#\left(\mu\right)|\left(\varphi\left(K_h\right)\right) \ge \varphi_\#\left(|\mu|\right) = |\mu|\left(K_h\right)$$

for any $h \in \mathbb{N}$. For any $\varepsilon > 0$ we can find mutually disjoint compacts sets $C_i \subset K_h$ such that

$$\sum_{i=1}^{+\infty} |\mu (C_i)| \geq |\mu| (K_h) - \varepsilon.$$

We have that $D_i = \varphi (C_i)$ are mutually disjoint compact subsets of $\varphi (K_h)$ because φ is one to one, hence

$$|\varphi_{\#} (\mu)| (\varphi (K_h)) \geq \sum_{i=1}^{+\infty} |\varphi_{\#} (\mu) (D_i)| = \sum_{i=1}^{+\infty} |\mu (C_i)| \geq |\mu| (K_h) - \varepsilon.$$

By arbitrarily of h and ε, we have the conclusion of this lemma. The inequality $|\varphi_{\#} (\mu)| \leq \varphi_{\#} (|\mu|)$ follows directly by the definition of total variation. $\square$

Definition 1.5 (Hausdorff metric of compacts sets)**.** *Let K_1 and K_2 be compact subsets of $\overline{\Omega}$. The Hausdorff distance between K_1 and K_2 is given by*

$$d_{\mathcal{H}} (K_1, K_2) := \max \left\{ \sup_{x \in K_1} dist\,(x, K_2), \sup_{y \in K_2} dist\,(y, K_1) \right\}.$$

Definition 1.6. *We say that a sequence (K_n) of compact subsets of $\overline{\Omega}$ converges in the Hausdorff metric to the compact set K if $d_{\mathcal{H}} (K_n, K) \to 0$.*

The following theorem gives us a compactness result, for further reference see [5], Theorem 6.1.

Theorem 1.2 (Blaschke)**.** *The collection of all nonempty closed subset of $\overline{\Omega}$ is a compact metric space, when endowed with the Hausdorff metric.*

Chapter 2

Space BV

In this chapter, we are going to enunciate and proof some important properties of the space BV.

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ and $u \in L^1(\Omega)$. We says that u has bounded variation if there exists a vector measure $Du = (D_1 u, D_2 u, \ldots, D_n u)$ with finite total variation in Ω, such that*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega \varphi d D_i u, \quad \forall \varphi \in C_0^1(\Omega).$$

The space of all bounded variation distributions will be denoted as $BV(\Omega)$.

Let $u \in BV(\Omega)$, the total variation of Du is

$$|Du|(\Omega) := \sup \left\{ \sum_{i=1}^n \int_\Omega u \frac{\partial \varphi}{\partial x_i} dx; \varphi \in C_c^1(\Omega), \|\varphi\|_\infty \leq 1 \right\},$$

Notation 2.1 ([17]). *Let $u \in BV(\Omega)$. By Lebesgue's decomposition theorem, we infer that $Du = D^a u + D^s u$ and that $D^a u << \mathcal{L}^n$ and $D^s u \perp \mathcal{L}^n$. We have also that ∇u is the density of $D^a u$ with respect to $\mathcal{L}^n$. Thus, we write*

$$D^a u = \mathcal{L}^n \llcorner Du = \nabla u \mathcal{L}^n$$

and then,

$$Du = \nabla u \mathcal{L}^n + D^s u.$$

Theorem 2.1. *For all* $f \in BV_{loc}(\mathbb{R}^n)$ *we have that* f *is approximately differentiable* $\mathcal{L}^n$*-a.e.*

Proof. By the Differentiability of BV functions (See Theorem 6.1 in [17]) and the Hölder inequality, we have that for $\mathcal{L}^n$-a.e $x \in \mathbb{R}^n$,

$$\fint_{B_r(x)} |f(y) - f(x) - (\nabla f(x), y - x)| dy = o(r) \quad as\ r \to \infty. \tag{2.1}$$

Let $x \in \mathbb{R}^n$ so that we have the property above. We proceed by contradiction, we suppose that there exists $\theta > 0$ such that f is not approximate differentiable at x, i.e.,

$$ap \limsup_{y \to x} \frac{|f(y) - f(x) - (\nabla f(x), y - x)|}{|y - x|} > \theta > 0.$$

Then, by definition of approximate differentiable, there exist a sequence (r_j) such that $r_j \to 0$ and a constant $\gamma > 0$ so that

$$\frac{\mathcal{L}^n\left(\left\{y \in B_{r_j}(x); |f(y) - f(x) - (\nabla f(x), y - x)| > \theta |y - x|\right\}\right)}{\omega_n r_j^n} \geq \gamma > 0.$$

Furthermore, by (2.1), there exists $\sigma > 0$ such that $\sigma r_j < r_j$, $\sigma^n < \frac{\gamma}{2}$, $\sigma \in [0, 1]$ and

$$\frac{\mathcal{L}^n\left(\left\{y \in B_{\sigma r_j}(x); |f(y) - f(x) - (\nabla f(x), y - x)| > \theta |y - x|\right\}\right)}{\omega_n r_j^n} \geq \frac{\gamma}{2}.$$

Then,

$$\frac{\mathcal{L}^n\left(\left\{y \in B_{r_j}(x) \setminus B_{\sigma r_j}(x); |f(y) - f(x) - (\nabla f(x), y - x)| > \theta |y - x|\right\}\right)}{\omega_n r_j^n} \geq \frac{\gamma}{2}$$

for $j \in \{1, 2, \ldots\}$. Since $|y - x| > \sigma r_j$ for any $y \in B(x, r_j) \setminus B(x, \sigma r_j)$, we can see that,

$$\frac{\mathcal{L}^n\left(\{y \in B(x, r_j); |f(y) - f(x) - (\nabla f(x), y - x)| > \theta \sigma r_j\}\right)}{\omega_n r_j^n} \geq \frac{\gamma}{2}, \tag{2.2}$$

for $j \in \{1,2,\ldots\}$. But, by (2.1), the expression on the left hand side of (2.2) is less or equal to $o(r_j)$, and then

$$\frac{o(r_j)}{\omega_n r_j^n} = o(1) \quad r_j \to 0,$$

which is a contradiction with (2.2). Then,

$$ap \limsup_{y \to x} \frac{|f(y) - f(x) - (\nabla f(x), y - x)|}{|y - x|} = 0.$$

□

Remark 2.1. *We have shown, in the Theorem 2.1, that* $apDf(x) = \nabla f(x)$.

For any $u \in BV(\Omega)$, by the dominated convergence theorem, the fact that u is approximately differentiable and the last remark, we have

$$\lim_{r \to 0^+} \frac{1}{r^n} \int_{B_r(x)} \frac{|u(y) - u(x) - (\nabla u(x), y - x)|}{|y - x|} dy = 0 \tag{2.3}$$

for $\mathcal{L}^n$-a.e $x \in \Omega$.

Definition 2.2. *Let* $E \subset \mathbb{R}^n$ *such that* $\mathbb{1}_E \in BV(\mathbb{R}^n)$. *We define the perimeter measure of* E, *denoted by* $|\partial E|$, *to the total variation of* $D\mathbb{1}_E$, *i.e,* $|\partial E| = |D\mathbb{1}_E|$. *We say that* E *is a set of finite perimeter if* $|\partial E| < +\infty$.

Remark 2.2. *From the Riez Representation theorem, we see that*

$$|\partial E|(V) = \sup\left\{\int_V div\varphi dx; \varphi \in C_c^1(V) \text{ and } \|\varphi\|_\infty \leq 1\right\}$$

for each open set $V \subset\subset \Omega$.

Notation 2.2. *For* $u : \Omega \to \mathbb{R}^n$, *and* $t \in \mathbb{R}^n$, *we define the level set of* u *as*

$$E_t := \{x \in \Omega; u(x) > t\}.$$

The next theorem is one of the most important results for the BV functions, there is another version for L^1_{loc} functions, for further references see [5], [15] and [17].

Theorem 2.2 (Coarea formula for BV functions)**.** *Let $u \in BV(\Omega)$. Let $U \subset \Omega$. Then,*

i) E_t *has finite perimeter for $\mathcal{L}^n$-a.e $t \in \mathbb{R}$,*

ii) $|Du|(B) = \int_{-\infty}^{+\infty} |\partial E_t|(B)\,dt$, *for all Borel set $B \subset \Omega$*

iii) Conversely, if $f \in L^1(\Omega)$ and $\int_{-\infty}^{+\infty} |\partial E_t|(\Omega)\,dt < +\infty$, then $u \in BV(\Omega)$.

Definition 2.3. *Let E a set of locally finite perimeter in $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$. We say $x \in \partial^* E$, the reduced boundary of E, if*

i) $|\partial E|(B(x,r)) > 0$ *for all $r > 0$,*

ii) $\lim_{r\to 0} \fint_{B(x,r)} \nu_E d|\partial E| = \nu_E(x)$ *and*

iii) $|\nu_E(x)| = 1$,

where $\nu_E(x) = \lim_{r\to 0} \dfrac{D\mathbb{1}_E(B_r(x))}{|D\mathbb{1}_E|(B_r(x))}$.

Let $E \subset \Omega$ be a set with finite perimeter, writing the polar decomposition of $D\mathbb{1}_E$ we have

$$D\mathbb{1}_E = \nu_E|\partial E|, \tag{2.4}$$

where ν_E is defined $|\partial E|-$a.e. in $\mathbb{R}^n$ and $|\nu_E| = 1$ $|\partial E|$-a.e. in $\mathbb{R}^n$.

Definition 2.4 (Locally perimeter set)**.** *Let $E \subset \Omega$ a Borel set such that $\mathbb{1}_E \in BV(\Omega)$. Then, E has finite perimeter in any open set $\Omega' \subset \Omega$. Then, we say that E is a locally perimeter set in Ω.*

Definition 2.5. *Let E a set of locally finite perimeter in $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$. We say $x \in \partial_* E$, the measure theoretic boundary of E, if*

$$\limsup_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\cap E)}{r^n} > 0$$

and

$$\limsup_{r\to 0} \frac{\mathcal{L}^n\left(B\left(x,r\right)\backslash E\right)}{r^n} > 0$$

Theorem 2.3 (Generalized Gauss-Green theorem or De Giorgi's theorem)**.** *Let $E \subset \mathbb{R}^n$ a locally finite perimeter set.*

i) Then $\mathcal{H}^{n-1}\left(\partial_* E \cap K\right) < \infty$ for each compact set $K \subset \mathbb{R}^n$.

ii) Furthermore, for $\mathcal{H}^{n-1}$-a.e $x \in \partial_* E$, there is a unique measure theoretic unit outer normal $\nu_E\left(x\right)$ such that

$$\int_E div\varphi dx = \int_{\partial_* E} (\varphi, \nu_E) d\mathcal{H}^{n-1}$$

for all $\varphi \in C_c^1\left(\mathbb{R}^n, \mathbb{R}^n\right)$.

Remark 2.3. *The measure theoretic unit outer normal $\nu_E\left(x\right)$ is the same as (2.4).*

By De Giorgi's theorem 2.3, for any set E such that $\mathbb{1}_E \in BV\left(\Omega\right)$, we have

$$|\partial E| = \mathcal{H}^{n-1} \llcorner \partial^* E = \mathcal{H}^{n-1} \llcorner \partial_* E,$$

and $\partial_* E$ (or $\partial^* E$) is $\mathcal{H}^{n-1}$-rectifiable.

By the last identity above, the Coarea formula, we have $\mathbb{1}_{E_t} \in BV\left(\Omega\right)$ for $\mathcal{L}^1$-almost every $t \in \mathbb{R}$ and

$$Du\left(F\right) = \int_{-\infty}^{+\infty} D\mathbb{1}_{E_t}\left(F\right) dt, \tag{2.5}$$

$$|Du|\left(F\right) = \int_{-\infty}^{+\infty} |D\mathbb{1}_{E_t}|\left(F\right) dt = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}\left(\partial_* E_t \cap F\right) dt \tag{2.6}$$

for any $u \in BV\left(\Omega\right)$ and any Borel set $F \in \Omega$.

Theorem 2.4 (Local approximation by smooth functions [17])**.** *Assume $f \in BV\left(\Omega\right)$. Then there exist functions $\left(f_n\right)_{n\in\mathbb{N}^*} \subset BV\left(\Omega\right) \cap C^\infty\left(\Omega\right)$ such that*

1. *$f_k \to f$ in $L^1\left(\Omega\right)$ and*

2. *$|Df_k|\left(\Omega\right) \to |Df|\left(\Omega\right)$ as $k \to +\infty$.*

Definition 2.6. *Let $u, u_h \in BV(\Omega)$. We say that (u_h) weakly* converges in $BV(\Omega)$ to u if (u_h) converges to u in $L^1(\Omega)$ and (Du_h) weakly* converges to Du in Ω, i.e.,*

$$\lim_{h\to\infty}\int_{\Omega}\phi dDu_h = \int_{\Omega}\phi dDu,$$

for all $\phi \in C_c(\Omega)$.

Chapter 3

Space SBV

We are going to construct the space SBV (Special Bounded Variation). The aim of this chapter is to prove the compactness theorem (Ambrosio's theorem) and the chain rule for the composition of a $u \in SBV(\Omega)$ and with a C^1 Lipschitz function.
We have already seen in the last chapter that for any $u \in BV(\Omega)$ we have $Du = \nabla u \mathcal{L}^n + D^s u$. First, we will study the singular part of Du.

3.1 Decomposition of Du

We will use the method of Ambrosio in [3] which is based in construct the jump set with the lower and upper approximate limits of a function.

Definition 3.1. *The upper and lower approximate limits u_+, u_- respectively, are defined by*

$$u_+(x) = \inf\{t \in [-\infty, +\infty]; \{x \in \Omega; u(x) > t\} \quad \textit{has density } 0 \textit{ at } x\}$$
$$u_-(x) = \sup\{t \in [-\infty, +\infty]; \{x \in \Omega; u(x) < t\} \quad \textit{has density } 0 \textit{ at } x\}$$

Definition 3.2 (Jump set)**.** *Let $u \in BV(\Omega)$. The jump set S_u is defined as*

$$S_u = \{x \in \Omega; u_-(x) < u_+(x)\}.$$

Observe that if x_0 is a Lebesgue point of u, then

$$u(x_0) = u_-(x_0) = u_+(x_0).$$

Also, the set of the points that are not Lebesgue points, is $\mathcal{L}^n$-negligible, we have that $\mathcal{L}^n(S_u) = 0$. If $x \notin S_u$, we denote by

$$\tilde{u}(x) = u_-(x) = u_+(x)$$

the common value.

Theorem 3.1. *The set S_u is $\mathcal{H}^{n-1}$-measurable (σ-finite) and rectifiable.*

Proof. Let E_t the level set of u. By the Coarea formula, E_t is a set of finite perimeter in $\mathbb{R}^n$ for $\mathcal{L}^1$-a.e. t. Furthermore, we observe that if $x \in S_u$ and $u_-(x) < t < u_+(x)$, then

$$\limsup_{r\to 0} \frac{\mathcal{L}^n(B(x,r) \cap \{u > t\})}{r^n} > 0$$

and

$$\limsup_{r\to 0} \frac{\mathcal{L}^n(B(x,r) \cap \{u < t\})}{r^n} > 0.$$

Hence,

$$\{x \in S_u; u_-(x) < t < u_+(x)\} \subset \partial_* E_t. \tag{3.1}$$

Choose $D \subset \mathbb{R}$ a countable, dense set such that E_t is of finite perimeter for each $t \in D$. By the Structure theorem for sets of finite perimeter (See [17] 5.7.3), for each $t \in D$, $\mathcal{H}^{n-1}$ almost all of $\partial_* E_t$ is contained in a countable union of C^1-hypersuperfaces.

