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Bloch Spaces on the open unit ball $\mathbb{B}_n \subset \mathbb{C}^n$.

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"I dedicate this thesis to my mother and my friends who unremittingly supported me during my years of study. They made this work possible."

I use this opportunity to express my gratitude to everyone who supported me throughout the course of my university career.
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Contents

Abstract

The Bloch space has a long history behind it and is a central object of study in several outstanding problems remain unresolved. It was introduced by the French mathematician Andre Bloch at the beginning of the 20-th century and, since then, has undergone a remarkable metamorphosis.
The basic idea was considered the class of holomorphic functions f on the unit disk of $\mathbb{C}$, denoted by $\mathbb{D}$, with normalisation $f'(0) = 1$, such that

$$(1 - |z|^2)f'(z) \tag{0.0.1}$$

is bounded. Equivalently, the Bloch space, on $\mathbb{D}$, consists on holomorphic functions whose derivative, on a fixed point $z \in \mathbb{D}$, grows no faster than a constant times the reciprocal of the distance from z to the boundary of $\mathbb{D}$.
One of the outstanding problems concerning the Bloch space, for instance, is to determine exactly what functions of the Bloch space have the property that the Taylor coefficients approach zero.
Many mathematicians payed attention to this functional space because of its intrinsic interest and because is the meeting place of several areas of Mathematical Analysis. In particular, the theory of Bloch space lies at the interface of Complex Analysis and Operator Theory and so it helps us to gain a deeper understanding of both of them. For instance, we show that the Bloch space is the largest space with certain natural properties and, thus, providing another justification for studying it.
However, we mention that some of such authors are, for example, L. Ahlfors, J.M. Anderson, J. Clunie, Ch. Pommerenke, P.L. Duren, B.W. Romberg and A.L. Shields.
During the period from 1925 through 1968 Bloch's result motivated works of various nature. In fact, in this period, one group of these mathematicians considered the generalisations of Bloch's result to balls in $\mathbb{R}^n$ to $\mathbb{C}^n$.
In the same period, another group, of the above mentioned mathematicians, were concentrated on the function theoretic implications for the case of the unit disc of $\mathbb{C}$. Furthermore, this holomorphic space plays an important role in classical geometric function theory mainly because of its Mobius invariance.
In fact, equipping the Bloch space, denoted by $\mathcal{B}$, with the following norm

$$|f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)|,$$

it turns out that the Bloch space is the maximal Mobius invariant Banach space of holomorphic functions. Actually, in this thesis we prove that this result holds in the several-variable case too.
In the period from 1969 to the present the Banach space $\mathcal{B}$ has been studied and become fairly active. Some progress has been made in establishing the Banach structure and, hence, the functional analytic properties of $\mathcal{B}$. In fact, the Banach space point of view has allowed a somewhat broader viewpoint and, consequently, has given rise to a new set of questions concerning the Bloch space.
During the period from 1970 through the middle of the 90's many mathematicians, such as for example J. Arazy, S. Fisher, J. M. Anderson and S. Axler, studied the theory of the Bloch spaces extending many results to the several-variable case. Namely, for functions defined on the open unit ball of $\mathbb{C}^n$ with values in $\mathbb{C}$. Hence, in higher dimensions, $\mathcal{B}$ is defined as the space of holomorphic functions $f : \mathbb{B}_n \longrightarrow \mathbb{C}$ such that

$$\sup_{z\in\mathbb{B}_n} |\nabla(f \circ \varphi_z)(0)| < \infty,$$

where φ_z is the involutive automorphism of $\mathbb{B}_n$. That is

$$\varphi_z(w) = \frac{z - \frac{<w,z>}{|z|^2} z - \sqrt{1-|z|^2}\left(z - \frac{<w,z>}{|z|^2} z\right)}{1- <w, z>}, \ z, w \in \mathbb{B}_n.$$

$<,>$ denotes the hermitian product of $\mathbb{C}^n$ and the complex gradient ∇ is defined as

$$\nabla f(z) := \left(\frac{\partial f}{\partial z_1}(z), ..., \frac{\partial f}{\partial z_n}(z)\right).$$

At the beginning of 90's, the concept of Bloch space was extended for functions defined on the open unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ with values in $\mathbb{C}^n$.
Finally, R.M. Timoney, on his thesis "Bloch functions in several complex variables", 1980, made a study of the Bloch space on bounded symmetric domains in $\mathbb{C}^n$. This work is quite expensive and deep, it would require material from areas which are not considered in the disc case.

The purpose of this manuscript is to give a wide collection of many results concerning the Bloch space for functions defined on the open unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ having values on $\mathbb{C}$. Namely, we shall show that many important properties for Bloch functions in one complex variable have analogs for functions in several complex variables. More is true, we will supply proofs of the major results and outline proofs of other ideas when they are not central to our interests.
In this thesis we will give an overview of the theory of the Bloch spaces and some applications.
The current thesis is organised as follows. In chapter 1, we first collect all needed ingredients from Functional and Complex Analysis in several variables. We start with some notation used through this thesis and the sets that will be used. We will introduce holomorphic functions of several complex variables and prove the $n-$dimensional analogues of several theorems well-known from the one-dimensional case.
We set the stage of many essential concepts for the rest of this thesis such as the Bergman metric, the Automorphism Group, Differential operators and the invariant Laplacian.
We present a concise review and introduction to the Lebesgue integral in the open unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ and, hence, the L^p spaces. These include the change of variables formula, integral operators and the basic integral estimate of the kernel functions.
This chapter ends recalling the notion of Subharmonic function and a technique used repeatedly in this thesis: Complex Interpolation of Banach spaces.

In chapter 2, we introduce the weighted Bergman spaces and concentrate on the general aspects of these spaces. Most results are concerned with the Banach (or metric) space structure of Bergman spaces. Our approach here is functional analyitc.
The theory of reproducing kernel Hilbert spaces interacts with many subjects in mathematics. In this chapter we provide an example of Hilbert space with reproducing kernel: the Bergman space of square Lebesgue integrable functions, that is still an important field in mathematical research, and some fruitful applications.
Some further topics related and studied in chapter 2 are, for instance, the characterisation in terms of derivatives and the atomic decomposition of the Bergman space. This consists on the decomposition of every function of the Bergman spaces into a very particular and nice family of functions.
We study the Bergman projection that provides the connection between the L^p spaces and the Bergman spaces. In this chapter we show that the duality of Bergman spaces is also similar to the L^p spaces. Moreover, the complex interpolation turns out to be similar to the L^p space interpolation illustrated in chapter 1.

In chapter 3 we will concentrate on the main topic of this thesis: The Bloch space on the open unit ball $\mathbb{B}_n$. Equipping $\mathcal{B}$ by the following seminorm

$$||f||_{\mathcal{B}} = \sup_{z\in\mathbb{B}_n} |\nabla(f \circ \varphi_z)(0)|,$$

we prove various characterisations of it, some functional and topological properties. One of these, that will be proved, is the lack the of separability.
We will focus in the following point of view: the Bloch space, in some sense, can be thought of as limit case of the Bergman spaces. In fact, we also show that the image, under the Bergman projection, of the space of bounded functions is the Bloch space.
In this chapter we study the Bloch space as a companion of Bergman spaces: we prove that the Bloch space can be identified with the dual space of the Bergman space given by the holomorphic functions Lebesgue-integrable.
However, the Bloch space is also interesting in its own right. In fact, the Bloch space has been studied much earlier, in geometric function theory, than the Bergman spaces.
The intimated relation to the Bergman metric is also proved. That is, the Bloch space consists on the set of holomorphic functions that are Lipschitz from the open unit ball $\mathbb{B}_n$ with the Bergman metric to $\mathbb{C}$ with the Euclidean metric.
In chapter 3 we illustrate and give a detailed description of a closed subspace of $\mathcal{B}$: the little Bloch space $\mathcal{B}_0$ defined as

$$\mathcal{B}_0 := \left\{ f \in \mathcal{B} \;\mid\; \lim_{|z| \to 1^-} |\nabla(f \circ \varphi_z)(0)| = 0 \right\}.$$

Hence, we investigate and gather some properties that are a direct consequence of such inclusion. For example, characterisations, convexity and density of polynomials. Special attention is given to the following aspects: construction of nontrivial functions, for both the Bloch and the little Bloch space, and connection with other functional spaces.
We also characterise the pointwise multipliers of the Bloch space and the little Bloch space.
Further results studied in this chapter, for both the Bloch and the little Bloch space, are the atomic decomposition and complex interpolation, where the relation of the Bloch space with the Bergman space is heavily used.

Chapter 1

Functional and Complex Analysis in several variables on $\mathbb{B}_n$

In this chapter we provide all the necessary and fundamental tools to study the holomorphic spaces of our interest.
We start discussing the n-dimensional complex number space $\mathbb{C}^n$, the open unit ball of $\mathbb{C}^n$ and the polydisc. After that, we introduce the notation used in this thesis. Next, the notion of holomorphic function in several variables is given in three different ways and the equivalence among them is also proved. Consequently, we study the main properties and some results of holomorphic functions, most of them are a natural extension from Complex Analysis in one variable. Then, we study the automorphism group on the open unit ball $\mathbb{B}_n$. In particular, we show that every automorphism is described in terms of a unitary transformation and an involution that interchanges a, where $a \in \mathbb{B}_n$, with the origin.
We will argue about an invariant operator under automorphisms of $\mathbb{B}_n$: the invariant Laplacian.
Regarding the Lebesgue integration, we talk about the most relevant peculiarities that are indispensable in our fieldwork. The description of these tools continues setting and exploring many techniques, such as change of variables formula, the fractional differential and integral operators.
After that, we provide some others instruments like the Bergman metric and Subharmonic functions.
Finally, in the last section, we introduce the notion of complex interpolation and present a version of the Marcinkiewicz interpolation theorem.
For this chapter the references are: Spaces of Holomorphic Functions in the Unit Ball, Zhu Kehe, Springer, 2010.
Holomorphic Functions and Integral Representations in Several Complex Variables, R. Michael Runge, Springer, 1986.
Function Theory of Several Complex Variables, Steven G. Krantz, Pacific Grove, 1992.

1.1 Preliminaries

In this section, we collect some basic notations, facts and terminology, which will be used throughout this thesis.
$\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers; $\mathbb{Z}$ and $\mathbb{Z}^+$ denote respectively the integers and nonnegative integers. Finally, $\mathbb{N}$ denotes the set of natural numbers.