Now, by (3.1), we have that $S_u \subset \bigcup_{t\in D} \partial_* E_t$. Then, the set S_u is $\mathcal{H}^{n-1}$-measurable (σ-finite) and rectifiable. □

Let $x \in \mathbb{R}^n$. Then $-\infty < u_-(x) \leq u_+(x) < +\infty$ for $\mathcal{H}^{n-1}$-a.e $x \in \mathbb{R}^n$. (See the proof in the Appendix, Theorem 3.2)

Remark 3.1. *We proved above that*

$$\{(x,t)\,;x\in S_u \text{ and } u_-(x)<t<u_+(x)\}\subset\{(x,t)\,;x\in S_u \text{ and } x\in\partial_* E_t\}\,.$$

Similarly, we can prove that

$$\{(x,t)\,;x\in S_u \text{ and } x\in\partial_* E_t\}\subset\{(x,t)\,;x\in S_u \text{ and } u_-(x)\le t\le u_+(x)\}$$

By Fubini and Coarea formula (See (2.6)), we deduce that

$$\begin{aligned}\int_{S_u}(u_+-u_-)\,d\mathcal{H}^{n-1} &= \int_{S_u}\left(\int_{u_-(x)}^{u_+(x)}\mathbb{1}dt\right)d\mathcal{H}^{n-1}\\ &= \int_{S_u\times\mathbb{R}}\mathbb{1}_{\{(x,t);x\in S_u,u_-(x)\le t\le u_+(x)\}}d\left(\mathcal{L}^1\times\mathcal{H}^{n-1}\llcorner S_u\right)\\ &= \int_{\mathbb{R}}\left(\int_{S_u}\mathbb{1}_{\{(x,t);x\in S_u\cap\partial_* E_t\}}d\mathcal{H}^{n-1}\right)dt\\ &= \int_{\mathbb{R}}\mathcal{H}^{n-1}(\partial_* E_t\cap S_u)\,dt\le|Du|(\Omega)<+\infty. \qquad (3.2)\end{aligned}$$

Theorem 3.2 ([17])**.** *Let $u\in BV(\Omega)$. Then $-\infty<u_-(x)\le u_+(x)<+\infty$ for $\mathcal{H}^{n-1}$-a.e $x\in\mathbb{R}^n$.*

Proof. Let $u\in BV(\Omega)$.

Claim 1: $\mathcal{H}^{n-1}(\{x;u_-(x)=+\infty\})=0$ and $\mathcal{H}^{n-1}(\{x;u_-(x)=-\infty\})=0$.

Proof of the Claim 1: We assume that $supp\,(u)$ is compact. Let $F_t=\{x\in\mathbb{R}^n;u_-(x)>t\}$. Since $u_+(x)=u_-(x)=u(x)$ $\mathcal{L}^n$-a.e, we have that $F_t=E_t$ $\mathcal{L}^n$-a.e, hence, $|\partial E_t|=|\partial F_t|$. By the Coarea formula for BV functions , we have that

$$\int_{-\infty}^{+\infty}|\partial F_t|(\mathbb{R}^n)\,dt=|Du|(\mathbb{R}^n)<+\infty,$$

and then for a subsequence $t_j\to+\infty$,

$$\liminf_{t_j\to\infty}\left|\partial F_{t_j}\right|(\mathbb{R}^n)=0. \qquad (3.3)$$

Since $supp\,(u)$ is compact, there exists $d > 0$ such that

$$\mathcal{L}^n\left(supp\,(u) \cap B\,(x,r)\right) \leq \frac{1}{8}\omega_n r^n \quad for\ all\ x \in supp\,(u)\ \ and\ for\ all\ r \geq d. \tag{3.4}$$

Let $t_j > 0$. By definition of u_- and F_{t_j},

$$\lim_{r\to 0} \frac{\mathcal{L}^n\left(F_{t_j} \cap B\,(x,r)\right)}{\omega_n r^n} = 1 \quad for\ x \in F_{t_j}.$$

Then for each $x \in F_{t_j}$, there exists $r > 0$ such that

$$\frac{\mathcal{L}^n\left(F_{t_j} \cap B\,(x,r)\right)}{\omega_n r^n} = \frac{1}{4}. \tag{3.5}$$

According (3.4), we have the same result for $r \leq d$.
We apply Vitali's covering theorem to find disjoint collection $\left(B\,(x_i, r_i)\right)_{i\in\mathbb{N}}$ of balls satisfying (3.5) for $x = x_i$, $r = r_i < d$ such that

$$F_{t_j} \subset \bigcup_{n=1}^{\infty} B\,(x_i, 5r_i)$$

Using (3.5) and the Relative Isoperimetric Inequality (See [17] 5.6), we see that

$$\left(\frac{\omega_n}{4}\right)^{\frac{n-1}{n}} \leq \frac{c\left|\partial F_{t_j}\right|\left(B\,(x_i, r_i)\right)}{r_i^{n-1}},$$

thus

$$r_i^{n-1} \leq c\left|\partial F_{t_j}\right|\left(B\,(x_i, r_i)\right).$$

So, we may calculate

$$\begin{aligned}\mathcal{H}^{n-1}_{10d}\left(F_{t_j}\right) &\leq \sum_{i=1}^{\infty} \omega_{n-1}\,(5r_i)^{n-1} \leq c\sum_{i=1}^{\infty}\left|\partial F_{t_j}\right|\left(B\,(x_i, r_i)\right)\\ &\leq c\left|\partial F_{t_j}\right|\left(\mathbb{R}^n\right).\end{aligned}$$

Since $\mathcal{H}^{n-1}_{10d}\left(\{x; u_-\,(x) = +\infty\}\right) = 0$, then $\mathcal{H}^{n-1}\left(\{u_+ = +\infty\}\right)$. The demonstration of $\mathcal{H}^{n-1}_{10d}\left(\{x; u_-\,(x) = -\infty\}\right) = 0$ is similar.

End of Claim 1

Claim 2: $\mathcal{H}^{n-1}\left(\{x; u_{+}(x) - u_{-}(x) = \infty\}\right) = 0$

Proof of Claim 2:

We have shown, in the Theorem 3.1, that the set $\{(x,t); u_{-}(x) < t < u_{+}(x)\} \subset S_u \times \mathbb{R}$ is σ-finite with respect to $\mathcal{H}^{n-1} \times \mathcal{L}^1$ in $\mathbb{R}^n$. Consequently, Fubini's theorem implies

$$\int_{-\infty}^{+\infty} \mathcal{H}^{n-1}\left(\{u_{-}(x) < t < u_{+}(x)\}\right) dt = \int_{\mathbb{R}^n} u_{+}(x) - u_{-}(x)\, d\mathcal{H}^{n-1}.$$

By the fact that $\{u_{-}(x) < t < u_{+}(x)\} \subset \partial_* E_t$, the Coarea formula and the Structure theorem of finite perimeter (See [17], Theorem 5.15), we can see that

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathcal{H}^{n-1}\left(\{u_{-}(x) < t < u_{+}(x)\}\right) dt &\leq \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}\left(\partial_* E_t\right) dt \\ &= \int_{-\infty}^{+\infty} |\partial E_t|\left(\mathbb{R}^n\right) dt = |Du|\left(\mathbb{R}^n\right) < +\infty.\end{aligned}$$

Consequently, $\mathcal{H}^{n-1}\left(\{x; u_{+}(x) - u_{-}(x) = \infty\}\right) = 0$.

End of the Claim 2. □

Remark 3.2. *If $u = \mathbb{1}_E$ for some Borel set $E \subset \Omega$, then S_u coincides with the essential boundary (or measure theoretical boundary).*

Combining the Theorem 5.1 and the theory developed until (2.6), we have that there exists a Borel map $\nu_u : S_u \to \mathbb{S}^{n-1}$ such that

$$\nu_{E_t}(x) = \nu_u(x) \quad for\ \mathcal{H}^{n-1} - a.e.\ x \in \partial^* E_t \cap S_u \tag{3.6}$$

for any t such that $\mathbb{1}_{E_t} \in BV(\Omega)$.

Definition 3.3. *We call jump part of derivative $D^j u$ the restriction of $D^s u$ to the jump set S_u. We call Cantor part of derivative $D^c u$ the restriction $D^s u$ to $\Omega \backslash S_u$.*

Definition 3.4 (SBV space)**.** *We say that $u \in SBV(\Omega)$ if $D^c u = 0$, or equivalently, if the singular part of Du is supported in S_u.*

Remark 3.3. *We have that $SBV(\Omega)$ is a proper subset of $BV(\Omega)$. On the contrary, we have that the Cantor function is an element of $BV(\Omega)$ but not in $SBV(\Omega)$, because, the distributional derivative of the Devil's staircase is supported in the Cantor set, which is also the jump set. Then, the derivative Cantor part D^c of the Devil's staircase functions is positive.*

The following proposition shows that, in some sense, $D^s u = D^j u + D^c u$ is the decomposition in absolutely continuous and singular part od D^s with respect to $\mathcal{H}^{n-1}$.

Proposition 3.3 ([3])**.** *Let $u \in BV(\Omega)$. Then, the jump part of derivative is absolutely continuous with respect to $\mathcal{H}^{n-1}$ and*

$$D^j u = (u_+ - u_-)\,\nu_u \mathcal{H}^{n-1} \llcorner S_u \tag{3.7}$$

where $\nu_u(x)$ is the unit vector in (3.6). Moreover, the Cantor part of derivative is orthogonal with respect to $\mathcal{H}^{n-1}$, i.e.

$$\mathcal{H}^{n-1}(F) < +\infty \implies |D^c u|(F) = 0$$

for any Borel Set $F \subset \Omega$.

Proof. Let $u \in BV(\Omega)$. We already proved that

$$\begin{aligned}\{(x,t)\,; x \in S_u, u_-(x) < t < u_+(x)\} &\subset \{(x,t)\,; x \in S_u; S_u \cap \partial_* E_t\} \\ &\subset \{(x,t)\,; u_-(x) \le t \le u_+(x)\}\,.\end{aligned}$$

Hence, we infer that $\mathcal{L}^n(\{t \in \mathbb{R}; x \in \partial_* E_t\}) = u_+(x) - u_-(x)$.
Let $F \subset S_u$ be any Borel set. Using the coarea formula, the definition and decomposition

of Du, (2.5), (3.2), (3.6) and the Fubini-Tonelli theorem, we obtain that

$$D^j u(F) = Du(F) = \int_{-\infty}^{+\infty} D\mathbb{1}_{E_t}(F)\,dt = \int_{-\infty}^{+\infty}\int_{F\cap\partial_* E_t} \nu_u(x)\,d\mathcal{H}^{n-1}(x)\,dt$$
$$= \int_F \left(\int_{-\infty}^{+\infty} \mathbb{1}_{\{t\in\mathbb{R};x\in\partial_* E_t\}}\right)\nu_u(x)\,d\mathcal{H}^{n-1} = \int_F (u_+ - u_-)\,\nu_u d\mathcal{H}^{n-1}(x).$$

And then, we can conclude (3.7).

Now, we are going to study the Cantor derivative part. Let $F \subset \Omega\backslash S_u$ be a Borel set with $\mathcal{H}^{n-1}(F) < +\infty$. Since $\mathcal{L}^n(F) = 0$ and by the coarea formula (2.6), we have

$$|D^c|u(F) = |D|u(F) = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(F\cap\partial_* E_t)\,dt.$$

By the fact that $F \subset \Omega\backslash S_u$

$$F\cap\partial_* E_t \subset \{x\in F; u_-(x) \le t \le u_+(x)\} = \{x\in F; \tilde{u}(x) = t\}.$$

And there are only countably many $t\in\mathbb{R}$ such that

$$\mathcal{H}^{n-1}(\{y\in F; \widetilde{u}(x) = t\}) > 0,$$

it follows that $|D^c|u = 0$. □

By the last proposition, for any $u \in BV(\Omega)$, we have

$$Du = D^a u + D^j u + D^c u = \nabla u\mathcal{L}^n + (u_+ - u_-)\,\nu_u \mathcal{H}^{n-1} \llcorner S_u + D^c u. \tag{3.8}$$

And the measures in the right hand side are mutually singular.

3.2 Chain rule

Let $u \in BV(\Omega)$ and $\psi \in C_c^1(\mathbb{R})$. The aim of this section is to obtain a chain rule, in the distributional sense, for $D(\psi(u))$. With the definition of $SBV(\Omega)$, the interest of this section is to give a chain rule for $\nabla\psi(u)$, $D^j\psi(u)$ and $D^c\psi(u)$ in terms of ∇u, $D^j u$ and $D^c u$.

Proposition 3.4. *Let $u \in BV(\Omega)$ and $\psi \in C_c^1(\mathbb{R})$, then $v = \psi(u) \in BV(\Omega)$ and*

$$D^j v = (\psi(u_+) - \psi(u_-))\,\nu_u \mathcal{H}^{n-1} \llcorner S_u, \tag{3.9}$$

$$\nabla v = \psi'(u)\,\nabla u, \quad D^c v = \psi'(\widetilde{u})\,D^c u. \tag{3.10}$$

Proof. **Claim 1:** Any function $\psi \in C_c^1(\mathbb{R})$ can be written as the difference of two Lipschitz functions $\psi_1, \psi_2 \in C^1(\mathbb{R})$ such that $\psi_i \geq 1$, $i = 1, 2$.
Proof of the Claim 1: Let $\psi \in C_c^1(\mathbb{R})$. We know that $m := \min \psi' \leq 0$, and we define ψ_2 Lipschitz function such that $\psi_2' \geq 1 - m$. The function ψ_1 will be defined as $\psi_1 = \psi + \psi_2$. It is easy to see that $\psi_1' = \psi' + \psi_2' \geq m + 1 - m = 1$.
End of the Claim 1.
After this claim, we will make the proof of this proposition for C^1, Lipschitz and increasing functions. Let $u \in BV(\Omega)$ and $\psi \in C^1$, Lipschitz and increasing function. By the approximation theorem, we can find a sequence $(u_h) \subset C^1(\mathbb{R})$ converging to $u \in L^1(\Omega)$ and such that $|Du_h|(\Omega)$ converges to $|Du|(\Omega)$ as $h \to +\infty$.
Claim 2: $v_h = \psi(u_h)$ converges to $v \in L^1(\Omega)$.
Proof of the Claim 2: It is easy to see that

$$\|v_h - v\|_1 = \|\psi(u_h) - \psi(u)\|_1 \leq Lip(\psi)\,\|u_h - u\|_1 \xrightarrow[h\to\infty]{} 0.$$

End of the Claim 2.