1.1.1 The space $\mathbb{C}^n$

Fix $n \in \mathbb{N}$, we denote by $\mathbb{C}^n$ the n-dimensional complex number space. It is defined as the cartesian product of n copies of $\mathbb{C}$, that is

$$\mathbb{C}^n := \{z = (z_1, ..., z_n), z_j \in \mathbb{C} \text{ for } 1 \leq j \leq n \ \} \, .$$

For the points of $\mathbb{C}^n$, we shall use the notation $z = (z_1, ..., z_n)$. Arguing as in the case of $\mathbb{R}^n$, the standard basis of $\mathbb{C}^n$ is

$$e_1 = (1, 0, ..., 0), \quad e_2 = (0, 1, ..., 0), \ ... \ , e_n = (0, 0, ...0, 1),$$

while the zero vector is given by $0 = (0, ..., 0)$.
We write $z_j = x_j + iy_j$ for the decomposition of the coordinates z_j into real and imaginary parts. Hence, the bijection

$$(z_1, ..., z_n) \in \mathbb{C}^n \longrightarrow (x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n} \tag{1.1.1}$$

establishes an $\mathbb{R}-$linear isomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. This isomorphism is used to introduce a norm and thus a topology on $\mathbb{C}^n$. Since all norms on $\mathbb{R}^{2n}$ are equivalent, all norms on $\mathbb{C}^n$ define the same topology.
The vector space structure of $\mathbb{C}^n$, over the field $\mathbb{C}$, is described as follows: addition and multiplication by a complex scalar λ are defined coordinate-by-coordinate : $z + w = (z_1 + w_1, ..., z_n + w_n)$ and $\lambda z = (\lambda z_1, ..., \lambda z_n)$.
The standard Hermitian inner product on $\mathbb{C}^n$ is defined by

$$< z, w >:= \sum_{j=1}^{n} z_j \overline{w}_j, \quad z, w \in \mathbb{C}^n,$$

where $\overline{w}_j$ indicates the complex conjugate of w_j. The norm associated is

$$|z| = \sqrt{< z, z >} = \sqrt{|z_1|^2 + ... + |z_n|^2},$$

which induces the Euclidean metric in the usual way: let $z, w \in \mathbb{C}^n$, we have that $\text{dist}(z, w) = |z - w|$.

1.1.2 The open unit ball $\mathbb{B}_n$

Fix $a \in \mathbb{C}^n$ and $r > 0$, the open ball centered at the point a of radius r is denoted by $B(a, r)$, namely

$$B(a, r) = \{z \in \mathbb{C}^n \mid |z - a| < r\}.$$

The topological boundary of $B(a, r)$, denoted by $\partial B(a, r)$, is $\partial B(a, r) = \{z \in \mathbb{C}^n \mid |z - a| = r\}$.
In particular, in these notes, the open unit ball $\mathbb{B}_n$ is

$$\mathbb{B}_n = \{z \in \mathbb{C}^n \mid \ |z| < 1\} \quad .$$

Through this thesis, we denote the unit sphere in $\mathbb{C}^n$ with $\mathbb{S}_n$, which is the boundary of the open unit ball, that is $\mathbb{S}_n = \{z \in \mathbb{C}^n \ \mid \ |z| = 1\}$. As a consequence, we easily can see that $\overline{\mathbb{B}}_n = \mathbb{B}_n \cup \mathbb{S}_n$.
For $n = 1$, in the remainder of this thesis, we will denote by

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\},$$

and by

$$\mathbb{S} := \{z \in \mathbb{C} \mid |z| = 1\}.$$

Finally, every set $\Omega \subseteq \mathbb{C}^n$ which is connected and open is said to be a domain.

1.1.3 Differential operators on $\mathbb{C}^n$

Now we introduce the partial differential operators on $\mathbb{C}^n$ given by

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \quad j = 1, ..., n. \tag{1.1.2}$$

The operators introduced in (1.1.2) are also known as Wirtinger derivatives. Moreover, we remark that these operators are going to be used in the Definition of holomorphic function in several variables. These operators satisfy the following

$$\frac{\partial z_k}{\partial z_j} = \frac{\partial \overline{z}_k}{\partial \overline{z}_j} = \delta_{kj}, \;\; k, j = 1, ..., n$$

$$\frac{\partial \overline{z}_k}{\partial z_j} = \frac{\partial z_k}{\partial \overline{z}_j} = 0, \quad k, j = 1, ..., n.$$

Likewise, we have the differentials of the coordinate functions, that is

$$dz_j = dx_j + idy_j, \quad j = 1, ..., n,$$

$$d\overline{z}_j = dx_j - idy_j, \quad j = 1, ..., n.$$

Since $\mathbb{C}^n$ is isomorphic to $\mathbb{R}^{2n}$, we can impose on $\mathbb{C}^n$ in a natural way any of the structures of $\mathbb{R}^{2n}$; for instance, the Lebesgue measure on $\mathbb{R}^{2n}$ becomes a measure on $\mathbb{C}^n$, which will be denoted by dV. The Lebesgue measure can be explicitely written as $dV(z) = \left(\frac{1}{2i}\right)^n d\overline{z}_1 dz_1 ... d\overline{z}_n dz_n$ or, equivalently, using the above differentials

$$dV(z) = dx_1 dy_1 \dots dx_n dy_n. \tag{1.1.3}$$

Actually, (1.1.3) is the form in which we use the Lebesgue measure.

1.1.4 Multi-index notation

Any theory of functions of several variables requires multi-index notation. A multi-index α is an element of $(\mathbb{Z}^+)^n$. The multi-index notation will be used to simplify formulas involving power series, polynomials and partial derivatives in several variables. Indeed, if $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}^+)^n$, $w = (w_1, \dots, w_n) \in \mathbb{C}^n$, then

$$w^\alpha = w_1^{\alpha_1} \;\cdots\; w_n^{\alpha_n},$$
$$\frac{\partial^\alpha}{\partial w} = \frac{\partial^{\alpha_1}}{\partial w_1} \;\cdots\; \frac{\partial^{\alpha_n}}{\partial w_n},$$
$$\frac{\partial^\alpha}{\partial \overline{w}} = \frac{\partial^{\alpha_1}}{\partial \overline{w}_1} \;\cdots\; \frac{\partial^{\alpha_n}}{\partial \overline{w}_n}.$$

Hence, in multi-index notation, a multi-variable power series can be written in the form

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha,$$

which is an abbreviation for $\displaystyle\sum_{\alpha_1=0}^{\infty} \cdots \sum_{\alpha_n=0}^{\infty} c_{\alpha_1,\dots,\alpha_n} z_1^{\alpha_1} \dots z_n^{\alpha_n}$, where $c_{\alpha_1,\dots,\alpha_n} \in \mathbb{C}$.
For the sake of simplicity, for $\alpha \in (\mathbb{Z}^+)^n$, we will use the following common notation

$$\alpha! = \alpha_1! \cdots \alpha_n! \;\; \textit{and} \;\; |\alpha| = \sum_{j=1}^{n} \alpha_j.$$

Furthermore, if $\alpha, \beta \in (\mathbb{Z}^+)^n$, $\alpha \leq \beta$ means that $\alpha_j \leq \beta_j$ for all j.
Finally, the multi-nomial formula is written by

$$(z_1 + ... + z_n)^N = \sum_{|\alpha|=N} \frac{N!}{\alpha!} z^\alpha.$$

1.1.5 C^k functions

Given $k \in \mathbb{Z}^+$ and an open set $\Omega \subseteq \mathbb{C}^n$, let $f : \Omega \longrightarrow \mathbb{C}$, it is easy to see that f can be considered as $f(z) = u(x_1, y_1, ..., x_n, y_n) + iv(x_1, y_1, ..., x_n, y_n)$, where $u, v : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$. So, when we write $f \in C^k(\Omega)$, we mean that u and v are k times continuously differentiable on Ω.
We gather some properties of the Wirtinger derivatives in the next proposition.

Proposition 1.1.1. *Let $\Omega \subseteq \mathbb{C}^n$ a domain, assume that $f, g \in C^1(\Omega)$. Then the following equalities hold*

$$\frac{\partial}{\partial z_i}(\alpha f + \beta g) = \alpha \frac{\partial f}{\partial z_i} + \beta \frac{\partial g}{\partial z_i}, \quad \frac{\partial}{\partial \overline{z}_i}(\alpha f + \beta g) = \alpha \frac{\partial f}{\partial \overline{z}_i} + \beta \frac{\partial g}{\partial \overline{z}_i}, \ \alpha, \beta \in \mathbb{C}, (\textit{Linearity}).$$

$$\frac{\partial}{\partial z_i}(f \cdot g) = \frac{\partial f}{\partial z_i} g + \frac{\partial g}{\partial z_i} f, \quad \frac{\partial}{\partial \overline{z}_i}(f \cdot g) = \frac{\partial f}{\partial \overline{z}_i} g + \frac{\partial g}{\partial \overline{z}_i} f, \quad (\textit{Product Rule}).$$

$$\overline{\frac{\partial f}{\partial z_i}} = \frac{\partial \overline{f}}{\partial \overline{z}_i}, \quad \overline{\frac{\partial f}{\partial \overline{z}_i}} = \frac{\partial \overline{f}}{\partial z_i}, \quad (\textit{Conjugation}).$$

1.1.6 Polydiscs and the distinguished boundary

To continue our short introduction, we mention the notion of polydisc and distinguished boundary. In spite of the fact that the principal topics of this thesis are holomorphic spaces on the open unit ball, the easiest approach to the most fundamental facts about holomorphic functions, in several complex variables, is based on polydiscs rather than balls. In fact, for instance, the polydisc is going to be used for what concerns the extension, to several variables, of the definition of holomorphic function. Moreover, we will use both the polydisc and the distinguished boundary for the generalisation of the Cauchy integral formula.
Let's start with the definition of polydisc.

Definition 1.1.2 (Polydiscs)**.** Fix $a = (a_1, ..., a_n) \in \mathbb{C}^n$ and $r = (r_1, ..., r_n)$ an n-tuple of real positive numbers, the open polydisc is defined as

$$P(a, r) := \{ z = (z_1, ..., z_n) \in \mathbb{C}^n \mid |z_i - a_i| < r_i, \ i = 1, ..., n \ \} = D_1(a_1, r_1) \times ... \times D_n(a_n, r_n), \quad (1.1.4)$$

where $D_j(a_j, r_j) = \{z_j \in \mathbb{C} \mid |z_j - a_j| < r_j\}$ for all $j = 1, ..., n$. The $n-$tuple $r = (r_1, ..., r_n)$ is called multiradius.

In other words, polydiscs in $\mathbb{C}^n$ are basically cartesian products of n discs in $\mathbb{C}$. More is true, the open polydiscs constitute a basis for the collection of open sets in the Cartesian product topology on $\mathbb{C}^n$ and, when $n = 1$, the polydisc and the ball coincide.
The circle, on $\mathbb{C}$, centered at the point $a_k \in \mathbb{C}$ with radius $r_k > 0$ is denoted by $C(a_k, r_k)$, that is

$$C(a_k, r_k) := \{ z \in \mathbb{C} \mid |z - a_k| = r_k \}.$$

The boundary of the polydisc is given by the disjoint union

$$\partial P(a, r) = \bigcup_{i=1}^{n} \left\{ C(a_i, r_i) \times \bigotimes_{j=1, j \neq i}^{n} \overline{D_j(a_j, r_j)} \right\},$$

where $\bigotimes_{j=1}^n \overline{D_j(a_j, r_j)} = \overline{D_1(a_1, r_1)} \times ... \times \overline{D_n(a_n, r_n)}$.
We introduce the distinguished boundary,

$$b_0 P(a, r) := C(a_1, r_1) \times C(a_2, r_2) \times ... \times C(a_n, r_n).$$

In many situations, $b_0 P(a, r)$ plays the same role as the unit circle in one complex variable. Hence, according to the previous remark about the boundary of the polydisc, we easily find that

$$b_0 P(a, r) \subset \partial P(a, r).$$

This means that $b_0 P(a, r)$ is strictly smaller than the topological boundary of the polydisc, when $n > 1$. In fact, we can notice that $b_0 P$ is of real dimension n, while the boundary of the polydisc has dimension $(2n - 1)$.

1.1.7 Power series

In order to introduce holomorphic functions of several variables, we must first discuss basic facts about multiple series. That is, formal expressions

$$\sum_{\alpha \in \mathbb{N}^n} b_\alpha, \quad b_\alpha = b_{\alpha_1,\ldots,\alpha_n} \in \mathbb{C}.$$

Of course, we start defining what we mean by the sum of a multiple series. First of all, if $n > 1$, the index set $\mathbb{N}^n$ does not carry any natural ordering, so that there is no canonical way to consider $\sum_{\alpha\in\mathbb{N}^n} b_\alpha$ as a sequence of finite partial sums as in the case $n = 1$. The ambiguity is avoided if one considers convergent series, defined as follows.

Definition 1.1.3. The multiple series $\sum_{\alpha\in\mathbb{N}^n} b_\alpha$ is called convergent if

$$\sup \left\{ \left| \sum_{\alpha\in\Lambda} b_\alpha \right|, \ \Lambda \text{ finite} \right\} < \infty.$$

Definition 1.1.4. Let $(c_\alpha)_{\alpha\in\mathbb{N}^n} \in \mathbb{C}$ and fix $a \in \mathbb{C}^n$, a power series in n complex variables $z_1, \ldots, z_n$ centered at the point a is a multiple series of the form

$$\sum_{\alpha\in\mathbb{N}^n} c_\alpha (z-a)^\alpha. \tag{1.1.5}$$

Definition 1.1.5. Fix $z^0 \in \mathbb{C}^n$, the above power series is said to be convergent at the point z^0 if the series

$$\sum_{\alpha\in\mathbb{N}^n} c_\alpha (z^0-a)^\alpha$$

converges.

Similarly, we easily deduce the definition of absolute and uniform convergence of a power series in n complex variables.

Definition 1.1.6. The domain of convergence of the power series is the interior of the set of points $z \in \mathbb{C}^n$ for which the series in formula (1.1.5) converges.

1.2 Holomorphic Functions

The objective of this part is to give a formal definition, study the behaviour and list the main properties of holomorphic functions in several variables. Some of the properties of holomorphic functions, like power series expansion, extend from one to several variables. However, they differ in many important aspects. For example, in one variable the zero set of a holomorphic function is a discrete set. The zero set of a holomorphic function in $\mathbb{C}^n$, $n \geq 2$, is never isolated: it has $2n-2$ real dimension. Another main difference is that there is no analog to the Riemann map theorem in higher dimensional spaces. Therefore, it is not correct to consider the theory of several complex variables as a straightforward generalisation of that of one complex variable.

1.2.1 Definition of Holomorphic Function

There are a number of possible ways to define what means for a complex valued function on an open set, in $\mathbb{C}^n$, to be holomorphic. We begin with the notion of separately holomorphic function.

Definition 1.2.1 (Separately Holomorphic)**.** Let $\Omega \subseteq \mathbb{C}^n$ be an open set, a function $f : \Omega \longrightarrow \mathbb{C}$ is said to be holomorphic, in Ω, in each variable separately, if for every $j = 1, ..., n$ and every fixed $z_1, ..., z_{j-1}, z_{j+1}, ..., z_n$ the map

$$\xi \longrightarrow f(z_1, ..., z_{j-1}, \xi, z_{j+1}, ..., z_n)$$

is holomorphic, in the classical one-variable sense, on the set

$$\{\xi \in \mathbb{C} \mid (z_1, ..., z_{j-1}, \xi, z_{j+1}, ..., z_n) \in \Omega\}.$$

We propose two different plausible definitions of holomorphic functions, which are equivalent to each other. The first one concerns the well known Cauchy-Riemann equations.

Definition 1.2.2 (Holomorphic Function)**.** Let $\Omega \subseteq \mathbb{C}^n$ be an open set, a function $f : \Omega \longrightarrow \mathbb{C}$ is called holomorphic on Ω if $f \in C^1(\Omega)$, namely continuously differentiable, and satisfies the Cauchy-Riemann equations in each variable separately, that is

$$\frac{\partial f}{\partial \bar{z}_j}(z) = 0, \text{ for } 1 \leq j \leq n \text{ and } z \in \Omega. \tag{1.2.1}$$

Let f be a holomorphic function, according to Definition 1.2.2, and suppose to fix $(n-1)$ variables $z_1, .., z_{j-1}, z_{j+1}, ..., z_n$. We notice that the map

$$\xi \longrightarrow f(z_1, .., z_{j-1}, \xi, z_{j+1}, ..., z_n)$$

is continuous differentiable and satisfies the Cauchy Riemann equations. Hence, we deduce that the map f is holomorphic in each variable separately. In other words, we've proved the following result.

Proposition 1.2.3. *A holomorphic function, according to Definition* 1.2.2*, is separately holomorphic.*

Remark 1.2.4. Since f can be considered as $f(z) = u(x_1, y_1, ..., x_n, y_n) + iv(x_1, y_1, ..., x_n, y_n)$, where $u, v : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$, using the operators introduced in (1.1.2), the Cauchy-Riemann equations can be equivalently written in the following form

$$\begin{cases} \dfrac{\partial u}{\partial x_j} = \dfrac{\partial v}{\partial y_j}, & \forall\, j = 1, ..., n, \\ \dfrac{\partial u}{\partial y_j} = -\dfrac{\partial v}{\partial x_j}, & \forall\, j = 1, ..., n. \end{cases} \tag{1.2.2}$$

The extension of the concept of holomorphic function, from one to several variables, is perhaps most naturally achieved using the basic property : the expansion in power series. We provide another definition of holomorphic function in several variables as follows.

Definition 1.2.5 (Holomorphic Function)**.** Let $\Omega \subseteq \mathbb{C}^n$ be an open set, a function $f : \Omega \longrightarrow \mathbb{C}$ is holomorphic on Ω if for each $z^0 \in \Omega$ there exists $(c_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{C}$, $r = r(z^0) > 0$ and a polydisc $P(z^0, r)$, where $\overline{P(z^0, r)} \subset \Omega$, such that f can be written as an absolutely convergent power series on $P(z^0, r)$. That is

$$f(z) = \sum_\alpha c_\alpha (z - z^0)^\alpha \ , \ \forall\, z \in P(z^0, r), \tag{1.2.3}$$

and the convergence is uniform on compact subsets of $P(z^0, r)$. In other words, every holomorphic function is locally the sum of a convergent power series.

In the next remark, we talk about homogenous expansion. For this aim, we recall the following two definitions.

Definition 1.2.6. Let $d \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}^n$, a polynomial of degree d, in $\mathbb{C}^n$, is an expression of the form

$$P(z) = \sum_{|\alpha| \leq d} c_\alpha z^\alpha$$

where $c_\alpha \in \mathbb{C}$.

Definition 1.2.7. A polynomial P is said to be homogenous of degree d, where $d \in \mathbb{N}_0$, if

$$P(\lambda z) = \lambda^d P(z), \quad \forall\, \lambda \in \mathbb{C}, z \in \mathbb{C}^n.$$

Remark 1.2.8. Assume $z^0 = 0$ in the expansion in equation (1.2.3), putting

$$f_k(z) := \sum_{|\alpha| \leq k} c_\alpha z^\alpha, \quad k \geq 0,$$

hence, we obtain

$$f(z) = \sum_{k=0}^{+\infty} f_k(z), \ \forall\ z \in P(0, r),$$

which is called homogenous expansion of f. There are several reasons to consider this type of expansion. For example, an advantage is that it is invariant under linear changes of variables: if L is a linear transformation on $\mathbb{C}^n$, then the composition $f_k \circ L$ is homogenous of degree k. Hence, for a suitable polydisc $P(0, r)$ such that $\overline{P(0, r)} \subset \Omega$, $(f \circ L)(z)$ admits the following locally expansion

$$(f \circ L)(z) = \sum_{k=0}^{+\infty} (f_k \circ L)(z), \ \forall z \in P(0, r),$$

where the convergence is uniform on compact subsets of $P(0, r)$.

Let $\Omega \subseteq \mathbb{C}^n$ be an open set and a holomorphic function $f : \Omega \longrightarrow \mathbb{C}$, according to Definition 1.2.5. Suppose that the coordinates $z_1, ..., z_{j-1}, z_{j+1}, ..., z_n$ are fixed. Let $(\tilde{c}_j)_{j \in \mathbb{N}_0} := (c_{1,...,j-1,j,j+1,...,n})_{j \in \mathbb{N}_0}$ be; that is, we fix $(1, ..., j-1, j+1, ..., n) \in \mathbb{N}^{n-1}$. Then we obtain that the power series in (1.2.3) can be arranged as a convergent power series in $z_j - z_j^0$. This means that the function is holomorphic in each variable separately on Ω; thus the ordinary complex derivative with respect to one of the variables z_j is well-defined. That is,

Proposition 1.2.9. *A holomorphic function of several variables, according to Definition* 1.2.5*, is holomorphic in each variable separately.*

Actually, considering Proposition 1.2.3 and 1.2.9, we deduce that

Proposition 1.2.10. *A holomorphic function, according to both Definitions* 1.2.2 *and* 1.2.5*, is holomorphic in each variable separately.*

1.2.2 The Cauchy integral formula for polydiscs and the Cauchy kernel

For functions that are holomorphic in each variable separately, there is a Cauchy integral representation formula which extends the well-known one variable formula. This representation formula is most easily established on polydiscs.