By the lower semi-continuity of $|Du_h|(\Omega)$ we have

$$\begin{aligned}|Dv|(\Omega) &\leq \liminf_{h\to\infty} |Dv_h|(\Omega) \leq \liminf_{h\to\infty} |D\psi(u_h)(\Omega)| \leq \|\psi'\|_\infty \liminf_{h\to\infty} |Du_h|(\Omega) \\ &= \|\psi'\|_\infty |Du|(\Omega)\end{aligned}$$

Using the Claim 2 and the last expression we can conclude that $v \in BV(\Omega)$.
Furthermore, we can see that $\nu_+ = \psi(u_+)$ and $\nu_- = \psi(u_-)$. Indeed, by monotony and injectivity of ψ,

$$\begin{aligned}\nu_+(x) &:= \inf\{t \in [-\infty, +\infty]; \{x \in \Omega; v(x) > t\} \text{ has } 0 \text{ density at } x\} \\ &= \inf\{t \in [-\infty, +\infty]; \{x \in \Omega; \psi(u(x)) > t\} \text{ has } 0 \text{ density at } x\} \\ &= \inf\left\{t \in [-\infty, +\infty]; \left\{x \in \Omega; u(x) > \psi^{-1}(t)\right\} \text{ has } 0 \text{ density at } x\right\} \\ &= \inf\{\psi(t) \in [-\infty, +\infty]; \{x \in \Omega; u(x) > t\} \text{ has } 0 \text{ density at } x\} = \psi(u_+(x)).\end{aligned}$$

In a similar way, we have the same result for $\psi(u_-(x))$. Once again, as ψ is one to one increasing function,

$$S_v = \{v_-(x) < v_+(x)\} = \{\psi(u_-) < \psi(u_+)\} = \{u_-(x) < u_+(x)\} = S_u.$$

By (3.6), the exterior normal coincides, i.e. $\nu_v = \nu_u$. Then, we infer that

$$Dv \llcorner S_v = (v_+ - v_-)\,\nu_v \mathcal{H}^{n-1} \llcorner S_v = (\psi(u_+) - \psi(u_-))\,\nu_u \mathcal{H}^{n-1} \llcorner S_u.$$

By the coaire formula (2.5), we have that for any Borel set $F \subset \Omega \backslash S_u$

$$\begin{aligned}Dv(F) &= \int_{-\infty}^{+\infty} D\mathbb{1}_{\{v>t\}}(F)\,dt = \int_{-\infty}^{+\infty} D\mathbb{1}_{\{u>\psi^{-1}(t)\}}(F)\,dt = \int_{-\infty}^{+\infty} D\mathbb{1}_{\{u>\tau\}}(F)\,\psi'(\tau)\,d\tau \\ &= \int_{-\infty}^{+\infty}\int_F \psi'(\tau)\, D\mathbb{1}_{\{u>\tau\}} d\tau.\end{aligned}$$

Since $x \in \partial_* \{u > \tau\} \backslash S_u$ implies $\widetilde{u}(x) = \tau$, we have that

$$Dv(F) = \int_{-\infty}^{+\infty}\int_F \psi'(\widetilde{u})\, D\mathbb{1}_{\{u>\tau\}} d\tau = \psi'(\widetilde{u})\, D^c u(F).$$

So we have shown the second part of (3.10). Since the sum of $D^a v$ and $D^c v$ is the restriction of Dv to $\Omega \backslash S_u$, so (3.10) is equivalent to

$$\begin{aligned} Dv \llcorner \Omega \backslash S_v &= D^a v + D^c v = \nabla (\psi(u)) \mathcal{L}^n + D^c \psi(u) = \psi'(u) \nabla u \mathcal{L}^n + D^c (\psi(u)) \\ &= \psi'(\widetilde{u}) \nabla u \mathcal{L}^n + \psi'(\widetilde{u}) D^c u = \psi'(\widetilde{u}) Du \llcorner \Omega \backslash S_u. \end{aligned}$$

Since the set of points which are not Lebesgue points is $\mathcal{L}^n$-negligible, we have last identity

$\square$

3.3 Compactness of SBV

We know that $BV(\Omega)$ has the compactness property, and this proof of the compactness is not needed the decomposition of the derivative. This section is devoted to prove a compactness of SBV using the decomposition of Du and the chain rule (Proposition 3.4), for any $u \in BV(\Omega)$.

Let $u \in BV(\Omega)$, and let $\psi \in C_c^1(\mathbb{R})$. By the chain rule, the distributional derivative of $\psi(u)$ is given by

$$D\psi(u) = \psi'(u) \nabla u \mathcal{L}^n + (\psi(u_+) - \psi(u_-)) \nu_u \mathcal{H}^{n-1} \llcorner S_u + \psi'(\widetilde{u}) D^c u.$$

In particular for all $\phi \in C_c^1(\mathbb{R})$, is easy to see that

$$\begin{aligned} \int_\Omega \left[\frac{\partial \phi}{\partial x_i} \psi(u) + \phi \psi'(u) \frac{\partial u}{\partial x_i} \right] dx &= - \int_\Omega \phi D\psi(u)\, dx + \int_\Omega \phi \psi'(u) \frac{\partial u}{\partial x_i} dx \\ &= - \int_\Omega \phi \psi'(u) \frac{\partial u}{\partial x_i} dx - \int_{S_u} \phi (\psi(u_+) - \psi(u_-)) \nu_{u,i} d\mathcal{H}^{n-1}(x) \\ &\quad - \int_\Omega \phi \psi'(\widetilde{u})\, dD^c u_i + \int_\Omega \phi \psi'(u) \frac{\partial u}{\partial x_i} dx \\ &= - \int_{S_u} \phi (\psi(u_+) - \psi(u_-)) \nu_{u,i} d\mathcal{H}^{n-1}(x) - \int_\Omega \phi \psi'(\widetilde{u})\, dD^c u_i, \end{aligned} \tag{3.11}$$

for any $i \in \{1, \ldots, n\}$. If $u \in SBV(\Omega)$ and $\mathcal{H}^{n-1}(S_u) < +\infty$, the above formula can be written as follows

$$\int_{\Omega} \left[\frac{\partial \phi}{\partial x_i} \psi(u) + \phi \psi'(u) \frac{\partial u}{\partial x_i} \right] dx = \int_{\Omega \times \mathbb{R}} \phi(x) \psi(s) \, d\mu_i(x, s), \tag{3.12}$$

where denoting by $\Phi^+, \Phi^- : \Omega \to \Omega \times \mathbb{R}$ the maps

$$\Phi^+(x) = (x, u_+(x)), \quad \Phi^-(x) = (x, u_-(x)), \tag{3.13}$$

and the measure $\mu = (\mu_1, \ldots, \mu_n)$ is defined by

$$\mu = \Phi^+_\# \left(\nu_u \mathcal{H}^{n-1} \llcorner S_u \right) - \Phi^-_\# \left(\nu_u \mathcal{H}^{n-1} \llcorner S_u \right). \tag{3.14}$$

Since the images of Φ^+ and Φ^- are disjoints (i.e. Φ^+ and Φ^- are one to one functions), by Lemma 1.1 and (2.6), we obtain

$$\begin{aligned} |\mu| &= \left| \Phi^+_\# \left(\nu_u \mathcal{H}^{n-1} \llcorner S_u \right) \right| + \left| \Phi^-_\# \left(\nu_u \mathcal{H}^{n-1} \llcorner S_u \right) \right| = \Phi^+_\# \left(\left| \mathcal{H}^{n-1} \llcorner S_u \right| \right) + \Phi^-_\# \left(\left| \mathcal{H}^{n-1} \llcorner S_u \right| \right) \\ &= \Phi^+_\# \left(\mathcal{H}^{n-1} \llcorner S_u \right) + \Phi^-_\# \left(\mathcal{H}^{n-1} \llcorner S_u \right) \end{aligned} \tag{3.15}$$

The useful of the next theorem is very technical, but, it gives another equivalence of SVB space with (3.12).

Theorem 3.5. *Let $u \in BV(\Omega)$, and let us assume that there exist $a \in L^1(\Omega, \mathbb{R}^n)$ and a vector measure $\mu = (\mu_1, \ldots, \mu_n)$ with finite total variation in $\Omega \times \mathbb{R}$ such that*

$$\int_{\Omega} \left[\frac{\partial \phi}{\partial x_i} \psi(u) + \phi \psi'(u) a_i \right] dx = \int_{\Omega \times \mathbb{R}} \phi(x) \psi(s) \, d\mu_i(x, s) \tag{3.16}$$

for any $\phi \in C^1_c(\Omega)$, $\psi \in C^1_c(\mathbb{R})$ and $i \in \{1, \ldots, n\}$. Then, $u \in SBV(\Omega)$, $a = \nabla u$ and

$$\mu = \Phi^+_\# \left(\nu_u \mathcal{H}^{n-1} \llcorner S_u \right) - \Phi^-_\# \left(\nu_u \mathcal{H}^{n-1} \llcorner S_u \right), \tag{3.17}$$

with Φ^+, Φ^- *are defined in (3.13). In particular, by (3.15),*

$$2\mathcal{H}^{n-1}(S_u) = |\mu|(\Omega \times \mathbb{R}) < +\infty.$$

Proof. Let us assume that (3.16) holds for some $a \in L^1(\Omega, \mathbb{R}^n)$ and some vector measure μ. We denote by θ the measure $\pi_\#(|\mu|)$, where $\pi : \Omega \times \mathbb{R} \to \Omega$ is the projection to the first variable. By (3.16), we infer that

$$\left| \int_\Omega \left[\nabla\phi\psi(u) + \phi\psi'(u)\, a\right] dx \right| \leq \|\psi\|_\infty \int_\Omega \phi(x)\, d\theta(x). \tag{3.18}$$

Let x_0 a Lebesgue point of a, such that we have (2.3) and

$$\lim_{r\to 0^+} r^{1-n}\theta(B_r(x_0)) = 0, \tag{3.19}$$

observe that $\mathcal{L}^n$-a.e. points in Ω satisfy the required properties.
We claim that $\nabla u(x_0) = a(x_0)$. Indeed, let B the unit open ball in $\mathbb{R}^n$ centered at 0, and let $\gamma(t) \in C_c^1(\mathbb{R})$, $\varphi \in C_c^1(B)$, inserting

$$\phi(x) = \varphi\left(\frac{x - x_0}{r}\right), \quad \psi(s) = \gamma\left(\frac{s - u(x_0)}{r}\right)$$

in (3.18) and changing variables, we get

$$\begin{aligned}
\int_{B_r(x_0)} (\nabla\phi)\,\psi(u)\, dx &= \frac{1}{r}\int_{B(x_0)} \nabla\varphi\left(\frac{x - x_0}{r}\right)\gamma\left(\frac{u(x) - u(x_0)}{r}\right) dx \\
&= \frac{r^n}{r}\int_B \nabla\varphi(y)\,\gamma\left(\frac{u(x_0 + ry) - u(x_0)}{r}\right) dy \\
&= r^{n-1}\int_B \nabla\varphi\gamma(u_r)\, dy,
\end{aligned}$$

where, $u_r(y) := \dfrac{u(x_0 + ry) - u(x_0)}{r}$. We infer also that

$$\begin{aligned}
\int_{B_r(x_0)} \phi\psi'(u)\, a dx &= \frac{1}{r}\int_{B_r(x_0)} \varphi\left(\frac{u(x) - u(x_0)}{r}\right)\gamma'\left(\frac{u(x) - u(x_0)}{r}\right) a(x)\, dx \\
&= r^{n-1}\int_B \varphi(y)\,\gamma'(u_r(y))\, a(x_0 + ry)\, dy = r^{n-1}\int_B \varphi(y)\,\gamma'(u_r)\, a_r dy,
\end{aligned}$$

where $a_r(y) = a(x_0 + ry)$. Then, by (3.18) we obtain

$$\left| \int_B \left[\nabla \varphi \gamma(u_r) + \varphi \gamma'(u_r) a_r \right] dy \right| \le r^{n-1} \|\gamma\|_\infty \int_{B_r(x_0)} \phi(x) \, d\theta(x).$$

Using (2.3), we infer that u_r converges in $L^1(B)$ to $u_0(y) = \nabla u(x_0) y$ as $r \to 0$. Passing to the limit as $r \to 0$ and using (3.19), we get

$$\int_B \left[\nabla \varphi \gamma(u_0) + \varphi \gamma'(u_0) a(x_0) \right] dy = 0.$$

On the other hand, since, $D\gamma(u_0) = \gamma'(u_0) \nabla u(x_0) \mathcal{L}^n$,

$$\int_B \left[\nabla \varphi \gamma(u_0) + \varphi \gamma'(u_0) \nabla u(x_0) \right] dy = 0.$$

Hence, differentiating the two expressions above, we have

$$\left[a(x_0) - \nabla u(x_0) \right] \int_B \varphi \gamma(u_0) \, dy = 0.$$

Since, γ and φ are arbitrary we conclude that $a(x_0)) = \nabla u(x_0)$. Furthermore, since the set of all points $x_0 \in \Omega$ such that (3.19) is σ-finite with respect to $\mathcal{H}^{n-1}$ (See [15] and [25]), it follows that the equality $a = \nabla u$ holds $\mathcal{L}^n$-a.e in Ω.

Now, we are going to prove that $D^c u = 0$. Indeed, let us fix $\psi \in C_c^1(\mathbb{R})$ and $\phi \in C_c^1(\Omega)$. By (3.11) and (3.18) and taking account that $a = \nabla u$, we obtain

$$\left| \int_{S_u} \phi \left(\psi(u_+) - \psi(u_-) \right) \nu_u d\mathcal{H}^{n-1} - \int_\Omega \phi \psi'(\widetilde{u}) \, dD^c u \right| \le \|\psi\|_\infty \int_\Omega \phi d\theta.$$

The inequality above does not contain derivatives of ϕ, hence, it holds for any Borel function ϕ. Since, $\mathcal{H}^{n-1} \llcorner S_u$ and $D^c u$ are mutually singular, we have

$$\left| \int_\Omega \phi \psi'(\widetilde{u}) \, dD^c u \right| \le \|\psi\|_\infty \int_\Omega \phi d\theta.$$

In particular, if $\phi = \mathbb{1}_F sign \psi'(\widetilde{u})$, where $F \subset \Omega \backslash S_u$ Borel set so that the triangle inequality becomes equality, we obtain

$$\left|\int_F \psi'(\widetilde{u})\, dD^c u\right| = \int_F |\psi'(\widetilde{u})| dD^c u \le \|\psi\|_\infty \theta(F). \tag{3.20}$$

Choosing $\psi_\varepsilon^1 = \sin\left(\frac{t}{\varepsilon}\right)$ and $\psi_\varepsilon^2 = \cos\left(\frac{t}{\varepsilon}\right)$, we have that

$$\frac{1}{\varepsilon}\int_F \left|\cos\left(\frac{\widetilde{u}}{\varepsilon}\right)\right| dD^c u \le \|\psi_\varepsilon^1\| \theta(F) = \theta(F)$$

and

$$\frac{1}{\varepsilon}\int_F \left|\sin\left(\frac{\widetilde{u}}{\varepsilon}\right)\right| dD^c u \le \|\psi_\varepsilon^2\| \theta(F) = \theta(F)$$

By the fact that $1 \le |\sin t| + |\cos t|$, we infer that

$$\frac{1}{\varepsilon}\left(\int_F dD^c u\right) \le \frac{1}{\varepsilon}\left(\int_F \left|\cos\left(\frac{\widetilde{u}}{\varepsilon}\right)\right| dD^c u + \int_F \left|\sin\left(\frac{\widetilde{u}}{\varepsilon}\right)\right| dD^c u\right) \le 2\theta(F).$$

Then,

$$\int_F |D^c u| \le 2\varepsilon\theta(F).$$

As F is arbitrarily, taking $\varepsilon \to 0$, we have then, $|D^c u| = 0$, and it implies that $D^c u = 0$. Now, we can conclude that $u \in SBV(\Omega)$.