Theorem 1.2.11 (Cauchy Formula for Polydiscs)**.** *Let $\Omega \subseteq \mathbb{C}^n$ an open set be, let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function in each variable separately on Ω and a polydisc $P(a, r)$, with multiradius $r = (r_1, ..., r_n)$, such that $\overline{P(a, r)} \subset \Omega$. Then, it holds that*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{C(a_n, r_n)} ... \int_{C(a_1, r_1)} \frac{f(\xi_1, ..., \xi_n)}{(\xi_1 - z_1)...(\xi_n - z_n)} d\xi_1 ... d\xi_n \ , \ \forall\ z \in P(a, r), \tag{1.2.4}$$

or equivalently,

$$f(z) = \frac{1}{(2\pi i)^n} \int_{b_0 P(a, r)} \frac{f(\xi_1, ..., \xi_n)}{(\xi_1 - z_1)...(\xi_n - z_n)} d\xi_1 ... d\xi_n \quad \forall z \in P(a, r).$$

Proof. We prove the theorem by induction over the number of variables n. For $n = 1$, this is Cauchy's integral formula, for the disc, for a holomorphic function of one complex variable. Let's suppose $n > 1$, and that the theorem has been proved for $(n-1)$ variables. Fix $z \in P(a, r)$, we apply the inductive hyphotesis with respect to $(z_2, ..., z_n)$ obtaining

$$f(z) = \frac{1}{(2\pi i)^{n-1}} \int_{C(a_n, r_n)} ... \int_{C(a_2, r_2)} \frac{f(z_1, ..., \xi_n)}{(\xi_2 - z_2)...(\xi_n - z_n)} d\xi_2 ... d\xi_n. \tag{1.2.5}$$

Fixing $\xi_2, ..., \xi_n$, the case $n = 1$ gives explicitly

$$f(z_1, \xi_2, ..., \xi_n) = \frac{1}{(2\pi i)} \int_{C(a_1,r_1)} \frac{f(\xi_1, ..., \xi_n)}{(\xi_1 - z_1)} d\xi_1, \tag{1.2.6}$$

we substitute (1.2.6) into (1.2.5) and, since f is measurable, we deduce that the integral makes perfectly sense as an iterated integral. Then, we immediately obtain the desired result. □

Remark 1.2.12. The Cauchy integral formula shows an important and subtle point about holomorphic functions in several variables: the value of the function f on $P(a, r)$ is completely determined by the values of f on the region of integration b_0P.

Definition 1.2.13 (Cauchy kernel)**.** The Cauchy kernel is the product that appears in the integrand of (1.2.4). Moreover, this kernel can be written as an absolutely convergent power series on $P(a, r)$, where $\overline{P(a, r)} \subset \Omega$. That is,

$$\prod_{j=1}^{n} \frac{1}{(\xi_j - z_j)} = \prod_{j=1}^{n} \frac{1}{(\xi_j - a_j) - (z_j - a_j)} = \sum_{\alpha \in \mathbb{N}^n} \frac{(z - a)^\alpha}{(\xi - a)^{\alpha+1}}, \forall\ z \in P(a, r),\ \xi \in b_0P(a, r), \tag{1.2.7}$$

where $\alpha + 1 = (\alpha_1 + 1, ..., \alpha_n + 1)$ and the convergence is uniform on compact subsets of $P(a, r)$.

Remark 1.2.14. In the next section, the Cauchy integral formula for polydiscs will be used to prove the equivalence between Definitions 1.2.2 and 1.2.5.

1.2.3 Equivalence between Definitions 1.2.2 and 1.2.5

The main object of this subsection is to show that the Definitions 1.2.2 and 1.2.5 are equivalent. We start proving 1.2.2 $\Longrightarrow$ 1.2.5. Let $\Omega \subseteq \mathbb{C}^n$ be an open set. A holomorphic function $f : \Omega \longrightarrow \mathbb{C}$, according to Definition 1.2.2, is separately holomorphic. Hence, for a polydisc $P(a, r)$ such that $\overline{P(a, r)} \subset \Omega$, we can apply the Cauchy integral formula, we use the expansion (1.2.7) of the Cauchy kernel and, thanks to the uniform convergence on compact subsets, we interchange the order of summation and integration as follows

$$\begin{aligned}
f(z) &= \frac{1}{(2\pi i)^n} \int_{b_0P(a,r)} \frac{f(\xi_1, ..., \xi_n)}{(\xi_1 - z_1)...(\xi_n - z_n)} d\xi_1...d\xi_n \\
&= \frac{1}{(2\pi i)^n} \int_{b_0P(a,r)} f(\xi_1, ..., \xi_n) \sum_{\alpha \in \mathbb{N}^n} \frac{(z - a)^\alpha}{(\xi - a)^{\alpha+1}} d\xi_1...d\xi_n \\
&= \sum_{\alpha \in \mathbb{N}^n} \frac{(z - a)^\alpha}{(2\pi i)^n} \int_{b_0P(a,r)} \frac{f(\xi_1, ..., \xi_n)}{(\xi - a)^{\alpha+1}} d\xi_1...d\xi_n, \quad \forall z \in P(a, r).
\end{aligned}$$

Hence, it follows immediately that the function f has a locally power series expansion of the form

$$f(z) = \sum_{\alpha} c_\alpha (z - a)^\alpha, \quad \forall z \in P(a, r), \tag{1.2.8}$$

where

$$c_\alpha(a_1, ..., a_n) = \frac{1}{(2\pi i)^n} \int_{b_0P(a,r)} \frac{f(\xi_1, ..., \xi_n)}{(\xi_1 - a_1)^{\alpha_1+1}...(\xi_n - a_n)^{\alpha_n+1}} d\xi_1...d\xi_n.$$

This proves that f is holomorphic according to Definition 1.2.5.
We prove that (1.2.5 $\Longrightarrow$ 1.2.2). Let $\Omega \subseteq \mathbb{C}^n$ be an open set and let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function according to Definition 1.2.5. Thanks to Proposition 1.2.9, f is holomorphic in each variable separately and, hence, the Cauchy-Riemann equations in each variable separately are satisfied. Finally, the locally expansion in power series clearly implies that $f \in C^1(\Omega)$.

After this argument, considering Proposition 1.2.11, as a consequence we have that

Theorem 1.2.15. *Every holomorphic function is holomorphic in each variable separately.*

Remark 1.2.16. Thanks to Hartogs theorem, we'll prove that the converse is also true.

1.2.4 The Hartogs Theorem

We now present Hartogs theorem, which states that a holomorphic function in each variable separately is holomorphic according to Definition 1.2.2 or equivalently 1.2.5. We start explaining what we do in this subsection. Let $\Omega \subseteq \mathbb{C}^n$ be a domain, suppose that $f : \Omega \longrightarrow \mathbb{C}$ is a holomorphic function in each variable separately and is bounded on compact subsets. Under these conditions, for a polydisc $P(a, r)$ such that $\overline{P(a, r)} \subset \Omega$, arguing in the same way that in subsection 1.2.3, we find that f can be written as an absolutely convergent power series on $P(a, r)$ as follows

$$f(z) = \sum_{\alpha} c_\alpha (z - a)^\alpha \ , \ \forall \ z \in P(a, r),$$

where the convergence is uniform on compact subsets of $P(a, r)$. Therefore, in order to prove the fundamental Hartogs theorem, this means that it suffices to check that if f is holomorphic in each variable separately, then it is bounded on compact subsets.
This subsection is organised as follows. We begin with the statement of Hartogs Lemma. Afterthat, we provide the Hartogs Theorem. The proof of Hartogs Lemma is omitted and can be found, for example, in the book of Joseph L. Taylor, Several Complex Variables with Connections to Algebraic Geometry and Lie Groups.

Lemma 1.2.17 (Hartogs Lemma). *Let f be holomorphic in $P(0, r) = D_1(0, r_1) \times ... \times D_n(0, r_n)$. Fix $z_1, ..., z_{n-1}$, let the power series expansion of f in the variable z_n be. That is*

$$f(z) = \sum_{k} f_k(z') z_n^k, \tag{1.2.9}$$

where the f_k are holomorphic in $P(0, r') := D_1(0, r_1) \times ... \times D_{n-1}(0, r_{n-1})$. If there exists $l > 0$, such that $l > r_n$, so that this series converges in $\overline{D_n(0, l)}$, for every $z' \in P(0, r')$, then (1.2.9) converges uniformly on each compact subset of $P(0, r') \times D_n(0, l)$. Therefore, f extends to be holomorphic on $P(0, r') \times D_n(0, l)$.

Thanks to this Lemma, we can prove Hartogs theorem.

Theorem 1.2.18 (Hartogs, 1906). *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $f : \Omega \longrightarrow \mathbb{C}$ holomorphic in each variable separately. Then f is holomorphic on Ω.*

Proof. This theorem is proved by induction on the dimension n. If $n = 1$, there is nothing to prove. Suppose that $n > 1$ and that the theorem is true for dimension $(n - 1)$. Let $a \in \Omega$ and a polydisc $P(a, r)$ such that $\overline{P(a, r)} \subset \Omega$. Using the following notation

$$P(a', r') = D_1(a_1, r_1) \times ... \times D_{n-1}(a_{n-1}, r_{n-1}),$$

we define

$$X_k := \left\{ z_n \in \overline{D_n(a_n, r_n/2)} : |f(z', z_n)| \leq k, \forall z' \in \overline{P(a', r')} \right\}.$$

We notice that, since $f(z', z_n)$ is continuous in z_n for each fixed z', X_k is closed for every k. Using the induction assumption, $f(z', z_n)$ is also continous in z' and, for every $z_n \in \overline{D_n(a_n, r_n/2)}$, bounded on $\overline{P(a', r')}$. Hence,

$$\overline{D_n(a_n, r_n/2)} \subset \bigcup_k X_k.$$

As a consequence of the Baire category theorem, the set X_k contains a $D_n(b_n, \delta)$, for some k, of some point $b_n \in D_n(a_n, r_n/2)$.
Since f is separately holomorphic and uniformly bounded in the polydisc $P(a', r') \times \overline{D_n(b_n, \delta)}$, we can conclude that f is holomorphic on $P(a', r') \times D_n(b_n, \delta)$.
We choose $s_n > r_n/2$ so that $D_n(b_n, s_n) \subset D_n(a_n, r_n)$. Then, $f(z', z_n)$ is holomorphic in z_n, on $D_n(b_n, s_n)$, for every $z' \in P(a', r')$. As a consequence, we deduce that the expansion in power series of $f(z', z_n)$ about (a', b_n) converges, for every fixed point $z' \in P(a', r')$, as a power series in z_n on $D_n(b_n, s_n)$. From Hartogs Lemma, it turns out that f is holomorphic on all of $P(a', r') \times D_n(b_n, s_n)$. By the arbitrariness of a, the proof is completed. □

Finally, in this thesis, we will deal with the following function spaces.

Definition 1.2.19.

$$H(\mathbb{B}_n) = \{f : \mathbb{B}_n \longrightarrow \mathbb{C} \mid f \text{ holomorphic } \}$$

$$H^\infty(\mathbb{B}_n) := \{f : \mathbb{B}_n \longrightarrow \mathbb{C} \mid f \text{ holomorphic and bounded } \}$$

$$A(\mathbb{B}_n) := C\left(\overline{\mathbb{B}}_n\right) \cap H(\mathbb{B}_n).$$

$$C_0(\mathbb{B}_n) := \left\{f \in C(\overline{\mathbb{B}_n}) \mid f|_{\mathbb{S}_n} = 0 \right\}$$

1.2.5 Consequences of Cauchy Integral Formula

The aim of this subsection is to summarise some elementary properties, for holomorphic functions of several variables, that are analogous to properties of functions of one variable. Most of them are deduced from the Cauchy Integral Formula. Since their proof follow the same lines as in the one variable case, we omit them. However, they can be found, for example, in the book of R. Michael Runge, Holomorphic Functions and Integral Representation in Several Complex Variables.