We are going to obtain (3.18). Comparing (3.15) and (3.16) with $a = \nabla u$

$$\int_{\Omega\times\mathbb{R}} \phi(x)\psi(s)\, d\mu(x,s) = \int_{S_u} \phi\left[\psi(u_+) - \psi(u_-)\right] \nu_u d\mathcal{H}^{n-1}, \tag{3.21}$$

for all $\psi \in C_c^1(\mathbb{R})$. Even more, the equality above holds for any bounded Borel function ϕ. Let $S \subset S_u$ be a Borel set such that $\mathcal{H}^{n-1}(S) < +\infty$ and

$$\tau_S := \Phi_\#^+\left(\nu_u \mathcal{H}^{n-1} \llcorner S\right) - \Phi_\#^-\left(\nu_u \mathcal{H}^{n-1} \llcorner S\right).$$

For any Borel set $F \subset S$, doing the same process to obtain (3.9) and (3.15), we have

$$\int_{\Omega\times\mathbb{R}} \mathbb{1}_F(x)\psi(s)\, d\mu(x,s) = \int_{\Omega\times\mathbb{R}} \mathbb{1}_F(x)\psi(s)\, d\tau_S(x,)$$

for any $\psi \in C_c^1(\mathbb{R})$. Since, F and ψ are arbitrary, we obtain

$$\mu \llcorner S \times \mathbb{R} = \tau_S \tag{3.22}$$

Similar arguments used in (3.15) gives

$$\begin{aligned}
2\mathcal{H}^{n-1}(S) &= 2\mathcal{H}^{n-1}(S_u \cap S) \\
&= \mathcal{H}^{n-1}\left(S_u \cap (\Phi^+)^{-1}(S \times \mathbb{R})\right) + \mathcal{H}^{n-1}\left(S_u \cap (\Phi^-)^{-1}(S \times \mathbb{R})\right) \\
&= |\tau_S|(S \times \mathbb{R}) \leq |\mu|(\Omega \times \mathbb{R}). \qquad (3.23)
\end{aligned}$$

Letting $S \to S_u$ (in the sense of Hausdorff metric) in (3.22) and (3.23), we infer

$$2\mathcal{H}^{n-1}(S_u) \leq \liminf_{S \to S_u} \mathcal{H}^{n-1}(S) \leq |\mu|(\Omega \times \mathbb{R}) < +\infty$$

and

$$\mu \llcorner S_u \times \mathbb{R} = \Phi_{\#}^{+}\big(\nu_u \mathcal{H}^{n-1} \llcorner S_u\big) - \Phi_{\#}^{-}\big(\nu_u \mathcal{H}^{n-1} \llcorner S_u\big).$$

On the other hand, (3.21) implies that $\mu \llcorner ((\Omega \backslash S_u) \times \mathbb{R}) = 0$, hence (3.17) follows. $\square$

The last theorem, is a important tool to show the compactness of $SBV(\Omega)$. This method was developed by L.Ambrosio in [3]).

Theorem 3.6 (Compactness of $SBV(\Omega)$)**.** *Let $\phi : [0, +\infty) \to [0, +)$ be a convex function such that*

$$\lim_{t \to +\infty} \frac{\phi(t)}{t} = +\infty.$$

Let $(u_h) \subset SBV(\Omega)$ be a sequence such that

$$\|u_h\|_\infty + \int_\Omega \phi(|\nabla u_h|)\, dx + \mathcal{H}^{n-1}(S_{u_h}) \leq c,$$

for some constant c independent of h. Then, there exist a sub-sequence $v_k = u_{h_k}$ converging to $v \in SBV(\Omega)$ in the $L^1_{loc}(\Omega)$ topology. Moreover, ∇v_k weakly converges to ∇v in

$L^1(\Omega,\mathbb{R}^n)$ *as* $k\to+\infty$ *and*

$$\mathcal{H}^{n-1}(S_v)\le \liminf_{k\to+\infty}\mathcal{H}^{n-1}(S_{v_k}).$$

Proof. Using the same process to obtain (3.12), we can find a vector measure μ_h in $\Omega\times\mathbb{R}$ such that

$$\int_\Omega\left[\frac{\partial\phi}{\partial x_i}\psi(u_h)+\phi\psi'(u_h)\frac{\partial u_h}{\partial x_i}\right]dx=-\int_{\Omega\times\mathbb{R}}\phi(x)\psi(s)\,d\mu_{h,i}(x,s)$$

for any $i\in\{1,\dots,\}$, any $\psi\in C_c^1(\mathbb{R})$ and any $\psi\in C_c^1(\mathbb{R})$. By (3.14), we have also

$$|\mu_h|(\Omega\times\mathbb{R})=2\mathcal{H}^{n-1}(S_{u_h}). \tag{3.24}$$

Suppose for simplicity that $\int_\Omega|\nabla u_h|dx\le c$. Since,

$$|Du_h|(\Omega)\le\int_\Omega|\nabla u_h|+2\|u_h\|_\infty\mathcal{H}^{n-1}(S_{u_h}),$$

by Theorem 3.23 in [5], we can assume, up to a subsequence $v_k=u_{h_k}$ converges in $L^1_{loc}(\Omega)$ to $v\in BV(\Omega)$. Now we will prove that $v\in SBV(\Omega)$

Possibly, extracting a further subsequence, we can assume that the measures $\sigma_k=\mu_{h_k}$ weakly* converges in $\Omega\times\mathbb{R}$ to σ and ∇v_k weakly converges in $L^1(\Omega,\mathbb{R}^n)$ to a (where a is defined as before).

Since ψ' is bounded and continuous and ∇v_k are equi-integrable, we can pass to the limit as $k\to+\infty$ in

$$\int_\Omega\left[\frac{\partial\phi}{\partial x_i}\psi(v_k)+\phi\psi'(v_k)\frac{\partial v_k}{\partial x_i}\right]dx=-\int_{\Omega\times\mathbb{R}}\phi(x)\psi(s)\,d\sigma_{k,i}(x,s)$$

to obtain

$$\int_\Omega\left[\frac{\partial\phi}{\partial x_i}\psi(v)+\phi\psi'(v)a_i\right]dx=-\int_{\Omega\times\mathbb{R}}\phi(x)\psi(s)\,d\sigma_i(x,s).$$

By the last theorem, we infer that $v\in SBV(\Omega)$, $a=\nabla v$ and

$$2\mathcal{H}^{n-1}(S_v)=|\sigma|(\Omega\times\mathbb{R}).$$

By (3.24) and the lower semi-continuity of the total variation with respect to weak* convergence of measures, we get

$$2\mathcal{H}^{n-1}(S_v) \leq \liminf_{k\to+\infty} |\sigma_k(\Omega \times \mathbb{R})| = 2\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{v_k}).$$

$\square$

To end this chapter, we introduce the following definition, which is very important to study minimizers and quasi-minimizers of Mumford-Shah functional.

Definition 3.5. *The space $SBV^p(\Omega)$, for $p > 1$, is a subset of $SBV(\Omega)$ made of all functions $u \in SBV(\Omega)$ such that $\nabla u \in L^p(\Omega)$ and $\mathcal{H}^{n-1}(S_u) < +\infty$.*

The main interest of this space, is to study coercivity and lower semi-continuous properties in functional like Mumford-Shah.

Chapter 4

Some regularity results for the jump set of minimizers and quasi-minimizers

In this chapter, we are going to study the existence of minimizers and quasi-minimizers of a functional similar to the Mumford-Shah functional. Also, we are going to see some regularity (Alfhors-David regularity) to the jumps sets related to the minimizers and quasi-minimizers.

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, with $\partial_D\Omega = \partial\Omega$ the Dirichlet boundary and $\Omega' \subset \mathbb{R}^n$ such that

1. $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary,

2. Ω' is a bounded open set with $\Omega \subset \Omega'$, $\Omega' \cap \partial\Omega = \partial_D\Omega$ and $diam\left(\Omega'\right) \leq 2diam\left(\Omega\right)$.

4.1 Existence of Minimizers and quasi-minimizers

For every open set $A \subset \Omega'$ and every $u \in SBV\left(A\right)$, let

$$F\left(u, A\right) := \begin{cases} \displaystyle\int_A |\nabla u|^2 dx + \mathcal{H}^{n-1}\left(S_u \cap A\right) & \text{if } u = 0 \text{ a.e. in } A \cap \left(\Omega' \backslash \overline{\Omega}\right), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.1}$$

The **minimal value** is

$$\mathbf{m}(u,A) := \inf\left\{F(v,A); v \in SBV^2(A), \{v \neq u\} \subset\subset A\right\},$$

and the deviation from minimality of u on A by setting, if $\mathbf{m}(u,A) < +\infty$,

$$Dev(u,A) := F(u,A) - \mathbf{m}(u,A).$$

Remark 4.1. *The role of Ω' in the definition of $F(u,A)$ is that of enforcing in a variational sense the Dirichlet condition on $\partial_D\Omega \cap A$. Indeed, it is immediately seen that*

$$S_u \cap A = (S_u \cap A \cap \Omega) \cup \{x \in \partial_D\Omega \cap A; u(x) \neq 0\},$$

where the value of u on $\partial_D\Omega \cap A$ is understood in the sense of traces.

Definition 4.1. *Let $A \subset \Omega'$ be an open set. We say that $u \in SBV^2(A)$ is a quasi-minimizer of $F(\cdot,A)$ if there exist constants $\omega > 0$ and $s \in (0,1)$ such that for every ball $B_\rho(x) \subset A$*

$$Dev(u, B_\rho(x)) \leq \omega\rho^{n-1+s}.$$

Let us intrduce for $\delta \geq 0$,

$$H_\delta := \{x = (x', x_n) \in \mathbb{R}^n; x_n > -\delta\},$$

and for every $r \leq 1$ and $u \in W^{1,p}(B_r)$, let

$$F_{0,\delta}(u, B_r) := \begin{cases} \displaystyle\int_{B_r} |\nabla u|^2 dx & \text{if } u = 0 \text{ a.e. in } B_r \backslash H_\delta, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.2}$$

Remark 4.2. *Note that if $r < \delta$, then the functional $F_{0,\delta}(\cdot, B_r)$ does not see the Dirichlet condition which therefore becomes irrelevant.*

Definition 4.2 (Local minimizers)**.** *Let $r > 0$. We say that $u \in W^{1,2}(B_r)$ is a local minimizer of $F_{0,\delta}(\cdot, B_r)$ if*

$$F_{0,\delta}(u, B_r) \leq F_{0,\delta}(v, B_r)$$

for every $v \in W^{1,2}(B_r)$ with $\{u \neq v\} \subset\subset B_r$.

The next theorems will be used to show the Alfhors-David regularity of the jump set related to minimizers and quasi-minimizers of the functional (4.1). For a general proof of these theorems, we refer the reader to [7].

Theorem 4.1 (Interior gradiend bound)**.** *Assume $\delta > \frac{1}{2}$. Let $u \in W^{1,2}\left(B_{\frac{1}{2}}\right)$ be a local minimizer of $F_{0,\delta}\left(\cdot, B_{\frac{1}{2}}\right)$. Then*

$$ess \sup_{B_{\frac{\rho}{2}}} |\nabla u|^2 \leq \frac{1}{\omega_n \rho^n} \int_{B_\rho} |\nabla u|^2 dx \quad \text{for every } \rho \leq \frac{1}{2}.$$

Proof. Let $\rho \leq \frac{1}{2}$. Let $u \in W^{1,2}\left(B_{\frac{1}{2}}\right)$ be a local minimizer of $F_{0,\delta}\left(\cdot, B_{\frac{1}{2}}\right)$. We know that u is an harminoc function, i.e., $\Delta u = 0$. Then, $-\Delta|\nabla u|^2 \leq 0$ and by the Mean Value Theorem, we have that

$$|\nabla u|^2(x) \leq \frac{1}{\omega_n \rho^n} \int_{B_\rho} |\nabla u|^2 dx.$$

We conclude

$$ess \sup_{B_{\frac{\rho}{2}}} |\nabla u|^2 \leq \frac{1}{\omega_n \rho^n} \int_{B_\rho} |\nabla u|^2 dx.$$

□

Theorem 4.2 (Boundary gradient bound)**.** *Assume that $\delta \in \left[0, \frac{1}{2}\right]$. Let $u \in W^{1,2}(B_1)$ be a local minimizer of $F_{0,\delta}(\cdot, B_1)$. Then*

$$ess \sup_{B_{\frac{\rho}{2}}} |\nabla u|^2 \leq \frac{2}{\omega_n \rho^n} \int_{B_\rho} |\nabla u|^2 dx \quad \text{for every } \rho \leq 1.$$

The proof of this theorem is easily established using the same proof of the last theorem, the harmonic reflection principle and the mean value theorem, because, u is harmonic on $B_1 \cap H_\delta$.

Let $A \subset \Omega'$ an open set, we will prove in the next theorem that there exist a minimizer for the next functional

$$F(u) := \begin{cases} \int_A |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u) & \text{if } u = g \text{ a.e. in } A \cap \left(\Omega' \backslash \overline{\Omega}\right), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.3}$$

Theorem 4.3. *There exist a minimizer $v \in SBV^2(A)$ for the functional (4.3).*

Proof. The idea of this proof is to use the Ambrosio compactness theorem (See Theorem 3.6) to obtain the existence of a minimizer of (4.3).
Let (u_h) a minimizing sequence of (4.3) so that

$$\lim_{h \to +\infty} F(u_h) = \inf_{u \in SBV^2(A), u = g a.e. in \left(\Omega' \backslash \overline{\Omega}\right) \cap A} F(u),$$

where F is defined in (4.1). Let $M = \|g\|_\infty$ and now we set $u_h = (u_h \wedge M) \vee (-M)$. It yields that $\|u_h\|_\infty \leq M$ and that $|\nabla u_h| \leq |\nabla u|$ $\mathcal{L}^n$-a.e on Ω for all $h \in \mathbb{N}$. We can see that there exists $c > 0$ such that

$$\int_A |\nabla u_h|^2 dx + \mathcal{H}^{n-1}(S_{u_h}) \leq cM$$

for all $h \in \mathbb{N}$ (This arguments is known as the $M-$truncation). By the Ambrosio Compactness Theorem (Theorem 3.6), we have that there exists a subsequence $(u_{h_k})_{k \in \mathbb{N}}$ in $SBV^2(A)$ and a function $v \in SBV^2(A)$ such that $u_{h_k} \to v$ in the $L^1_{loc}(\Omega)-$topology, $\nabla u_{h_k} \to \nabla v$ weakly and $\mathcal{H}^{n-1}(S_v) \leq \liminf_{k \to +\infty} \mathcal{H}^{n-1}(S_{h_k})$. Then,

$$F(v) \leq \liminf_{k \to +\infty} F(u_{h_k})$$

and we conclude that $v \in SBV^2(A)$ is a minimizers of the problem (4.3). □

Let us consider an open set $A \subset \Omega'$, and let $u \in SBV^2(A)$ be a solution of

$$\min \left\{ \int_A |\nabla v|^2 dx + \int_{S_v} d\mathcal{H}^{n-1}(x); v \in SBV^2(A), v = g \text{ a.e. in } \left(\Omega' \backslash \overline{\Omega}\right) \cap A \right\}, \tag{4.4}$$

where $g \in W^{1,\infty}(\Omega')$. With this assumptions, the following energy upper bound estimate holds:

Lemma 4.4. *Let $A \subset \Omega'$, $g \in W^{1,\infty}(\Omega')$, and let $u \in SBV^2(A)$ be a solution of (4.4). There exists a constant $c_0 > 0$ (depending only on $n, \|\nabla g\|_\infty, diam(\Omega)$ and the Lipschitz constant of $\partial\Omega$) such that for any ball $B_\rho(x) \subset A$*

$$\|\nabla u\|^2_{L^2(B_\rho(x),\mathbb{R}^n)} + \mathcal{H}^{n-1}(S_u \cap B_\rho(x)) \leq c_0 \rho^{n-1}.$$

Proof. Let $u \in SBV^2(A)$ be a solution of (4.4). Let us compare u with $v := u\mathbb{1}_{A\setminus(\Omega\cap B_\rho(x))}$. We can observe that $v \in SBV^2(A)$ with $A\setminus(\Omega \cap B_\rho(x))$ set with Lipschitz boundary hence it has finite perimeter, indeed,

$$\begin{aligned}
Dv &= \nabla v \mathcal{L}^n + (v_+ - v_-)\nu\mathcal{H}^{n-1} \llcorner S_u \\
&= \nabla u \mathcal{L}^n \llcorner A\setminus(\Omega \cap B_\rho(x)) + (u_+ - u_-)\nu\mathcal{H}^{n-1} \llcorner (S_u \cap A\setminus(\Omega \cap B_\rho(x))) \\
&= \nabla u \mathcal{L}^n \llcorner A\setminus(\Omega \cap B_\rho(x)) + (u_+ - u_-)\nu\mathcal{H}^{n-1} \llcorner (\partial A\setminus(\Omega \cap B_\rho(x))).
\end{aligned}$$

$Dv = \mathbb{1}_{A\setminus(\Omega\cap B_\rho(x))} Du + \left(u\nabla\mathbb{1}_{A\setminus(\Omega\cap B_\rho(x))}\right)\mathcal{L}^n$. Also, we have that $v = g$ on $(\Omega'\setminus\underline{\Omega}) \cap A$, in fact, it is easy to see that $\left(\Omega'\setminus\overline{\Omega}\right) \cap A \subset (\Omega\setminus B_\rho(x)) \cap A$. Moreover

$$S_v \cap \overline{B_\rho(x)} \subset [\partial B_\rho(x) \cap \Omega] \cup [\partial_D\Omega \cap B_\rho(x)].$$

Now, we have that v is a competitor of u ($v = u = g$ a.e. in $B_\rho(x) \cap \left(\Omega'\setminus\overline{\Omega}\right)$, then, by minimality and the last identity

$$\begin{aligned}
\int_{B_\rho(x)} |\nabla u|^2 dy + \mathcal{H}^{n-1}(S_u \cap B_\rho(x)) &\leq \int_{B_\rho(x)} |\nabla v|^2 dy + \mathcal{H}^{n-1}\left(S_v \cap \overline{B_\rho(x)}\right) \\
&\leq \|\nabla g\|^p_\infty \omega_n \rho^n + \omega_n \rho^{n-1} + \mathcal{H}^{n-1}(\partial_D\Omega \cap B_\rho(x)),
\end{aligned}$$

so that the result follows by the Lipschitz regularity of $\partial\Omega$, and the fact that $\rho \leq diam(\Omega') \leq 2diam(\Omega)$, actually, we have that $\mathcal{H}^{n-1}(\partial_D\Omega) \leq c_1\rho^{n-1}$ where c_1 depends on $Lip(\partial\Omega)$ and $diam(\Omega)$. $\square$

The next result shows the existence of quasi-minimizer of the problem (4.4).

Theorem 4.5. *Assume (4.1). Let $A \subset \Omega'$, $g \in W^{1,\infty}(\Omega')$, and let $u \in SBV^2(A)$ be a solution of (4.4). Then the function*

$$\hat{u} := u - g \in SBV^2(A)$$

is a quasi-minimizer of $F(\cdot, A)$ for $s = \frac{1}{2}$ and $\omega > 0$ wich only depends on n (dimension), $\|\nabla g\|_\infty$, diam (Ω) and the Lipschitz constant of $\partial\Omega$.

Proof. Since $\hat{u} = u - g$ and $g \in W^{1,\infty}(\Omega')$, we have that $S_{\hat{u}} = S_u$, then $\hat{u}$ minimizes

$$v \mapsto \int_A |\nabla v|^2 dy + 2\int_A \nabla v \cdot \nabla g dy + \int_a |\nabla g|^2 dy + \mathcal{H}^{n-1}(S_v \cap A),$$

for all $v \in SBV^2(A)$ with $v = 0$ a.e. in $\left(\Omega' \backslash \overline{\Omega}\right) \cap A$. This holds, replacing $\hat{u} + g$ in (4.4). Now, we will try to construct the setting of the lemma above. Let $B_\rho(x) \subset A$ and $v \in SBV^2(A)$ with $\{v \neq \hat{u}\} \subset B_\rho(x)$. From the previous minimality lemma, we deduce that

$$F(\hat{u}, B_\rho(x)) \leq F(v, B_\rho(x)) + \int_{B_\rho(x)} (\nabla v - \nabla \hat{u}) \cdot \nabla g dy. \tag{4.5}$$

By Cauchy-Schwartz inequality, the definition of $\hat{u}$, Minkoski's inequality and the previous lemma, we infer that

$$\begin{aligned}
\left|\int_{B_\rho(x)} \nabla \hat{u} \cdot \nabla g dy\right| &\leq \|\nabla \hat{u}\|_{L^2(B_\rho(x))} \|\nabla g\|_{L^2(B_\rho(x))} \leq \|\nabla u - \nabla g\|_{L^2(B_\rho(x))} \|\nabla g\|_{L^2(B_\rho(x))} \\
&\leq \|\nabla u\|_{L^2(B_\rho(x))} \|\nabla g\|_{L^2(B_\rho(x))} + \|\nabla g\|^2_{L^2(B_\rho(x))} \\
&\leq c_0 \rho^{\frac{n-1}{2}} \|\nabla g\|_\infty \omega_n \rho^{\frac{n}{2}} + \omega_n \rho^n \|\nabla g\|^2_\infty \leq c_1 \rho^{n-1+\frac{1}{2}},
\end{aligned} \tag{4.6}$$

where c_1 is a constant which depends on n, $\|\nabla g\|_\infty$, $diam\,(\Omega)$ and the Lipschitz constant of $\partial\Omega$. On the other hand, to estimate the other term, we use Young's inequality to get that

$$\left|\int_{B_\rho(x)} \nabla v \cdot \nabla g dy\right| \leq \int_{B_\rho(x)} \varepsilon^2 |\nabla v|^2 + \frac{1}{2\varepsilon^2}|\nabla g|^2 dy \leq c_2 \left(\varepsilon^2 F\left(v, B_\rho\left(x\right)\right) + \frac{1}{2\varepsilon^2}\rho^n\right), \tag{4.7}$$

for some constant c_2 depending on $\|\nabla g\|_\infty$ and for $\varepsilon > 0$ to be fixed later. Hence, gathering (4.5), (4.6) and (4.7), we obtain that

$$F\left(\hat{u}, B_\rho\left(x\right)\right) \leq \left(1 + c_2\varepsilon^2\right) F\left(v, B_\rho\left(x\right)\right) + c_1\rho^{n-1+\frac{1}{2}} + c_2\varepsilon^{-2}\rho^n$$

Taking the infimum with respect to such v, to have the definition of deviation of minimality, we infer that

$$Dev\left(\hat{u}, B_\rho\left(x\right)\right) \leq c_2\varepsilon^2 F\left(\hat{u}, B_\rho\left(x\right)\right) + c_1\rho^{n-1+\frac{1}{2}} + c_2\varepsilon^{-2}\rho^n.$$

As a consequence, using Lemma 4.4, we infer that

$$Dev\left(\hat{u}, B_\rho\left(x\right)\right) \leq c_3\left(\varepsilon^2\rho^{n-1} + \rho^{n-1+\frac{1}{2}} + \varepsilon^{-2}\rho^n\right),$$

where $c_3 > 0$ depends only on $n, \|\nabla g\|_\infty, diam\left(\Omega\right)$ and the Lipschitz constant of $\partial\Omega$. If we choose $\varepsilon = \rho^{\frac{1}{4}}$, we finally get that

$$Dev\left(\hat{u}, B_\rho\left(x\right)\right) \leq \omega\rho^{n-1+\frac{1}{2}},$$

for some constant $\omega > 0$ so that $\hat{u}$ is a quasi-minimizer for $F\left(\cdot, A\right)$ with the choice $s = \frac{1}{2}$. $\square$

4.2 Density lower bound for the jump set of quasi-minimizers under homogeneous Dirichlet boundary conditions

The case for the Mumford-Shah functional was studied by De Giorgi, Carreiro and Leaci in [16], and after, extended to the nonlinear case by Ambrosio, Fusco, Pallara and Fonseca (See [5, 19]). We are going to develop the density lower bound for the jump set, treated by Carreiro and Leaci in [12], but, we will do the same as Babadjian and Giacomini in [7]. Furthermore, our analysis is for balls with possibly intersection $\partial\Omega$ but with center inside Ω. The aim of this section is prove the next theorem.

Theorem 4.6. *Assume that Ω satisfies (4.1), and that*

$$\partial\Omega \text{ is of class } C^1.$$

Let $A \subset \Omega'$ be an open set, and let $u \in SBV^p(A)$ be a quasi-minimizer of $F(\cdot, A)$ with constants $\omega > 0$ and $s \in (0,1)$. Then, there exist $\vartheta_0 > 0$ and $\rho_0 > 0$ (depending only on n, s, ω) such that

$$\mathcal{H}^{n-1}(S_u \cap B_\rho(x)) \geq \vartheta_0 \rho^{n-1},$$

for all balls $B_\rho(x) \subset A$ with center $x \in \overline{S_u}$, radius $\rho \leq \rho_0$.

For the proof of this theorem, we will follow closely ([7] and [5] Section 7.1 and 7.2). At the end of this chapter, we will prove that the jump set is essentially closed.

4.2.1 Problems on the unit ball

First, we reduce, a similar problem, to the case of unit ball.

Given $D \subset B_1$ a Borel set, $c > 0$, $u \in SBV(B_1)$ and $\rho \leq 1$, let us set

$$F_D(u, c, B_\rho) := \begin{cases} \displaystyle\int_{B_\rho} |\nabla u|^2 dx + c\mathcal{H}^{n-1}(S_u \cap B_\rho) & \text{if } u = 0 \ a.e \ in \ D, \\ +\infty & \text{otherwise.} \end{cases}$$

We denote by

$$\mathbf{m}_D(u, c, B_\rho) := \inf\left\{F_D(u, c, B_\rho); v \in SBV^2(B_1), \{v \neq u\} \subset\subset B_\rho\right\}$$

the minimal value, and by $Dev_D(u, c, B_\rho)$ the deviation from minimality of u on B_ρ for F_D, defined, if $\mathbf{m}_D(u, c, B_\rho) < +\infty$, by

$$Dev_D(u, c, B_\rho) = F_D(u, c, B_\rho) - \mathbf{m}_D(u, c, B_\rho).$$

The next theorem gives us a compactness property and it is a variant of [7], Proposition 7.5.

Lemma 4.7. *Let $D_h \subset B_1$ be a sequence of Borel sets such that for some $d_0 > 0$, we have $|D_h| \geq d_0$ for every $h \in \mathbb{N}$. Let $(v_h) \subset SBV^2(B_1)$ be such that*

$$\sup_{h \in \mathbb{N}} \int_{B_1} |\nabla v_h|^2 dx < +\infty, \quad \mathcal{H}^{n-1}(S_{v_h}) \to 0, \quad v_h = 0 \ a.e. \ in \ D_h.$$

Let us define $\overline{v}_h = (v_h \wedge \tau^+(v_h, B_1)) \vee \tau^-(v_h, B_1)$, where

$$\begin{cases} \tau^-(v_h, B_1) := \inf\left\{t \in [-\infty, +\infty]; |\{v_h < t\}| \geq \left[2\gamma_n \mathcal{H}^{n-1}(S_{v_h})\right]^{\frac{n}{n-1}}\right\}, \\ \tau^+(v_h, B_1) := \inf\left\{t \in [-\infty, +\infty]; |\{v_h < t\}| \geq |B_1| - \left[2\gamma_n \mathcal{H}^{n-1}(S_{v_h})\right]^{\frac{n}{n-1}}\right\} \end{cases} \tag{4.8}$$

and γ_n is the dimensional constant of the isoperimetric inequality.

Then there exists $h_0 \in \mathbb{N}$ (depending only on d_0 and n) such that $\overline{v}_h = 0$ a.e. in D_h for any

$h \geq h_0$. *Moreover there exist a subsequence* $\left(v_{h_j}\right) \subset (v_h)$, *and a function* $v \in W^{1,p}(B_1)$ *such that* $\overline{v}_h \to v$ *strongly in* $L^2(B_1)$, $v_{h_j} \to v$ *a.e. in* B_1, *and*

$$\int_{B_\rho} |\nabla v|^2 dx \leq \liminf_{j\to+\infty} \int_{B_\rho} \left|\nabla \overline{v}_{h_j}\right|^2 dx \quad \textit{for every } \rho \leq 1.$$

Before the proof of Lemma 4.7 we state a theorem which is a consequence of Poincaré inequality for $SBV(\Omega)$ (See Theorem 4.14, [5]) and the Ambrosio Compactness Theorem.

Proposition 4.8. *Let* $B \subset \mathbb{R}^n$ *a ball and* $(u_h) \subset SBV(B)$ *be a sequence such that*

$$\sup_{h\in\mathbb{N}} \int_B |\nabla u|^2 dx < +\infty \qquad \lim_{h\to+\infty} \mathcal{H}^{n-1}(S_{u_h})$$

and let m_h *be medians of* u_h *in* B, *i.e.,* $m_h := \inf\left\{t \in [-\infty,+\infty]\,; |\{v_h > t\}| \leq \frac{|B|}{2}\right\}$. *Then there exist a subsequence* $\left(u_{h_j}\right)$ *and a function* $u \in W^{1,2}(B)$ *such that functions* $\overline{u}_{h_j} - m_{h_j}$ *converge in* $L^2(B)$ *to* u *and*

$$\int_B |\nabla u|^2 dx \leq \liminf_{j\to+\infty} \int_B \left|\nabla u_{h_j}\right|^2 dx.$$

As remark, we can see that $\overline{u}_{h_j} - m_{h_j}$ converge also in $\mathcal{L}^n$-a.e. in B to u (See Theorem 7.5 and Remark 7.6 in [5])

Proof of Lemma 4.7. First, we will show that $\overline{v}_h = 0$ a.e. in D_h holds to h large enough. Since $v_h = 0$ a.e. in D_h, we have that $\overline{v}_h = 0$, as a consequence of the fact that for h large enough

$$\tau^-(v_h, B_1) \leq 0 \quad \text{and} \quad \tau^+(v_h, B_1) \geq 0. \tag{4.9}$$

Indeed, given $\varepsilon > 0$, since $\mathcal{H}^{n-1}(S_{v_h}) \to 0$ we have for $h \geq h_0$ independent of ε

$$\begin{aligned} |\{v_h < \varepsilon\}| &= |\{v_h < \varepsilon\} \cap D_h| + |\{v_h < \varepsilon\} \cap D_h^c| \geq |\{v_h < \varepsilon\} \cap D_h| \\ &= |D_h| - |D_h \cap \{v_h \geq \varepsilon\}| \geq d_0 > \left[2\gamma_n \mathcal{H}^{n-1}(S_{v_h})\right]^{\frac{n}{n-1}}. \end{aligned}$$

The value ε is admissible for the computation of $\tau^-(v_h, B_1)$ for $h \geq h_0$ so that $\tau^-(v_h, B_1) \leq \varepsilon$. Then, we infer that $\tau^-(v_h, B_1) \leq 0$. Using the fact that

$$\tau^+(v_h, B_1) \geq \sup\left\{t \in [-\infty, +\infty]\,; |\{v_h > t\}| \geq \left[2\gamma_n \mathcal{H}^{n-1}(S_{v_h})\right]^{\frac{n}{n-1}}\right\} \tag{4.10}$$

and proceeding as before with $-\varepsilon$, we can show that $-\varepsilon$ is admissible in (4.10) and then $\tau^+(v_h, B_1) \geq -\varepsilon$; it implies $\tau^+(v_h, B_1) \geq 0$. Thus (4.9) follows. Now in view of Poincare's inequality in $BV(\Omega)$ (See Theorem 4.14, [5]), denoting by

$$m_h := \inf\left\{t \in [-\infty, +\infty]\,; |\{v_h > t\}| \leq \frac{|B_1|}{2}\right\},$$

a median for v_h, there exists a subsequence $\left(v_{h_j}\right)$ and a function $v \in W^{1,2}(B_1)$ such that

$$\overline{v}_{h_j} - m_{h_j} \to v \quad \text{strongly in } L^2(B_1), \quad v_{h_j} - m_{h_j} \to v \quad \text{a.e. in } B_1, \tag{4.11}$$

and, applying Ambrosio's compactness theorem to the truncation at level M (See the proof of Theorem 4.3) of the previous sequence, and letting $M \to +\infty$, we get for every $\rho \leq 1$

$$\int_{B_\rho} |\nabla v|^2 dx \leq \liminf_{j\to\infty} \int_{B_\rho} \left|\nabla \overline{v}_{h_j}\right|^2 dx$$

The proof is complete if we show that $\left(m_{h_j}\right)$ is bounded; since we can consider (up to extracting a further sequence) $v+m$ as limit function, where m is a limit point for $\left(m_{h_j}\right)$. Since $\overline{v}_h = 0$ a.e. in D_h for $h \geq h_0$, by (4.11), we obtain

$$\limsup_{j\to+\infty} \left|m_{h_j}\right|^2 \left|D_{h_j}\right| \leq \|v\|^2_{L^2(B_1)},$$

and the results follows since $\left|D_{h_j}\right| \geq d_0$. $\square$

The following proposition show that the limit of one Sobolev function is a local minimizer of $|\cdot|^2$. Furthermore, this result is an adaptation of Theorem 7.7 in [5] to the case of homogeneous Dirichlet boundary condition.