Corollary 1.2.20. *Suppose $f \in H(\Omega)$, where $\Omega \subseteq \mathbb{C}^n$ is a domain. Then $f \in C^\infty(\Omega)$. Furthermore, for any multiindex α, we have that*

$$\frac{\partial^\alpha f}{\partial z} \in H(\Omega).$$

Moreover, the mean value property holds. That is, for a polydisc $P(a,r)$, such that $\overline{P(a,r)} \subset \mathbb{B}_n$, we have

$$f(a) = \frac{1}{(2\pi)^n}\int_0^{2\pi} \dots \int_0^{2\pi} f(r_1 exp(it_1), \dots, r_n exp(it_1))dt_1 \dots dt_n$$

Proof. We basically use the Cauchy integral formula on any closed polydisc which is contained in Ω. Hence, we can differentiate under the integral sign and the result follows easily. After that, to prove the pointwise integral formula, we just substitute $z = a$ in (1.2.4) to obtain the desired formula. Finally, putting $z = 0$ and $a = 0$ in (1.2.4), we get the following nice formula

$$\begin{aligned} f(0) &= \frac{1}{(2\pi i)^n}\int_{C(0,r_n)} \dots \int_{C(0,r_1)} \frac{f(\xi_1, \dots, \xi_n)}{\xi_1 \dots \xi_n} d\xi_1 \dots d\xi_n \\ &= \int_{r\mathbb{S}_n} f(\xi) d\sigma(\xi) \\ &= \int_{\mathbb{S}_n} f(r\xi) d\sigma(\xi). \end{aligned}$$

This completes our proof. □

Corollary 1.2.21 (Cauchy estimates)**.** *Let $\Omega \subset \mathbb{C}^n$ a domain, $f \in H(\Omega)$ and a polydisc $P(a,r)$, such that $\overline{P(a,r)} \subset \Omega$. Then, for every $\alpha \in (\mathbb{Z}^+)^n$, we have*

$$\left|\frac{\partial^\alpha f}{\partial z}(a)\right| \leq \frac{\alpha!}{r^\alpha} \sup_{z \in b_0 P(a,r)} |f(z)|. \tag{1.2.10}$$

Proof. Fixed $0 < \rho < r$, applying the Cauchy formula to the polydisc $P(a,\rho)$, so that $\overline{P(a,\rho)} \subset P(a,r)$, and differentiating under the integral sign, we obtain

$$\frac{\partial^\alpha f}{\partial z}(a) = \frac{\alpha!}{(2\pi i)^n}\int_{b_0 P(a,\rho)} \frac{f(\xi)}{(\xi - a)^{\alpha+1}} d\xi_1 \dots d\xi_n. \tag{1.2.11}$$

We perform the following estimation,

$$\left|\frac{\partial^\alpha f}{\partial z}(a)\right| \leq \frac{\alpha!}{(2\pi)^n}\int_{b_0P(a,\rho)}\frac{|f(\xi)|}{|(\xi-a)^{\alpha+1}|}|d\xi_1\dots d\xi_n|$$

$$= \sup_{z\in b_0P(a,\rho)}|f(z)|\frac{\alpha!}{(2\pi)^n}\int_{C(a_1,\rho_1)}\frac{|d\xi_1|}{{\rho_1}^{\alpha_1+1}}\ \cdots\ \int_{C(a_n,\rho_n)}\frac{|d\xi_n|}{{\rho_n}^{\alpha_n+1}}$$

$$= \sup_{z\in b_0P(a,\rho)}|f(z)|\frac{\alpha!}{(2\pi)^n}\frac{2\pi\rho_1}{{\rho_1}^{\alpha_1+1}}\ \cdots\ \frac{2\pi\rho_n}{{\rho_n}^{\alpha_n+1}}$$

$$= \frac{\alpha!}{\rho^\alpha}\sup_{z\in b_0P(a,\rho)}|f(z)|.$$

Hence, taking the limit as ρ approaches r, the result follows. □

In order to proceed, we need to recall the following definition.

Definition 1.2.22 (Locally bounded)**.** Let $\Omega\subset\mathbb{C}^n$ a domain be and $\Gamma := \{f:\Omega\longrightarrow\mathbb{C}\mid f\in C(\Omega)\}$ a family of functions. We say that Γ is locally bounded if for every $z_0\in\Omega$ and $r>0$ such that $\overline{B(z_0,r)}\subset\Omega$, there exists $M>0$ that satisfies

$$\sup_{z\in\overline{B(z_0,r)}}|f(z)|\leq M,\ \forall f\in\Gamma.$$

Theorem 1.2.23 (Liouville)**.** *Asssume that $f\in H(\mathbb{C}^n)$ is bounded in $\mathbb{C}^n$, then f is constant.*

Theorem 1.2.24 (Weierstrass)**.** *Let $\Omega\subseteq\mathbb{C}^n$ be a domain, suppose that $(f_n(z))_{n\in\mathbb{N}}\in H(\Omega)$ converges to a function f uniformly on each compact subset. Then $f\in H(\Omega)$ and*

$$\lim_{n\to+\infty}\frac{\partial^\alpha f_n}{\partial z}=\frac{\partial^\alpha f}{\partial z} \tag{1.2.12}$$

on every compact subset $K\subset\Omega$ and for any multiindex α.

Proposition 1.2.25. *(Montel's theorem) Let a domain $\Omega\subset\mathbb{C}^n$ be and $\Gamma := \{f:\Omega\longrightarrow\mathbb{C}|f\in H(\Omega)\}$. Suppose that Γ is locally bounded, then Γ is normal.*

Theorem 1.2.26 (Identity Theorem)**.** *Let a domain $\Omega\subset\mathbb{C}^n$ be, if $f,g\in H(\Omega)$ satisfy $f(z)=g(z)$ for all points z in a non empty open subset $U\subset\Omega$. Then $f(z)=g(z)$ for all points $z\in\Omega$.*

Theorem 1.2.27 (Maximum Modulus Principle)**.** *Let $f\in H(\Omega)$, where $\Omega\subset\mathbb{C}^n$ is a domain . If there exists $w\in\Omega$ such that $|f(z)|\leq|f(w)|$ for all z in some open neighbourhood of w, then $f(z)=f(w)$ for all points $z\in\Omega$.*

1.2.6 Composition of Holomorphic Functions

One of the fundamental properties of holomorphic functions of one complex variable is that the composition of two holomorphic functions is also holomorphic; we extend this property to functions of several complex variables as follows. First of all, we give the notion of holomorphicity to functions of the form $F:\mathbb{C}^n\longrightarrow\mathbb{C}^m$. Suppose that $\Omega\subset\mathbb{C}^n$ and $\Omega'\subset\mathbb{C}^m$ are two domains; the variables in Ω will be written as $z=(z_1,...,z_n)$ and variables in Ω' will be written as $w=(w_1,...,w_m)$. Hence, every map $F:\Omega\longrightarrow\Omega'$ can be described by m functions

$$w_1=f_1(z_1,...,z_n),...,w_m=f_m(z_1,...,z_n). \tag{1.2.13}$$

The map F is called a holomorphic map in Ω if the m functions $f_1,...,f_m$ are holomorphic functions in Ω. Equivalently, if every component is holomorphic. Finally, if $f(w_1,...,w_m)=f(w)$ is a function defined in Ω', the composition $f(F(z))$ is then a well-defined function in Ω.

Theorem 1.2.28 (Composition Theorem). *If $f(w)$ is a holomorphic function in Ω' and if $F : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^m$ is a holomorphic map, then the composition $f(F(z))$ is holomorphic in Ω.*

Proof. Separate $f_j(z)$ into their real and imaginary parts by writing $f_j(z) = u_j(z) + iv_j(z)$. Since all the maps involved are differentiable in the underlying real coordinates, we apply the usual chain rule:

$$\frac{\partial f(F(z))}{\partial \overline{z}_j} = \sum_{k=1}^{m} \left(\frac{\partial f}{\partial w_k} \frac{\partial f_k}{\partial \overline{z}_j} + \frac{\partial f}{\partial \overline{w}_k} \frac{\partial \overline{f}_k}{\partial \overline{z}_j} \right). \tag{1.2.14}$$

If the function f and the map F are both holomorphic, then $\dfrac{\partial f}{\partial \overline{w}_k} = 0$ and $\dfrac{\partial f_k}{\partial \overline{z}_j} = 0$ for all k; hence $\dfrac{\partial f(F(z))}{\partial \overline{z}_j} = 0$ for all j. From the Cauchy-Riemann equations and since $f(F(z)) \in C^1(\Omega)$, then follows that the function $f(F(z))$ is holomorphic. $\square$

1.2.7 Holomorphic maps of the form $F : \Omega \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}^m$.

The purpose of this subsection is to list some properties, for holomorphic functions of the form $F : \Omega \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}^m$, that will be necessary in the description of the automorphism group of $\mathbb{B}_n$. We start recalling the following definition.

Definition 1.2.29. A map $F : \Omega \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}^m$, that is written as $F = (f_1, ..., f_m)$, is called holomorphic if its components $f_1, ..., f_m$ are holomorphic functions on Ω.

Remark 1.2.30. Every holomorphic map $F : \Omega \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}^m$ admits a locally expansion in power series: fix $a \in \Omega$, there exists a polydisc $P(a, r)$, such that $\overline{P(a, r)} \subset \Omega$, so that F is written as an absolutely convergent power series on $P(a, r)$. That is, there exists $(c_\alpha)_{\alpha \in \mathbb{N}^n} := (c^1_\alpha, ..., c^m_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{C}^m$, where $c^j_\alpha \in \mathbb{C}$ and $j = 1, ..., m$, such that

$$F(z) = \sum_\alpha c_\alpha (z-a)^\alpha := \left(\sum_\alpha c^1_\alpha (z-a)^\alpha, ..., \sum_\alpha c^m_\alpha (z-a)^\alpha \right), \forall z \in P(a, r).$$

Moreover, the convergence is uniform on compact subsets of $P(a, r)$.
As a consequence, for holomorphic maps of the form $F : \mathbb{B}_n \longrightarrow \mathbb{C}^m$ and for a polydisc $P(0, r)$, such that $\overline{P(0, r)} \subseteq \mathbb{B}_n$, there exists a sequence of vector-valued functions $(F_k(z))_{k \in \mathbb{N}} := \left(F^1_k(z), ..., F^m_k(z)\right)_{k \in \mathbb{N}}$, where the components $F^j_k(z) : \Omega \longrightarrow \mathbb{C}$ are homogenous polynomials having degree k, such that the following locally absolutely expansion in power series holds:

$$F(z) = \sum_{k=0}^{+\infty} F_k(z) := \left(\sum_{k=0}^{+\infty} F^1_k(z), ..., \sum_{k=0}^{+\infty} F^m_k(z) \right), \forall z \in P(0, r),$$

where $\overline{P(0, r)} \subset \mathbb{B}_n$. Of course, the convergence is uniform on compact subsets of $P(0, r)$.

Definition 1.2.31. Let $F : \Omega \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}^m$ be a holomorphic map, where $\Omega \subseteq \mathbb{C}^n$ is a domain, the jacobian matrix of F in z is the complex linear map $J_\mathbb{C} F : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ that admits this representation:

$$J_\mathbb{C} F(z) = \left(\frac{\partial f_i}{\partial z_j}(z) \right)_{m \times n} = \begin{bmatrix} \dfrac{\partial f_1}{\partial z_1} & \cdots & \dfrac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial z_1} & \cdots & \dfrac{\partial f_m}{\partial z_n} \end{bmatrix}.$$

We call $J_\mathbb{C} F(z)$ the complex Jacobian matrix of the holomorphic map F at z.

Remark 1.2.32. When we write $J_\mathbb{C} F(0)z$, we mean the matrix multiplication between the matrix $J_\mathbb{C} F(0)$ and the column vector $z \in \mathbb{C}^n$. Hence, for a polydisc $P(0, r)$ such that $\overline{P(0, r)} \subset \Omega$, the homogenous expansion of F in $P(0, r)$ begins as follows

$$F(z) = F(0) + J_\mathbb{C} F(0) z + ..., \forall\ z \in P(0, r).$$

We now consider in more detail the equidimensional case $m = n$.
From the theory of Complex Analysis of one variable, it is a well-known fact that any holomorphic map $F : \Omega_1 \subseteq \mathbb{C} \longrightarrow \Omega_2 \subseteq \mathbb{C}$ induces a map between domains of $\mathbb{R}^2$ using the canonical identification $z = x + iy \equiv (x, y) \longrightarrow (u(x, y), v(x, y))$ and, thanks to the Cauchy-Riemann equations, its Jacobian determinant is

$$det(J_{\mathbb{R}}\ F) = |F'|^2.$$

Regarding the several variables case, let $F : \Omega \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a holomorphic map, where $\Omega \subset \mathbb{C}^n$ is a domain, denoting by $J_{\mathbb{C}}F$ its jacobian matrix, by $detJ_{\mathbb{C}}F(z)$ its determinant and by $detJ_{\mathbb{R}}F(z)$ the real Jacobian determinant of the induced map. Then, proceeding similarly as in the previous case, we can prove the following lemma.

Lemma 1.2.33. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $F : \Omega \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a holomorphic map. Then*

$$detJ_{\mathbb{R}}F(z) = |detJ_{\mathbb{C}}F(z)|^2, \ \forall\ z \in \Omega. \tag{1.2.15}$$

Proof. After a permutation of the rows and columns, we can write

$$detJ_{\mathbb{R}}F = det \begin{bmatrix} \dfrac{\partial u_k}{\partial x_j} & \dfrac{\partial u_k}{\partial y_j} \\ \dfrac{\partial v_k}{\partial x_j} & \dfrac{\partial v_k}{\partial y_j} \end{bmatrix},$$

where each one of the four blocks on the right are real $n \times n$ matrices. Consider i times the bottom blocks and adding it to the top, it turns out that

$$detJ_{\mathbb{R}}F = det \begin{bmatrix} \dfrac{\partial u_k}{\partial x_j} + i\dfrac{\partial v_k}{\partial x_j} & \dfrac{\partial u_k}{\partial y_j} + i\dfrac{\partial v_k}{\partial y_j} \\ \dfrac{\partial v_k}{\partial x_j} & \dfrac{\partial v_k}{\partial y_j} \end{bmatrix}.$$

Using Cauchy-Riemann equations, we obtain

$$detJ_{\mathbb{R}}F = det \begin{bmatrix} \dfrac{\partial u_k}{\partial x_j} + i\dfrac{\partial v_k}{\partial x_j} & -\dfrac{\partial v_k}{\partial x_j} + i\dfrac{\partial u_k}{\partial x_j} \\ \dfrac{\partial v_k}{\partial x_j} & \dfrac{\partial u_k}{\partial x_j} \end{bmatrix}.$$

Now we substract i times the left blocks from the right side, it follows that

$$detJ_{\mathbb{R}}F = det \begin{bmatrix} \dfrac{\partial u_k}{\partial x_j} + i\dfrac{\partial v_k}{\partial x_j} & 0 \\ \dfrac{\partial v_k}{\partial x_j} & \dfrac{\partial u_k}{\partial x_j} - i\dfrac{\partial v_k}{\partial x_j} \end{bmatrix}.$$

Then, since $\dfrac{\partial f_j}{\partial z_j} = \dfrac{\partial f_j}{\partial x_j}$, where $j = 1, ..., n$, and recalling that $\overline{f}_k = u_k - iv_k$, we find that

$$detJ_{\mathbb{R}}F = det \begin{bmatrix} \dfrac{\partial f_k}{\partial x_j} & 0 \\ \dfrac{\partial v_k}{\partial x_j} & \dfrac{\partial \overline{f}_k}{\partial x_j} \end{bmatrix} = detJ_{\mathbb{C}}F\ \overline{detJ_{\mathbb{C}}F} = |detJ_{\mathbb{C}}F(z)|^2,$$

and we are done. $\square$

1.3 The Automorphism Group

In this section, we compute the group of automorphisms of $\mathbb{B}_n$. After a short review about the unitary mapping in $\mathbb{C}^n$, fixed $a \in \mathbb{B}_n$, we construct an automorphism that interchanges a and the origin 0. Since the set of holomorphic automorphisms of $\mathbb{B}_n$ is a group under composition and, hence, the composition of the previous two maps is an automorphism too, we'll prove that these are all the automorphisms of the open unit ball.

1.3.1 Introduction about Automorphisms

The main purpose of this initial part is a brief review of the biholomorphic maps of $\mathbb{B}_n$ onto $\mathbb{B}_n$. Let's start giving the definition of biholomorphic map.

Definition 1.3.1 (Biholomorphic Map)**.** A map $F : \mathbb{B}_n \longrightarrow \mathbb{B}_n$ is said to be bi-holomorphic if the following hold :
1) F is a bijection.
2) F is holomorphic.
3) F^{-1} is holomorphic.

Definition 1.3.2 (Automorphism Group)**.** We introduce the following set

$$Aut(\mathbb{B}_n) := \{F : \mathbb{B}_n \longrightarrow \mathbb{B}_n \mid F \text{ biholomorphism}\}.$$

The afore-mentioned space has the algebraic structure of a group under composition .
As in the one dimensional case, these biholomorphisms will be called automorphisms of $\mathbb{B}_n$.

1.3.2 The unitary mapping

In the current subsection, we study a class of automorphism of $\mathbb{B}_n$: the unitary mappings. First of all, we start recalling its definition. Secondly, we discuss a crucial property concerning the automorphism group: every automorphism that fixes the origin can be identified with an unitary mapping.

Definition 1.3.3 (Unitary mapping)**.** Given an $n \times n$ matrix $U = (u_{ij})$, where $u_{ij} \in \mathbb{C}$, we can associate a linear map $L_U : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ by $L_U(z) := Uz$. The matrix U is called unitary if it preserves the inner product of $\mathbb{C}^n$, namely

$$< Uz, Uw >=< z, w > \tag{1.3.1}$$

for $z, w \in \mathbb{C}^n$. Furthermore, L_U is said to be a unitary mapping.

Remark 1.3.4. We deduce from (1.3.1) that a unitary mapping is an isometry and, clearly, an isomorphism. In particular, $L_U : \mathbb{B}_n \longrightarrow \mathbb{B}_n$ is an automorphism of the open unit ball $\mathbb{B}_n$.

In the following Lemma, we prove that the automorphisms that fix the origin of $\mathbb{C}^n$ coincide exactly with the unitary transformations.

Lemma 1.3.5. *Let $\varphi \in Aut(\mathbb{B}_n)$, we have that*

$$\varphi \textit{ is an unitary transformation of } \mathbb{B}_n \iff \varphi(0) = 0.$$

Proof. ($\Longleftarrow$) Chosen a complex number $\lambda \in C(0,1)$, we define the following holomorphic map

$$F(z) := \varphi^{-1}\left(\overline{\lambda}\varphi(\lambda z)\right), \quad z \in \mathbb{B}_n.$$

It turns out that

$$F(0) = \varphi^{-1}\left(\overline{\lambda}\varphi(\lambda 0)\right) = \varphi^{-1}\left(\overline{\lambda}\varphi(0)\right) = \varphi^{-1}\left(\overline{\lambda}0\right) = \varphi^{-1}(0) = 0.$$

Furthermore,

$$J_{\mathbb{C}}F(0) \underbrace{=}_{chain\ rule} J_{\mathbb{C}}\varphi^{-1}(\overline{\lambda}\varphi(0))\overline{\lambda}\lambda J_{\mathbb{C}}\varphi(0) \underbrace{=}_{|\lambda|=1} J_{\mathbb{C}}\varphi^{-1}(0)J_{\mathbb{C}}\varphi(0) = (J_{\mathbb{C}}\varphi(0))^{-1}J_{\mathbb{C}}\varphi(0) = I,$$

where I is the identity matrix. If F is the identity map of $\mathbb{B}_n$, we don't have anything to prove. Otherwise, we could write F using its homogenous expansion. That is,

$$F(z) = (f_1(z), ..., f_n(z)) = z + \sum_{k=l}^{+\infty} F_k(z) := \left(z_1 + \sum_{k=l}^{+\infty} F_k^1(z), ..., z_n + \sum_{k=l}^{+\infty} F_k^n(z)\right), \forall z \in P(0,r),$$

where $P(0,r)$ is a polydisc such that $\overline{P(0,r)} \subset \mathbb{B}_n$ and F_k^j are homogenous polynomial of degree k, for $j = 1, ..., n$, so that, for $l \geq 2$, $F_l^j(z)$ is not zero on $P(0,r) \setminus \{0\}$ for $j = 1, ..., n$.
Fix $j \in \{1, ..., n\}$ and define

$$\phi(z) := \underbrace{f_j(z) \circ ... \circ f_j(z)}_{N\ times}$$

since $f_j(0) = 0$ and $f_j'(0) = 1$, we deduce that $\phi(0) = 0$ and

$$\phi'(0) = f_j'\underbrace{(f_j \circ .... \circ f_j)}_{N-1\ times}(0) f_j' \underbrace{(f_j \circ .... \circ f_j)}_{N-2\ times}(0)\ ...\ f_j'(0) = 1.$$