Proposition 4.9. *Let $f_h : \mathbb{R}^{n-1} \to \mathbb{R}$ be a sequence of continuous functions, and let*

$D_h := \{x = (x', x_n) \in B_1; x_n \leq f_h(x')\}$. Assume that (f_h) is locally uniformly converging to the constant function $-\delta$, with $\delta \in (0,1)$. Let $c_h > 0$ and $v_h \in SBV^2(B_1)$ be such that

$$\sup_{n\in\mathbb{N}} F_{D_h}(v_h, c_h, B_1) < +\infty,$$

$$\lim_{h\to+\infty} Dev_{D_h}(v_h, c_h.B_1) = 0,$$

$$\lim_{h\to+\infty} \mathcal{H}^{n-1}(S_{v_h}) = 0,$$

$$v_h \to v \in W^{1,2}(B_1) \quad \textit{a.e. in } B_1.$$

Then v is a local minimizer of $F_{0,\delta}(\cdot, B_1)$ and

$$\lim_{h\to+\infty} F_{D_h}(v_h, c_h, B_\rho) = \int_{B_\rho} |\nabla v|^2 dx \quad \textit{for every } \rho \in (0,1). \tag{4.12}$$

Proof. Since $\rho \mapsto F_{D_h}(v_h, c_h, B_\rho)$ is increasing on $[0,1]$, by Helly's theorem we may assume that up to a subsequence

$$\lim_{h\to+\infty} F_{D_h}(v_h, c_h, B_\rho) = \alpha(\rho) \quad \text{for every } \rho \in [0,1]$$

for some increasing function $\alpha : [0,1] \to [0,\infty)$.
By a geometric argument, and the fact that $f_h \to -\delta$ locally uniform, and $\delta < 1$, we can show that $|D_h| \geq d_0$ for some constant $d_0 > 0$. Hence, by Lemma 4.7 we have that (for a not relabeled subsequence) $\overline{v}_h \to v$ strongly in $L^2(B_1)$, $\overline{v}_h = 0$ a.e. in D_h for h large enough, and for all $\rho \leq 1$

$$\int_{B_\rho} |\nabla v|^2 dx \leq \liminf_{h\to+\infty} \int_{B_\rho} |\nabla \overline{v}_h|^2 dx. \tag{4.13}$$

We claim that

$$\lim_{h\to+\infty} F_{D_h}(\overline{v}_h, c_h, B_\rho) = \alpha(\rho) \quad \text{for a.e. } \rho \in (0,1), \tag{4.14}$$

and that

$$\lim_{h\to+\infty} Dev_{D_h}(\overline{v}_h, c_h, B_\rho) = 0 \quad \text{for every } \rho \in (0,1) \tag{4.15}$$

Assuming as true (4.14) and (4.15), we are going to show that $v \in W^{1,2}(B_1)$ is a local minimizer of the functional $F_{0,\delta}(\cdot, B_1)$ given in (4.2). Let $w \in W^{1,2}(B_1)$ with $\{w \neq v\} \subset\subset B_1$ and $w = 0$ a.e. in $B_1 \backslash H_\delta$. By the Lemma 6.3 in [7], we can find $w_h \in W^{1,2}(B_1)$ satisfying $w_h = 0$ a.e. in D_h, and $w_h \to w$ strongly in $W^{1,2}(B_1)$.

Let $0 < \rho' < \rho < 1$ be such that α is continuous at ρ, $\{w \neq v\} \subset\subset B_{\rho'}$ and (4.14) holds for both ρ and ρ'. Let η be a cut-off function between $B_{\rho'}$ and B_ρ; comparing $\overline{v}_h$ with $\eta w_h + (1-\eta)\overline{v}_h$ (observe that $\eta w_h + (1-\eta)\overline{v}_h = 0$ a.e. in D_h, then an admissible competitor), we have the inequality

$$\mathbf{m}_{D_h}(\overline{v}_h, c_h, B_\rho) \leq F_{D_h}(\eta w_h + (1-\eta)\overline{v}_h, c_h, B_\rho)$$

By the definition of Deviation from minimality, we have

$$\begin{aligned} F_{D_h}(\overline{v}_h, c_h, B_\rho) &\leq F_{D_h}(\eta w_h + (1-\eta)\overline{v}_h, c_h, B_\rho) + Dev_{D_h}(\overline{v}_h, c_h, B_\rho) \\ &= \int_{B_\rho} |\nabla(\eta w_h + (1-\eta)\overline{v}_h)|^2 dx + c_h \mathcal{H}^{n-1}\left(S_{\eta w_h + (1-\eta)\overline{v}_h} \cap B_\rho\right) \\ &\quad + Dev_{D_h}(\overline{v}_h, c_h, B_\rho). \end{aligned}$$

We know that there exists $C > 0$ such that $|\nabla \eta| \leq \dfrac{C}{\rho - \rho'}$ (See Lemma 7.4 in [5]), then we infer that

$$\begin{aligned} \int_{B_\rho} |\nabla(\eta w_h + (1-\eta)\overline{v}_h)|^2 dx &\leq \int_{B_\rho} |\eta \nabla w_h|^2 dx + \int_{B_\rho} |(1-\eta)\nabla \overline{v}_h|^2 dx \\ &\quad + \int_{B_\rho} |(\overline{v}_h - w_h)\nabla \eta|^2 dx \\ &\leq \int_{B_{\rho'}} |\nabla w_h|^2 dx + C\int_{B_\rho \backslash B_{\rho'}} |\nabla w_h|^2 dx + C\int_{B_\rho \backslash B_{\rho'}} |\nabla \overline{v}_h|^2 dx \\ &\quad \frac{C}{(\rho - \rho')^2} \int_{B_\rho \backslash B_{\rho'}} |\overline{v}_h - w_h|^2 dx, \end{aligned}$$

and

$$\mathcal{H}^{n-1}\left(S_{\eta w_h + (1-\eta)\overline{v}_h} \cap B_\rho\right) = \mathcal{H}^{n-1}(S_{w_h} \cap B_\rho \backslash B_{\rho'}) + \mathcal{H}^{n-1}(S_{\overline{v}_h} \cap B_\rho \backslash B_{\rho'})$$

Finally, combining the three last expresion, we see that

$$\begin{aligned}
F_{D_h}\left(\overline{v}_h, c_h, B_\rho\right) \leq\ & F_{0,\delta}\left(w_h, B_{\rho'}\right) + Dev_{D_h}\left(\overline{v}_h, c_h, B_\rho\right) \\
& + C\left[F_{D_h}\left(\overline{v}_h, c_h, B_\rho \backslash B_{\rho'}\right) + F_{0,\delta}\left(w_h, B_\rho \backslash B_{\rho'}\right)\right] \\
& + \frac{C}{\left(\rho-\rho'\right)^2} \int_{B_\rho \backslash B_{\rho'}} \left|\overline{v}_h - w_h\right|^2 dx.
\end{aligned}$$

For $h \to +\infty$, using the fact that $\rho' < \rho$, and thanks to (4.14) and (4.15) we obtain that

$$\alpha\left(\rho\right) \leq \int_{B_{\rho'}} |\nabla w|^2 dx + C\left[\alpha\left(\rho\right) - \alpha\left(\rho'\right) + \int_{B_\rho \backslash B_{\rho'}} |\nabla w|^2 dx\right] + \frac{C}{\left(\rho-\rho'\right)^2} \int_{B_\rho \backslash B_{\rho'}} |v-w|^2 dx.$$

Since $w = v$ on $B_\rho \backslash B_{\rho'}$, letting $\rho' \to \rho$ we get

$$\alpha\left(\rho\right) \leq \int_{B_\rho} |\nabla w|^2 dx.$$

Choosing $w = v$ in the previous relation, from (4.13) and (4.14) we obtain

$$\alpha\left(\rho\right) = \int_{B_\rho} |\nabla v|^2 dx. \tag{4.16}$$

Hence, $F_{0,\delta}\left(v, B_1\right) \leq F_{0,\delta}\left(w, B_1\right)$, we conclude the local minimality. Furthermore, by the equality above, the monotone functions α and $\rho \mapsto \int_{B_\rho} |\nabla v|^2 dx$ coincides a.e. on $(0,1)$, and since the latter is continuous, they coincides everywhere on $(0,1)$. This finishes the proof of (4.12).

It remains to prove (4.14) and (4.15) to complete the proof.

Let $\widetilde{v}_h$ and $\widetilde{\overline{v}}_h$ be the Lebesgue representatives of v_h and $\overline{v}_h$ respectively. By definition of $\overline{v}_h$ and $\tau^{\pm}\left(v_h, B_1\right)$, the coarea formula for Lipschitz functions, the isoperimetric inequality (See (3.43) in [5]) and the fact that $c_h \mathcal{H}^{n-1}\left(S_{v_h}\right)$ is uniformly bounded, we conclude

$$c_h \int_0^1 \mathcal{H}^{n-1}\left(\left\{\widetilde{v}_h \neq \widetilde{\overline{v}}_h\right\} \cap \partial B_\rho\right) d\rho = c_h \left|\left\{\widetilde{v}_h \neq \widetilde{\overline{v}}_h\right\}\right| \leq \left[2\gamma_n \mathcal{H}^{n-1}\left(S_{v_h}\right)\right]^{\frac{n}{n-1}} \to 0.$$

It yields, up to a further subsequence, that

$$\lim_{h\to+\infty} c_h\mathcal{H}^{n-1}\left(\left\{\widetilde{v}_h \neq \widetilde{\overline{v}}_h\right\}\cap\partial B_\rho\right)=0 \quad \text{for a.e. } \rho\in(0,1).$$

For $\rho<1$, by truncation we deduce that

$$\begin{aligned}F_{D_h}(\overline{v}_h,c_h,B_\rho) &:= \int_{B_\rho}|\nabla\overline{v}_h|^2dx + c_h\mathcal{H}^{n-1}(S_{\overline{v}_h}\cap B_\rho)\\ &\leq \int_{B_\rho}|\nabla v_h|^2dx + c_h\mathcal{H}^{n-1}(S_{v_h}\cap B_\rho) \leq F_{D_h}(v_h,c_h,B_\rho).\end{aligned}$$

For $\rho<\rho'<1$, comparing v_h and $\overline{v}_h\mathbb{1}_{B_\rho}+v_h\mathbb{1}_{B_{\rho'}\setminus B_\rho}$ (note that the last function vanishes on D_h, so that is is an admissible competitor), and by the definition of deviation we have

$$F_{D_h}(v_h,c_hB_\rho)\leq F_{D_h}(\overline{v}_h,c_hB_\rho)+c_h\mathcal{H}^{n-1}\left(\left\{\widetilde{v}_h\neq\widetilde{\overline{v}}_h\right\}\cap\partial B_\rho\right)+Dev_{D_h}(v_h,c_hB_{\rho'}).$$

Letting $h\to+\infty$ and combining the two expressions above, we obtain (4.14).
In a similar way, by taking as competitor $w\mathbb{1}_{B_\rho}+v_h\mathbb{1}_{B_{\rho'}\setminus B_\rho}$, where $w\in SBV^2(B_\rho)$ is such that $\{w\neq\overline{v}_h\}\subset\subset B_\rho$ and $w=0$ a.e. in D_h, taking the minimum in the expression above to obtain the definition of deviation of minimality, we can prove that

$$\begin{aligned}Dev_{D_h}(\overline{v}_h,c_hB_\rho)\leq F_{D_h}(\overline{v}_h,c_hB_\rho)-F_{D_h}(v_h,c_hB_\rho)+c_h\mathcal{H}^{n-1}\left(\left\{\widetilde{v}_h\neq\widetilde{\overline{v}}_h\right\}\cap\partial B_\rho\right)\\ +Dev_{D_h}(v_h,c_hB_{\rho'}).\end{aligned}$$

Letting $h\to+\infty$, we deduce that (4.15) holds for a.e. $\rho\in(0,1)$. Since the deviation is an increasing function of ρ, we infer that (4.15) actually holds for every $\rho\in(0,1)$. □

4.2.2 The density lower bound estimate

In order to prove Theorem 4.6, we will need the Decay lemma. First we are going to use the following geometric fact. We follow the method developed in [7].

Lemma 4.10. *Let (x_h) be a sequence in $\Omega\cap\partial\Omega$ such that $x_h\to\overline{x}\in\partial\Omega$, and let $\rho_h\to 0$*

be such that $B_{\rho_h}(x_h) \subset \Omega'$ and $B_{\rho_h}(x_h) \cap \partial\Omega \neq \emptyset$. Let us re-scale $B_{\rho_h}(x_h)$ to the unit ball B_1 by means of the change of variable $x := x_h + \rho_h y$ with $y \in B_1$, and let $D_h \subset B_1$ be the region corresponding to $\left(\Omega' \backslash \overline{\Omega}\right) \cap B_{\rho_h}(x_h)$.
Then, there exists a coordinate system such that, up to a subsequence,

$$D_h = \{y = (y', y_n) \in B_1; y_n \leq f_h(y')\}$$

for some $f_h \in C^1(\mathbb{R}^{n-1})$ locally uniform converging to a constant $-\delta \in [0, 1]$.