Our aim is to prove that $\phi''(0) = N f_j''(0)$. Hence, we proceed by induction: for $N = 2$, we have

$$\phi''(0) = 2 f_j''(0),$$

suppose the statement true for $N-1$, then

$$\phi''(0) = ((f_j \underbrace{(f_j \circ ... \circ f_j)}_{N-1})')' \mid_{z=0} = ((f_j' \underbrace{(f_j \circ ... \circ f_j)}_{N-1}) \underbrace{(f_j \circ ... \circ f_j)}_{N-1}')' \mid_{z=0}$$

$$= f_j'' \underbrace{(f_j \circ ... \circ f_j)}_{N-1} ((\underbrace{f_j \circ ... \circ f_j}_{N-1})')^2 \mid_{z=0} + \underbrace{(f_j \circ ... \circ f_j)''}_{N-1} f_j' \underbrace{(f_j \circ ... \circ f_j)}_{N-1} \mid_{z=0}$$

$$= f_j''(0) + (N-1) f_j''(0) = N f_j''(0).$$

Moreover, a similar computation shows that

$$\phi'''(0) = N f_j'''(0) + 2N(f_j''(0))^2.$$

We give a formula that generalises the previous one,

$$\phi^{(k)}(0) = N f_j^{(k)}(0) + (k-1) N (f_j^{(k-1)}(0))^{k-1}.$$

This means that, for a polydisc $P(0,r)$ such that $\overline{P(0,r)} \subset \mathbb{B}_n$, we have the following local expansion

$$f_j^N(z) := \underbrace{f_j \circ f_j \circ ... \circ f_j}_{N\ times}(z) = z_j + N F_l^j(z) + ...,\ \forall z \in P(0,r),$$

where the omitted terms consist of polynomials of degree greater than l. Hence, as a consequence, we compose F with itself N times and we get

$$F^N(z) := \underbrace{F \circ F \circ ... \circ F}_{N\ times}(z) = z + N F_l(z) + ...,\ \forall z \in P(0,r),$$

where, again, we omitted polynomials of degree greater than l. Since the following limit

$$\lim_{N \to +\infty} F^N(z) =: G(z)$$

defines a biholomorphism, this implies that in the right part of the previous equality we have

$$F_j(z) = 0,\ \forall j \geq l,$$

which is a contradiction with that $F_l \neq 0$. Then, $F(z) = z$ or, equivalently, $\varphi(\lambda z) = \lambda \varphi(z), \forall z \in \mathbb{B}_n$. In other words, the homogenous expansion of φ is just given by the linear term, i.e. φ is a linear transformation. Hence, there exists a $n \times n$ matrix $U = (u_{ij})_{ij}$, where $u_{ij} \in \mathbb{C}$, such that

$$\varphi(z) = Uz,\ \forall z \in \mathbb{B}_n.$$

Since $J_{\mathbb{C}}\varphi(z) = U, \forall z \in \mathbb{B}_n$, φ maps $\mathbb{B}_n$ onto itself and using the change of variables formula we have

$$|det(U)| = 1.$$

Our conclusion is that φ is a unitary transformation.
($\Longrightarrow$) Every unitary transformation is an isometry:

$$0 \underbrace{=}_{z=0} |z| \overbrace{=}^{isometry} |\varphi(z)| \underbrace{=}_{z=0} |\varphi(0)|,$$

and we obtain $\varphi(0) = 0$. $\square$

1.3.3 The involutive automorphism

We know that for every α in the unit disc of $\mathbb{C}$ corresponds an automorphism φ_α of the disc that interchanges α and 0. Explicitly, $\varphi_\alpha(z) = \dfrac{\alpha - z}{1 - \overline{\alpha} z}$. The same can be done in the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$. In this subsection, we introduce an automorphism on $\mathbb{B}_n$, that acts in the same way. To achieve this goal, in the next definition, we recall some tools from linear algebra.

Definition 1.3.6. Let $a \in \mathbb{B}^n \setminus \{0\}$, the orthogonal projection from $\mathbb{C}^n$ onto the subspace generated by a, which is indicated by $[a]$, is

$$P_a(z) := \frac{< z, a >}{|a|^2} a, \quad z \in \mathbb{C}^n.$$

Q_a is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus [a]$, that is

$$Q_a(z) := z - \frac{< z, a >}{|a|^2} a, \quad z \in \mathbb{C}^n.$$

We study another important family of automorphisms. They are usually known as involutive automorphisms or involutions.

Definition 1.3.7. Let $a \in \mathbb{B}_n \setminus \{0\}$, denoting by $s_a = \sqrt{1 - |a|^2}$, the involutive automorphism is:

$$\varphi_a(z) := \frac{a - P_a(z) - s_a Q_a(z)}{1 - < z, a >}, \quad z \in \mathbb{B}_n. \tag{1.3.2}$$

Moreover, if $a = 0$, we just put $\varphi_a(z) = -z$.

Remark 1.3.8. The denominator of this formula does not vanish in $\mathbb{B}_n$: applying the Schwarz inequality we find that $| < z, a > | \leq |a||z| < 1$. Hence, the map φ_a is holomorphic on $\mathbb{B}_n$.

As a consequence of the following formula, we prove that φ_a maps $\mathbb{B}_n$ into $\mathbb{B}_n$.

Lemma 1.3.9. *Let $a \in \mathbb{B}_n$, we have that*

$$1 - |\varphi_a(z)|^2 = \frac{\left(1 - |a|^2\right)\left(1 - |z|^2\right)}{|1 - < z, a > |^2}, \quad z \in \mathbb{B}_n. \tag{1.3.3}$$

where φ_a is the involutive automorphism in (1.3.2).

Proof. Since the case $a = 0$ is trivial, we can suppose that $a \neq 0$.
We use the orthogonality between $a - P_a(z)$ and $Q_a(z)$ in $\mathbb{C}^n$, with the identities

$$|a|^2 |P_a(z)|^2 = | < z, a > |^2, \qquad < P_a(z), a > = < z, a > .$$

We easily calculate that:

$$\begin{aligned}
|a - P_a(z) - s_aQ_a|^2 &=< a - P_a(z) - s_aQ_a,\ a - P_a(z) - s_aQ_a > \\
&=< a - P_a, a - P_a - s_aQ_a > -s_a < Q_a, a - Pa - s_aQ_a > \\
&=< a - P_a, a - P_a > -s_a \underbrace{< a - P_a, Q_a >}_{=0} + s_a^2 < Q_a, Q_a > -s_a \underbrace{< Q_a, a - P_a >}_{=0} \\
&= |a - P_a(z)|^2 + (1 - |a|^2)|Q_a(z)|^2 \\
&= |a|^2 - 2Re < P_a(z), a > +|P_a(z)|^2 + (1 - |a|^2)(|z|^2 - |P_a(z)|^2),
\end{aligned}$$

that is

$$|a - P_a(z) - s_aQ_a|^2 = |1- < z, a > |^2 - (1 - |a|^2)(1 - |z|^2)$$

dividing both sides by $|1- < z, a > |^2$, we find

$$\frac{|a - P_a(z) - s_aQ_a|^2}{|1- < z, a > |^2} = 1 - \frac{(1 - |a|^2)(1 - |z|^2)}{|1- < z, a > |^2}$$

thus

$$1 - \frac{|a - P_a(z) - s_aQ_a|^2}{|1- < z, a > |^2} = \frac{(1 - |a|^2)(1 - |z|^2)}{|1- < z, a > |^2}.$$

Finally, recalling the definition of φ_a, the previous formula turns out to be

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1- < z, a > |^2}.$$

□

Corollary 1.3.10. *The map φ_a sends $\mathbb{B}_n$ into itself. Moreover, $\varphi_a : \partial\mathbb{B}_n \longrightarrow \partial\mathbb{B}_n$.*

Proof. From (1.3.3), since $z, a \in \mathbb{B}_n$, we obtain

$$|\varphi_a(z)|^2 = 1 - \underbrace{\frac{(1 - |z|^2)(1 - |a|^2)}{|1- < z, a > |^2}}_{\geq 0} \leq 1.$$

Finally, if $z \in \partial\mathbb{B}_n$, then $|\varphi_a(z)| = 1$. That is, the map φ_a sends the boundary $\partial\mathbb{B}_n$ into itself. □

As a consequence of the following result, we'll prove that φ_a is a bijection and an automorphism.

Proposition 1.3.11 (Involution Property)**.** *Fix $a \in \mathbb{B}_n$, let φ_a be the involutive automorphism in* (1.3.2)*. Then, the involution property holds, that is*

$$\varphi_a \circ \varphi_a(z) = z,\ \forall\ z \in \mathbb{B}_n.$$

Proof. First of all, we can easily see that

$$\begin{aligned}
1- < \varphi_a(z), a > &\doteq 1- < \frac{a - P_a(z) - s_aQ_a(z)}{1- < z, a >}, a > \\
&= \frac{1- < z, a > -|a|^2 + < P_a(z), a > +s_a < Q_a(z), a >}{1- < z, a >} \\
&= \frac{1 - |a|^2}{1- < z, a >}.
\end{aligned}$$

Now

$$P_a(\varphi_a(z)) = \frac{< a - P_a(z) - s_a Q_a(z), a >}{|a|^2(1- < z, a >)} a$$

$$= \frac{< a - z + z - P_a(z) - s_a Q_a(z), a >}{|a|^2(1- < z, a >)} a$$

$$= \frac{< a - z, a >}{|a|^2(1- < z, a >)} + \underbrace{\frac{< z - P_a(z) - s_a Q_a(z), a >}{|a|^2(1- < z, a >)}}_{=0}$$

$$= \frac{a}{|a|^2} \frac{|a|^2 - < z, a >}{1- < z, a >}$$

$$= \frac{a - P_a(z)}{1- < z, a >},$$

namely,

$$P_a(\varphi_a(z)) = \frac{a - P_a(z)}{1- < z, a >}.$$

We easily deduce

$$Q_a(\varphi_a(z)) = \varphi_a(z) - \frac{a}{|a|^2} \frac{|a|^2 - < z, a >}{1- < z, a >}$$

$$= \frac{a - P_a(z) - s_a Q_a(z)}{1- < z, a >} - \frac{a}{|a|^2} \frac{|a|^2 - < z, a >}{1- < z, a >}$$

$$= \frac{a|a|^2 - < z, a > a - s_a z|a|^2 + s_a < z, a > a - a|a|^2 + a < z, a >}{|a|^2(1- < z, a >)}$$

$$= \frac{-s_a z|a|^2 + s_a < z, a > a}{|a|^2(1- < z, a >)}$$

$$= -s_a \left\{ \frac{z}{(1- < z, a >)} - \frac{< z, a > a}{|a|^2(1- < z, a >)} \right\}$$

$$= -s_a \frac{Q_a(z)}{1- < z, a >},$$

that is

$$Q_a(\varphi_a(z)) = -s_a \frac{Q_a(z)}{1- < z, a >}.$$

Since

$$1- < \varphi_a(z), a > = \frac{1 - |a|^2}{1- < z, a >},$$

hence, we find

$$\begin{aligned}
\varphi_a \circ \varphi_a(z) &= \frac{a - P_a(\varphi_a(z)) - s_a Q_a(\varphi_a(z))}{1 - <\varphi_a(z), a>} \\
&= \frac{a - P_a(\varphi_a(z)) - s_a Q_a(\varphi_a(z))}{1 - |a|^2}(1 - <z, a>) \\
&= \frac{a - a <z, a> - a + P_a(z) + (1 - |a|^2) Q_a(z)}{(1 - |a|^2)} \\
&= \frac{-a <z, a> + P_a(z) + (1 - |a|^2)(z - P_a(z))}{(1 - |a|^2)} \\
&= \frac{-a <z, a> + P_a(z) + z - P_a(z) - |a|^2 z + |a|^2 P_a(z)}{(1 - |a|^2)} \\
&= \frac{-a <z, a> + z - |a|^2 z + <z, a> a}{(1 - |a|^2)} \\
&= \frac{z - |a|^2 z}{(1 - |a|^2)} = z,
\end{aligned}$$

and we obtain

$$\varphi_a \circ \varphi_a(z) = z, \ \forall z \in \mathbb{B}_n.$$

□

Corollary 1.3.12. *Fix $a \in \mathbb{B}_n$, let φ_a be an involutive automorphism. Then, φ_a is an automorphism of $\mathbb{B}_n$ that exchanges the origin 0 with a.*