Proof. Using the fact that $\partial\Omega$ is of class C^1, we can consider the orthogonal coordinate system relative to $\overline{x}$ such that

$$\Omega \cap B_r(\overline{x}) = \{x = (x', x_n) \in B_r(\overline{x}); x_n < f(x')\},$$

where $r > 0$ and $f \in C^1(\mathbb{R}^{n-1})$ is such that $f(\overline{x}') = \overline{x}_n$ and $\nabla f(x') = 0$.
Since $x_h \to \overline{x}$ and $\rho_h \to 0$, then $B_{\rho_h}(x_h) \subset B_r(\overline{x})$ for h large enough. Let us use the coordinate system in $\overline{x}$ also for defining the blow up, i.e., the region D_h is then given for h large by those $y = (y', y_n) \in B_1$ such that

$$(x_h)_n + \rho_h y_n \leq f(x'_h + \rho_h y')$$

and we can see immediately that

$$y_n \leq f_h(y') \leq \frac{f(x'_h + \rho_h y') - (x_h)_n}{\rho_h}.$$

Let $z_h = (z'_h, (x_h)_n) \in B_{\rho_h}(x_h) \cap \partial\Omega$ so that we can write, as $(z_h)_n = f(z'_h)$, and

$$f_h(y') = \frac{f(x'_h + \rho_h y') - f(x'_h) + f(x'_h) - f(z'_h) + (z_h)_n - (x_h)_n}{\rho_h}.$$

Since we have $|f(x'_h) - f(z'_h)| \leq C|x'_h - x'_h| \leq C\rho_h$ and $|(z_h)_n - (x_h)_n| \leq \rho_h$, then up to a subsequence,

$$\frac{f(x'_h) - f(z'_h) + (z_h)_n - (x_h)_n}{\rho_h} \to c \in \mathbb{R}.$$

Moreover, the sequence of functions

$$g_h(y) := \frac{f(x'_h + \rho_h y') - f(x'_h)}{\rho_h}$$

is uniformly bounded and equicontinuous, by Arzela-Ascoli Theorem, we have that this sequence converges locally uniformly to zero since

$$\frac{f(x'_h + \rho_h y') - f(x'_h)}{\rho_h} \to \nabla f(\overline{x}') \cdot y' = 0.$$

Note that we have actually proved that $f_h \to c$ locally uniformly on $\mathbb{R}^{n-1}$.
Since $f_h(0) \leq 0$, we infer that $c \leq 0$. On the other hand, since $D_h \neq \emptyset$, there exists $\xi_h = (\xi'_h, (x_h)_n) \in B_1$ with $f_h(\xi'_h) \geq (\xi_h)_n > -1$, which implies $c \geq -1$. We can conclude taking $\delta = -c$. □

The following lemma, is a key point to obtain a density lower bound for the jump set of quasi-minimizers of (4.1). We are going to make little modifications to the method developed in [7].

Lemma 4.11 (Decay lemma)**.** *There exists a constant $C_1 > 0$ (depending only on n) with the following property: for every $\tau \in (0,1)$, there exist $\varepsilon(\tau) > 0$, $\vartheta(\tau) > 0$ and $r(\tau) > 0$ such that for every ball $B_\rho(x) \subset \Omega'$ with $x \in \overline{\Omega}$, $\rho \leq r(\tau)$, and for every $u \in SBV^2(B_\rho(x))$ satisfying $u = 0$ a.e. in $\left(\Omega' \backslash \overline{\Omega}\right) \cap B_\rho(x)$ with*

$$\mathcal{H}^{n-1}(S_u \cap B_\rho(x)) \leq \varepsilon(\tau)\rho^{n-1}, \quad Dev(u, B_\rho(x)) \leq \vartheta(\tau) F(u, B_\rho(x)),$$

then we have

$$F(u, B_{\tau\rho}(x)) \leq C_1 \tau^n F(u, B_\rho(x)).$$

Proof. Suppose the assertion of the lemma is false. For this purpose, we proceed by contradiction by showing that the result holds true for any choice of

$$C_1 > \max\left\{4^n, \frac{2}{\omega_n \rho^n}(2^n + \omega_n)\right\},$$

where $\frac{2}{\omega_n \rho^n}$ is the constant given by Theorem 4.2.
It suffices to assume that $\tau < \frac{1}{4}$; indeed, if $\tau \geq \frac{1}{4}$, we can see

$$C_1 \tau^n F(u, B_\rho(x)) \geq 4^n \frac{1}{4^n} F(u, B_\rho(x)) \geq F(u, B_\rho(x)).$$

Assume by contradiction that there exist sequences of positive numbers $\varepsilon_h, \vartheta_h, r_h \to 0$, of points $x_h \in \Omega \cup \partial\Omega$, of radii $\rho_h \leq r_h$ such that $B_{\rho_h}(x_h) \subset \Omega'$, and of functions $u_h \in SBV^2(B_{\rho_h}(x_h))$ with $u_h = 0$ a.e. in $\left(\Omega' \backslash \overline{\Omega}\right) \cap B_{\rho_h}(x_h)$, with the properties

$$\mathcal{H}^{n-1}\left(S_{u_h} \cap B_{\rho_h}(x_h)\right) = \varepsilon_h \rho_h^{n-1},$$

$$Dev\left(u_h, B_{\rho_h}(x_h)\right) = \vartheta_h F\left(u_h, B_{\rho_h}(x_h)\right),$$

and

$$F\left(u_h, B_{\rho_h}(x_h)\right) > C_1 \tau^n F\left(u_h, B_{\rho_h}(x_h)\right).$$

The following method, the general case, is developed in [5]. We re-scale B_{ρ_h} to B_1 and u_h to $v_h \in SBV^2(B_1)$ by setting

$$v_h(y) := \rho^{-\frac{1}{2}} c_h^{\frac{1}{2}} u_h(x_h + \rho_h y)$$

with

$$c_h := \frac{\rho^{n-1}}{F\left(u_h, B_{\rho_h}(x_h)\right)}.$$

Let $D_h \subset B_1$ be the set associated to $\left(\Omega' \backslash \overline{\Omega}\right) \cap B_{\rho_h}(x_h)$ under such re-scaling. If $D_h = \emptyset$, by the Lemma 7.14 in [1] we can deduce the result of the Decay Lemma (because the Dirichlet boundary condition will not play a role). Then we are going to assume that $D_h \neq \emptyset$. Consequently, since $\rho_h \to 0$, we can suppose, up to a subsequence, that $x_h \to \overline{x} \in \partial\Omega$. Then by Lemma 4.10, up to a further subsequence, we can find a coordinate system in which

$$D_h = \left\{x = (x', x_n) \in B_1; x_n \leq f_h(x')\right\},$$

for some $f_h \in C^1\left(\mathbb{R}^{n-1}\right)$ with $f_h \to -\delta$ locally uniformly, where $\delta \in [0, 1]$.

With the notation of the beginning of the Subsection 4.2.1, we get that

$$F_{D_h}(v_h, c_h, B_1) = 1, \quad Dev_{D_h}(v_h, c_h, B_1) = \vartheta_h, \quad \mathcal{H}^{n-1}(S_{v_h} \cap B_1) = \varepsilon_h \tag{4.17}$$

and

$$F_{D_h}(v_h, c_h, B_\tau) > C_1 \tau^n. \tag{4.18}$$

Let $\delta \in [0,1]$. For a consequence of Poincaré inequality in SBV (See [5], Propostion 7.5 and Remark 7.6), there exists $v \in W^{1,2}\left(B_{\frac{1}{2}}\right)$ such that, if m_h is a median of v_h in $B_{\frac{1}{2}}$, up to a subsequence

$$v_h - m_h \to v \quad \text{a.e. in } B_{\frac{1}{2}}.$$

There exists $h_0 \in \mathbb{N}$ such that $D_h \cap B_{\frac{1}{2}} = \emptyset$ for all $h \geq h_0$. In particular, since the boundary condition disappears, from (4.17) we obtain

$$F\left(v_h, c_h, B_{\frac{1}{2}}\right) \leq 1, \quad Dev\left(v_h, c_h, B_{\frac{1}{2}}\right) \to 0, \quad \mathcal{H}^{n-1}\left(S_{v_h} \cap B_{\frac{1}{2}}\right) \to 0.$$

By Proposition 4.9, v is a local minimizer of $F_{0,\delta}\left(\cdot, B_{\frac{1}{2}}\right)$ and for all $\rho < \frac{1}{2}$

$$\int_{B_\rho} |\nabla v|^2 dx = \lim_{h \to +\infty} F(v_h, c_h, B_\rho) = \lim_{h \to +\infty} F_{D_h}(v_h, c_h, B_\rho) \leq 1. \tag{4.19}$$

Using the Interior gradient bound (Theorem 4.1) and (4.19), the function v turns out to be locally Lipschitz on $B_{\frac{1}{2}}$ with the estimate

$$ess \sup_{B_{\frac{1}{4}}} |\nabla v|^2 \leq \frac{1}{\omega_n} \int_{B_{\frac{1}{2}}} |\nabla v|^2 dx \leq \frac{1}{\omega_n}.$$

But, since $\tau < \frac{1}{4}$

$$\lim_{h \to +\infty} F_{D_h}(v_h, c_h, B_\tau) = \int_{B_\tau} |\nabla v|^2 dx \leq 2^n \tau^n,$$

which is a contradiction to (4.18), since $C_1 > 2^n$.

Now, we study the case when $0 \leq \delta \leq \frac{1}{2}$, with this assumption, we have that $|D_h| \geq d_0 > 0$. By Lemma 4.7 and Proposition 4.9 there exists a local minimizer $v \in W^{1,2}(B_1)$ of $F_{0,\delta}(\cdot, B_1)$ such that, up to a subsequence, $v_h \to v$ a.e. in B_1. Furthermore, for every

$\rho \in (0,1)$

$$\int_{B_\rho} |\nabla v|^2 dx = \lim_{h\to+\infty} F_{D_h}(v_h, c_h, B_\rho) \leq 1. \tag{4.20}$$

Then using the gradient bound given by Theorem 4.2 and (4.20), we have that

$$ess\sup_{B_{\frac{1}{4}}} |\nabla v|^n \leq \frac{2^n}{\omega_n \rho^n} \int_{B_{\frac{1}{2}}} |\nabla v|^2 dx + 1 \leq C_0' \left(\frac{2^n}{\omega_n \rho^n} + 1\right),$$

Then, we get the contradiction by arguing as in the previous case. □

Now, we have all the tools to prove Theorem 4.6.

Proof of Theorem 4.6. We will follow, in our context, the prove given in [7], Theorem 3.4. Let $A \subset \Omega'$ be an open set, and let $u \in SBV^2(A)$ be a quasi-minimizer of $F(\cdot, A)$ with constants s and ω. For any ball $B_\rho \subset A$, we have

$$F(u, B_\rho(x)) \leq n\omega_n \rho^{n-1} + \omega \rho^{n-1+s}. \tag{4.21}$$

Indeed, let $\rho' < \rho$, comparing u with $u\mathbb{1}_{B_\rho(x)\setminus B_{\rho'}(x)}$, by the quasi-minimality we have

$$\begin{aligned}\int_{B_{\rho'}(x)} |\nabla u|^2 dx + \mathcal{H}^{n-1}\left(S_u \cap \overline{B}_{\rho'}\right) &\leq \mathcal{H}^{n-1}\left(S_u \cap \overline{B}_{\rho'}\right) + Dev(u, B_\rho(x)) \\ &\leq n\omega_n \rho^{n-1} + \omega \rho^{n-1+s}.\end{aligned}$$

Letting $\rho' \to \rho$, we conclude (4.21).

Following the notation of Decay lemma, let $\tau \in (0,1)$ be such that $C_1 \tau^n \leq \tau^{n+1-s}$ and let $\sigma \in (0,1)$ be such that $C_1 \sigma (n\omega_n + 1) \leq \varepsilon(\tau)$. Let us define

$$\rho_0 := \min\left\{\frac{1}{\omega}, \frac{\varepsilon(\tau)\,\tau^n \vartheta(\tau)}{\omega}, \frac{\varepsilon(\tau)\,\sigma^{n-1}\vartheta(\tau)}{\omega}, r(\sigma)^s, r(\tau)^s\right\}^{\frac{1}{s}}, \tag{4.22}$$

and

$$\vartheta_0 := \varepsilon(\tau).$$

Note that ϑ_0 and ρ_0 are constants depending on n, s and ω.

Let suppose that the assertion of this theorem is false. Define the set

$$I := \left\{ x \in A; \limsup_{\rho \to 0} \frac{1}{\omega_n \rho^n} \int_{B_\rho(x)} |u(y)|^{\frac{n}{n-1}} dy = +\infty \right\},$$

and assume that we have

$$\mathcal{H}^{n-1}\left(S_u \cap B_\rho(x)\right) < \vartheta_0 \rho^{n-1}$$

for some $x \in S_u \backslash I$, and for some $\rho < \rho_0$ with $B_\rho(x) \subset A$. We claim that

$$F\left(u, B_{\sigma\tau^h\rho}(x)\right) \leq \varepsilon(\tau)\, \tau^{hs} \left(\sigma\tau^h\rho\right)^{n-1} \quad \text{for all } h \in \mathbb{N}. \tag{4.23}$$

We now proceed by induction to prove (4.23).

Let $h = 0$, and assume, and assume first that $Dev\left(u, B_\rho(x)\right) \leq \vartheta(\tau)\, F\left(u, B_\rho(x)\right)$. Since we have $\rho < r(\tau)$ by our choice (4.22) of ρ_0, we deduce from Decay lemma (Lemma 4.11) and (4.21) that

$$\begin{aligned} F\left(u, B_{\sigma\rho}(x)\right) &\leq C_1 \sigma^n F\left(u, B_\rho(x)\right) \leq C_1 \sigma^n \left[n\omega_n \rho^{n-1} + \omega\rho^{n-1+s}\right] \\ &\leq C_1 \sigma \left(\sigma\rho\right)^{n-1} \left(n\omega_n + 1\right) \leq \varepsilon(\tau) \left(\sigma\rho\right)^{n-1}. \end{aligned}$$

On the contrary, if $Dev\left(u, B_\rho(x)\right) > \vartheta(\tau)\, F\left(u, B_\rho(x)\right)$, then using the definition of quiasi-minimizers

$$\begin{aligned} F\left(u, B_{\sigma\rho}(x)\right) &\leq F\left(u, B_\rho(x)\right) \leq \frac{1}{\vartheta(\tau)} Dev\left(u, B_\rho(x)\right) \\ &\leq \frac{\omega\rho^{n-1+s}}{\vartheta(\tau)} \leq \varepsilon(\tau) \left(\sigma\rho\right)^{n-1}, \end{aligned}$$

we conclude the validity of (4.23) for $h = 0$.

We assume that (4.23) holds for some $h \in \mathbb{N}$, and let us prove that it still holds for $h+1$. We repeat the process as $h = 0$, if $Dev\left(u, B_{\sigma\tau^h\rho}(x)\right) \leq \vartheta(\tau)\, F\left(u, B_{\sigma\tau^h\rho}(x)\right)$, since

$\sigma\tau^h\rho \leq r(\tau)$, applying the Decay lemma and from the induction hypothesis, we get

$$F\left(u, B_{\sigma\tau^{h+1}\rho}(x)\right) \leq C_1\tau^n F\left(u, B_{\sigma\tau^h\rho}(x)\right) \leq \varepsilon(\tau) C_1\tau^n\tau^{hs}\left(\sigma\tau^h\rho\right)^{n-1}$$
$$\leq \varepsilon(\tau)\tau^{(h+1)s}\left(\sigma\tau^{h+1}\rho\right)^{n-1}.$$

On the other hand, if $Dev\left(u, B_{\sigma\tau^h\rho}(x)\right) > \vartheta(\tau) F\left(u, B_{\sigma\tau^h\rho}(x)\right)$, by definition of quasi-minimizer, we infer

$$F\left(u, B_{\sigma\tau^{h+1}\rho}(x)\right) \leq F\left(u, B_{\sigma\tau^h\rho}(x)\right) \leq \frac{1}{\vartheta(\tau)} Dev\left(u, B_{\sigma\tau^h\rho}(x)\right)$$
$$\leq \frac{\omega\left(\sigma\tau^h\rho\right)^{n-1+s}}{\vartheta(\tau)} \leq \varepsilon(\tau)\tau^{(h+1)s}\left(\sigma\tau^{h+1}\rho\right)^{n-1},$$

thus, we have the proof of (4.23).

We have as a consequence of (4.23), for every $x \in S_u \backslash I$, that

$$\lim_{r\to 0}\frac{1}{r^{n-1}}\left(\int_{B_r(x)}|\nabla u|^2 dy + \mathcal{H}^{n-1}\left(S_u \cap B_r(x)\right)\right) = 0,$$

which is a contradiction to Theorem 7.8 in [5]. Then, we have that Theorem 4.6 holds for any $x \in S_u \backslash I$, and by density, for any $x \in \overline{S_u \backslash I}$.

It remains to prove that $\overline{S_u \backslash I} = \overline{S_u}$. Let $x \notin \overline{S_u \backslash I}$. By Lemma 3.75, $\mathcal{H}^{n-1}(I) = 0$ and then it is possible to find a neighborhood $U \subset A$ containing x, for which $\mathcal{H}^{n-1}(S_u \cap U) = 0$, and thus (by extension) $u \in W^{1,2}(U)$ (See (4.2) in [5]).

From the Poincaré inequality and (4.21), for any ball $B_r(x_0) \subset U$, we have

$$\int_{B_r(x_0)} \left|u(y) - \overline{u}_{x_0,r}\right|^2 dy \leq cr^2 \int_{B_r(x_0)} |\nabla u|^2 dy \leq c' r^{n+1}$$

where $\overline{u}_{x_0,r}$ is the average of u over $B_r(x_0)$. Thanks to the Campanato theorem, (see e.g. Theorem 7.51 in [5]), we infer that $u \in \mathcal{C}^{0,\frac{1}{2}}_{loc}(U)$ which shows that $U \subset A \backslash S_u$. Hence, $x \notin \overline{S_u}$, which completes the proof. $\square$

Remark 4.3. *Let $u \in SBV^2(A)$ be a quasi-minimizer of $F(\cdot, A)$ for $A \subset \Omega'$. By an application of a classical theorem of Geometric Measure Theory (See Theorem 2.53 and*

Section 2,9 in [5]), we have, for $\mathcal{H}^{n-1}-a.e.$ $x \in \left(\overline{\Omega} \cap A\right) \setminus S_u$, that

$$\lim_{\rho \to 0} \frac{\mathcal{H}^{n-1}\left(S_u \cap B_\rho(x)\right)}{\rho^{n-1}} = 0.$$

We deduce, from the Theorem 4.6, that there exists $\rho_0, \vartheta > 0$ so that $\mathcal{H}^{n-1}\left(S_u \cap B_rho\right) > \vartheta$ for all $\rho < \rho_0$. Combining these two consequences, we can deduce that

$$\mathcal{H}^{n-1}\left(\left(\overline{\Omega} \cap A\right) \cap \left(\overline{S_u} \setminus S_u\right)\right) = 0, \tag{4.24}$$

then, the set S_u is essentialy closed in $\left(\overline{\Omega} \cap A\right)$. This important remark was developed by De Giorgi in [16].

4.3 Existence of minimizer for the strong formulation of the Mumford-Shah functional

Let Ω be an open subset of $\mathbb{R}^n$, and $g \in L^\infty(\Omega)$. The Mumford Shah functional is

$$J(u, K) = \int_{\Omega \setminus K} |\nabla u|^2 + (u-g)^2 \, dx + \mathcal{H}^{n-1}(K \cap \Omega), \tag{4.25}$$

The aim is to minimize J in the following set,

$$\mathcal{A} = \left\{(u, K); K \subset \overline{\Omega} \text{ closed and } u \in W^{1,2}_{loc}(\Omega \setminus K)\right\}. \tag{4.26}$$

The problem of minimize J is that $K \mapsto \mathcal{H}^{n-1}(K)$ is not lower-semicontinuous with respect to the Hausdorff measure, for more references, see Section 6.1 in [5].

The idea of De Giorgi, Carreiro and Leacci to to approach to the solution of this problem is to solve the weak formulation; in the last two sections, we have proven that there exists minimizers, quasi-minimizers and we have a regularity over its jump set for the weak formulation of Mumford-Shah functional.

Now, we are going to see that the pair $\left(u, \overline{S_u}\right)$ given in the Theorem 4.6 is also a strong solution.

Theorem 4.12 ([5])**.** *If $u \in SBV_{loc}(\Omega)$ is a minimizer of $F(\cdot,\Omega)$ then the pair $\left(u,\overline{S_u}\right)$ is a minimizer of J, i.e. $J\left(u,\overline{S_u}\right) \leq J(v,K)$ for any closed set $K \subset \overline{\Omega}$ and any $v \in W^{1,2}_{loc}(\Omega \setminus K)$.*

Proof. First, we can note, as a consequence of Theorem 4.6, that if u is a minimizer of $F(\cdot,\Omega)$, then $F(u,\Omega) < +\infty$. It yields that $\nabla u \in L^2(\Omega)$, and we deduce that $u \in W^{1,2}_{loc}\left(\Omega \setminus \overline{S_u}\right)$. Let $v \in W^{1,2}_{loc}\left(\Omega \setminus \overline{S_u}\right)$ such that $J(v,K) < \infty$.
Without loss of generality, we suppose that v is bounded. By the Proposition 4.4 in [5], we have that $v \in SBV(\Omega)$ and $\mathcal{H}^{n-1}(S_v \setminus K) = 0$. Therefore, by the minimality of u and using the fact that S_u is essentially closed (See 4.24), we have that

$$J\left(u,\overline{S_u}\right) = F(u,\Omega) = \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u \cap \Omega) \leq \int_\Omega |\nabla v|^2 dx + \mathcal{H}^{n-1}(K \cap \Omega) \leq J(v,K).$$

We conlude from the expresion above the result of the theorem. □

Chapter 5

Annexes

5.1 Annex A

Definition 5.1 ([5], k-dimensional densities)**.** *Let μ be a positive Radon measure in a open set $\Omega \subset \mathbb{R}^n$ and $k \geq 0$. The upper and lower k-dimensional densities of μ at x are respectively defined by*

$$\Theta_k^*(\mu, x) := \limsup_{r \to 0} \frac{\mu(B_r(x))}{\omega_k r^k}, \quad \Theta_{*k}(\mu, x) := \liminf_{r \to 0} \frac{\mu(B_r(x))}{\omega_k r^k}.$$

We denote $\Theta(\mu, x)$ the common value of $\Theta_k^*(\mu, x)$ and $\Theta_{*k}(\mu, x)$.

Definition 5.2 ([18])**.** *Let $A \subset \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. We say that ν is an **exterior normal** of A at $x_0 \in \mathbb{R}^n$ if and only if $\nu \in \mathbb{S}^{n-1}$,*

$$\theta^n [\mathcal{L}^n \llcorner \{x; (x - x_0, \nu) > 0\} \cap A, x_0] = 0$$

and

$$\theta^n [\mathcal{L}^n \llcorner \{x; (x - x_0, \nu) < 0\} \setminus A, x_0] = 0.$$

We define $n(A, x_0) = \nu$ if ν is an exterior normal of A at x_0, $n(A, x_0) = 0$ if there exists no exterior normal of A at x_0.

Remark 5.1. *Let $A \subset \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. If ν_1 ad ν_2 are exterior normal of A at $x_0 \in \mathbb{R}^n$,* *then,* $\nu_1 = \nu_2$. *(See [18], 4.5.5)*

Theorem 5.1 ([18])**.** *If f is a real valued $\mathcal{L}^n-$measurable function such that*

$$f_-(x) = ap \liminf_{z \to x} f(z) \in \overline{\mathbb{R}} \ for \ x \in \mathbb{R}^n,$$

$$f_+(x) = ap \limsup_{z \to x} f(z) \in \overline{\mathbb{R}} \ for \ x \in \mathbb{R}^n,$$

where $ap \liminf$ *and* $ap \limsup$ *are defined in Definition 1.1. Then for* $\mathcal{H}^{n-1}$ *almost all* $x_0 \in S_f$*, there exists* $\nu \in \mathbb{S}^{n-1}$ *such that,*

$$n\left[\{x; f(x) \geq s\}, x_0\right] = \nu$$

for all $s \in \left[f_-\left(x_0\right), f_+\left(x_0\right)\right]$.

Proof. We define $Q_s = \{x; f_-(x) \geq s\}$. Let $\partial^* Q_s$ the reduced boundary of Q_s (See Definition 2.3) and $\partial_* Q_s$ the measure theoretic boundary (See Definition 2.5). By the Lemma 5.5 in [17], we have that $\partial^* Q_s \subset \partial_* Q_s$ and $\mathcal{H}^{n-1}\left(\partial_* Q_s \backslash \partial^* Q_s\right) = 0$. Let Σ be a dense countable subset of $\mathbb{R}$ (See [17], pg. 211 or [18] pg. 490).
Let $x_0 \in S_f \backslash \bigcup \{\partial_* Q_r \backslash \partial^* Q_r; r \in \Sigma\}$, we see

$$\mathcal{H}^{n-1}\left(\bigcup \{\partial_* Q_r \backslash \partial^* Q_r; r \in \Sigma\}\right) = 0.$$

Fix $r, t \in \Sigma$ such that $f_-(x_0) < r < t < f_+(x_0)$. By definition of the measure theoretic boundary, we can see that $x_0 \in \partial_* Q_r \cap \partial_* Q_t$.
Now, we are going to show that for Q_r and Q_t, its exteriors normal coincide. Let $\nu_1 = n\left(Q_r, x_0\right) \in \mathbb{S}^{n-1}$, $\nu_2 = n\left(Q_t, x_0\right) \in \mathbb{S}^{n-1}$.
As a consequence of the Theorem 3.61 (Federer's theorem) in [5], we can see that $\theta^n\left(\mathcal{L}^n \llcorner Q_t, x_0\right) = \frac{1}{2} = \theta^n\left(\mathcal{L}^n \llcorner Q_r, x_0\right)$ and it is easy to see that

$$Q_t \subset Q_r. \tag{5.1}$$

Indeed, as $r < t$, we have

$$Q_t = \{x; f_-(x) \geq t\} \subset \{x; f_-(x) \geq r\} = Q_r.$$

Hence, by the definition of f_-, we have

$$\theta^n \left(\mathcal{L}^n \llcorner Q_r \backslash Q_t, x_0\right) = 0. \tag{5.2}$$

We can see that $\nu_1 = n\,(Q_t, x_0)$, indeed, by (5.1) and the fact that $\nu_1 = n\,(Q_r, x_0)$, we have

$$\theta^n \left[\mathcal{L}^n \llcorner \{x; (x - x_0, \nu_1) > 0\} \cap Q_t, x_0\right] \leq \theta^n \left[\mathcal{L}^n \llcorner \{x; (x - x_0, \nu_1) > 0\} \cap Q_r, x_0\right] = 0,$$

and also, by (5.1) and (5.2), we can see that

$$\theta^n \left[\mathcal{L}^n \llcorner \{x; (x - x_0, \nu_1) < 0\} \backslash Q_t, x_0\right] \leq \theta^n \left[\mathcal{L}^n \llcorner \{x; (x - x_0, \nu_1) > 0\} \backslash Q_r, x_0\right] = 0.$$

Then, ν_1 is an exterior normal to Q_t. By the same argument, we can conclude that ν_2 is the exterior normal to Q_r and then $\nu_1 = \nu_2$.

For all numbers s such that $r < s < t$, we have that $Q_t \subset Q_s \subset Q_r$ and $\theta^n \left(\mathcal{L}^n \llcorner Q_s \backslash Q_t, x_0\right)$. By the argument above, we have that $n\,(Q_s, x_0) = \nu_1$. By the fact that Q_s differs to $\{f \geq s\}$, we can conclude the result of the theorem. $\square$

Bibliography

[1] L. Ambrosio, *A compactness theorem for a special class of functions of bounded variation*, Preprints di matematica, Scuola normale superiore, 1988.

[2] ———, *A compactness theorem for a new class of functions of bounded variation*, Bollettino della Unione Matematica Italiana **3** (1989), no. 4, 857–881.

[3] ———, *A new proof of the sbv compactness theorem*, Calc. Var (1995), no. 3, 127–137.

[4] L. Ambrosio, N. Fusco, and D. Pallara, *Partial regularity of free discontinuity sets, ii*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **Ser. 4, 24** (1997), no. 1, 39–62 (en). MR 1475772

[5] ———, *Functions of bounded variation and free discontinuity problems*, Oxford Science Publications, Clarendon Press, 2000.

[6] L. Ambrosio and D. Pallara, *Partial regularity of free discontinuity sets, i*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **Ser. 4, 24** (1997), no. 1, 1–38 (en). MR 1475771

[7] J.F Babadjian and A. Giacomini, *Existence of strong solutions for quasi-static evolution in brittle fracture*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze **13** (2014), 925–974.

[8] A. Bonnet, *Sur la regularite des bords des minima de la fonctionelle de mumford-shah.*, C. R. Acad. Sc;. Paris. Ser. I Math.. **321** (199), 1275–1219.

[9] ———, *Partial regularity of free discontinuity sets, i*, C. R. Acad. Sc;. Paris. Ser. I Math.. **311** (1995), 1121–1126.

[10] ———, *On the regularity of edges in image segmentation*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis **13** (1996), no. 4, 485–528.

[11] J.M. Bony, *Cours d'analyse: Ecole polytechnique, Département de mathématiques*, Ecole polytechnique, 1991.

[12] M. Carriero and A. Leaci, *Existence theorem for Dirichlet problem with free discontinuity set*, Nonlinear Anal. **15** (1990), 661–677.

[13] G. David, *C^1-arcs for minimizers of the mumford-shah functional*, SIAM Journal on Applied Mathematics **56** (1996), no. 3, 783–888.

[14] ———, *Singular sets of minimizers for the mumford-shah functional*, 1 ed., Progress in Mathematics, Birkhäuser Basel, 2005.

[15] ———, *Thecniques d'analyse*, Lecture notes, 2006.

[16] E. de Giorgi, M. Carriero, and A. Leaci, *Existence theorem for a minimum problem with free discontinuity set*, Archive for Rational Mechanics and Analysis **108** (1989), 195–218.

[17] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, Taylor & Francis, 1991.

[18] H. Federer, *Geometric measure theory*, Grundlehren der mathematischen Wissenschaften, Springer, 1969.

[19] I. Fonseca and N. Fusco, *Regularity results for anisotropic image segmentation models*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **Ser. 4, 24** (1997), no. 3, 463–499 (en). MR 1612389

[20] G. Lebeau, *Théorie des distributions et analyse de Fourier*, École Polytechnique. Département de Mathématiques, École polytechnique, Département de Mathématiques, 1998.

[21] P. Mattila, *Geometry of sets and measures in euclidean spaces: Fractals and rectifiability*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999.

[22] F. Morgan, *Geometric measure theory*, Academic Press, fifth edition ed., 2016.

[23] D. Mumford and J. Shah, *Optimal approximations by piecewise smooth functions and associated variational problems*, Communications on Pure and Applied Mathematics **42** (1989), no. 5, 577–685.

[24] S. Semmes, *Some elementary aspects of hausdorff measure and dimension*, 2010.

[25] L. M. Simon, *Lectures on geometric measure theory*, Canberra : Centre for Mathematical Analysis, Australian National University, 1984 (English), Geometric measure theory (ANB/PRECIS SIN 0735329).