Proof. Fix $w \in \mathbb{B}_n$, our first goal is to prove that there exists $z \in \mathbb{B}_n$ such that $\varphi_a(z) = w$. From the involution property follows that

$$\varphi_a(w) = z,$$

and, since φ_a sends $\mathbb{B}_n$ into itself, we've proved that φ_a is surjective. Furthemore, from the involution property, we deduce that φ_a is invertible on $\mathbb{B}_n$, whose inverse function is φ_a itself and, since φ_a is holomorphic, this proves that φ_a is an automorphism of $\mathbb{B}_n$. Finally,

$$\varphi_a(0) = a, \qquad \varphi_a(a) = 0.$$

□

We give an alternative detailed description of the orthogonal projection P_a and, hence, of Q_a as well. Such description will be used to prove two formulas regarding the complex Jacobian matrix of the involutive automorphism φ_a.
The orthogonal projection from $\mathbb{C}^n$ onto the one-dimensional subspace generated by z, denoted by P_z, can be written as

$$P_z(w) = \frac{1}{|z|^2} \begin{pmatrix} z_1\overline{z}_1 & \cdots & z_1\overline{z}_n \\ \vdots & \ddots & \vdots \\ z_n\overline{z}_1 & \cdots & z_n\overline{z}_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \frac{<w, z>}{|z|^2} z. \tag{1.3.4}$$

Now, we introduce the following auxiliary matrix

$$A(z) = (z_i \overline{z}_j)_{n \times n} = \begin{pmatrix} z_1\overline{z}_1 & \cdots & z_1\overline{z}_n \\ \vdots & \ddots & \vdots \\ z_n\overline{z}_1 & \cdots & z_n\overline{z}_n \end{pmatrix}, \quad z = (z_1, ...z_n) \in \mathbb{C}^n, \tag{1.3.5}$$

and, if we identify linear transformations on $\mathbb{C}^n$ with $n \times n$ matrices via the standard basis of $\mathbb{C}^n$, we easily obtain that P_z can be written as follows

$$P_z = \frac{A(z)}{|z|^2}, \ z \neq 0. \tag{1.3.6}$$

Moreover, proceeding similarly, it turns out that

$$Q_z = I - \frac{A(z)}{|z|^2}, \ z \neq 0. \tag{1.3.7}$$

In the next Lemma, for a fixed $a \in \mathbb{B}_n$, we provide two formulas concerning the complex Jacobian matrix of the involution automorphism, respectively, in the origin and in a.

Lemma 1.3.13. *Let $a \in \mathbb{B}_n$, the following formulas hold*

$$J_{\mathbb{C}}\varphi_a(0) = -(1-|a|^2)P_a - \sqrt{1-|a|^2}Q_a \tag{1.3.8}$$

and

$$J_{\mathbb{C}}\varphi_a(a) = -\frac{P_a}{(1-|a|^2)} - \frac{Q_a}{\sqrt{1-|a|^2}}, \tag{1.3.9}$$

where φ_a is the involution automorphism in (1.3.2).

Proof. Since, for every $z \in \mathbb{B}_n$ and $a \in \mathbb{B}_n$, we have that $| < z, a > | < 1$, hence

$$\varphi_a(z) = (a - P_a(z) - s_a Q_a(z)) \sum_{k=0}^{\infty} < z, a >^k$$

$$= a + \underbrace{|a|^2 \frac{< z, a >}{|a|^2}}_{=P_a(z)} a - P_a(z) - s_a Q_a(z) + \mathcal{O}(|z|^2)$$

$$= a + (|a|^2 - 1)P_a(z) - s_a Q_a(z) + \mathcal{O}(|z|^2),$$

Hence, since $s_a^2 = 1 - |a|^2$, it turns out that

$$\varphi_a(z) = \underbrace{a}_{=\varphi_a(0)} - \underbrace{s_a^2 P_a(z) - s_a Q_a(z)}_{linear\ terms} + \mathcal{O}(|z|^2).$$

Using the notation introduced above, it follows that

$$J_{\mathbb{C}}\varphi_a(0) = -s_a^2 P_a - s_a Q_a.$$

For what concerns $J_{\mathbb{C}}\varphi_a(a)$, our aim is to expand $\varphi_a(z)$ in power series in a suitable neighborhood $B(a, r)$, such that $\overline{B(a, r)} \subset \mathbb{B}_n$. First of all,

$$\overline{B(a, r)} \subset \mathbb{B}_n, \quad r \in [0, 1 - |a|).$$

This means that

$$| < z - a, a > | \underbrace{\leq}_{C-S} |z - a||a| \underbrace{\leq}_{z \in B(a,r)} r|a| < (1 - |a|)|a| < (1 - |a|^2),$$

that is

$$\frac{| < z - a, a > |}{(1 - |a|^2)} < 1.$$

Hence, we proceed as follows,

$$\varphi_a(z) = \frac{a - \frac{<z,a>}{|a|^2}a - s_a\left(z - \frac{<z,a>}{|a|^2}a\right)}{1- < z-a, a > - < a, a >}$$

$$= \frac{-\frac{<z-a,a>}{|a|^2}a - s_a\left(z - a - \frac{<z-a,a>}{|a|^2}a\right)}{1 - |a|^2 - < z-a, a >}$$

$$= \frac{-s_a(z-a) + (s_a - 1)\frac{<z-a,a>}{|a|^2}a}{(1-|a|^2)\left(1 - \frac{<z-a,a>}{1-|a|^2}\right)}$$

$$= \left(-s_a(z-a) + (s_a-1)\frac{< z-a, a >}{|a|^2}a\right)\sum_{k=0}^{\infty}\frac{< z-a, a >^k}{(1-|a|^2)^{k+1}}$$

$$= \frac{1}{1-|a|^2}\left(-s_a(z-a) + (s_a-1)\frac{< z-a, a >}{|a|^2}a\right)\left\{\frac{1}{1-|a|^2} + \sum_{k=1}^{\infty}\frac{< z-a, a >^k}{(1-|a|^2)^{k+1}}\right\}$$

$$= \frac{1}{s_a^2}\left(-s_a(z-a) + (s_a-1)\frac{< z-a, a >}{|a|^2}a\right) + \mathcal{O}(|z-a|^2)$$

$$= -\frac{1}{s_a^2}\frac{< z-a, a >}{|a|^2}a - \frac{1}{s_a}\left((z-a) - \frac{< z-a, a >}{|a|^2}a\right) + \mathcal{O}(|z-a|^2)$$

which means

$$\varphi_a(z) = -\frac{1}{s_a^2}\underbrace{\frac{< z-a, a >}{|a|^2}a}_{=P_a(z-a)} - \frac{1}{s_a}\underbrace{\left((z-a) - \frac{< z-a, a >}{|a|^2}a\right)}_{=Q_a(z-a)} + \mathcal{O}(|z-a|^2).$$

Then, we deduce that

$$J_{\mathbb{C}}\varphi_a(a) = -\frac{P_a}{s_a^2} - \frac{Q_a}{s_a}.$$

□

Lemma 1.3.14. *Let $a \in \mathbb{B}_n$, we have*

$$det J_{\mathbb{R}}\varphi_a(z) = \left(\frac{1-|a|^2}{|1- < z, a > |^2}\right)^{n+1}, \forall z \in \mathbb{B}_n, \tag{1.3.10}$$

where φ_a is the involution automorphism in (1.3.2).

Proof. Let $z \in \mathbb{B}_n$, putting $w := \varphi_a(z)$, we consider the automorphism

$$L_U := \varphi_w \circ \varphi_a \circ \varphi_z,$$

so that $L_U(0) = 0$. Hence, by Lemma 1.3.5, L_U is a unitary transformation. Using the involution property, we can write

$$\varphi_a = \varphi_w \circ L_U \circ \varphi_z;$$

then, applying the chain rule, we obtain

$$J_{\mathbb{C}}\varphi_a(z) = J_{\mathbb{C}}\varphi_w(0)J_{\mathbb{C}}L_U(z)J_{\mathbb{C}}\varphi_z(z)$$

$$= J_{\mathbb{C}}\varphi_w(0)UJ_{\mathbb{C}}\varphi_z(z),$$

where U is a unitary matrix.
From (1.3.8), the linear map $J_{\mathbb{C}}\varphi_w(0)$ has a one-dimensional eigenspace with eigenvalue $-(1-|w|^2)$ and an $(n-1)-$ dimensional eigenspace with eigenvalue $-\sqrt{1-|w|^2}$. This means that

$$det(J_{\mathbb{C}}\varphi_w(0)) = -(1-|w|^2)(-1)^{n-1}(1-|w|^2)^{1/2(n-1)}$$

$$= (-1)^n(1-|w|^2)^{(n+1)/2}.$$

Similarly, from (1.3.9) we deduce that the linear map $J_{\mathbb{C}}\varphi_z(z)$ has a one-dimensional eigenspace with eigenvalue $-\dfrac{1}{(1-|w|^2)}$ and an $(n-1)-$ dimensional eigenspace with eigenvalue $-\dfrac{1}{\sqrt{1-|w|^2}}$, hence

$$det(J_{\mathbb{C}}\varphi_z(z)) = -\frac{1}{(1-|z|^2)}(-1)^{n-1}\frac{1}{(1-|z|^2)^{1/2(n-1)}}$$

$$= \frac{(-1)^n}{(1-|z|^2)^{(n+1)/2}}.$$

Now, since $|det(U)| = 1$ and using Lemma 1.3.9, it turns out that

$$detJ_{\mathbb{R}}\varphi_a(z) = |detJ_{\mathbb{C}}\varphi_a(z)|^2$$

$$= \left(\frac{1-|w|^2}{1-|z|^2}\right)^{n+1}$$

$$= \left(\frac{1-|a|^2}{|1-<z,a>|^2}\right)^{n+1},$$

where we recall that $w = \varphi_a(z)$. □

1.3.4 The characterisation of $Aut(\mathbb{B}_n)$

In the next theorem, we show that all the automorphisms of $\mathbb{B}_n$ are obtained from the maps φ_a and the unitary transformations.

Theorem 1.3.15. *Given $\varphi \in Aut(\mathbb{B}_n)$, then there exist L_U, L_V, unitary transformations of $\mathbb{C}^n$, and φ_a, φ_b, involutive automorphisms, such that*

$$\varphi = L_U\varphi_a = \varphi_b L_V\ .$$

Proof. Let $\varphi \in Aut(\mathbb{B}_n)$, indicating by $a = \varphi^{-1}(0)$, since $Aut(\mathbb{B}_n)$ is a group, the map $\psi := \varphi\circ\varphi_a$ is an automorphism that satisfies $\psi(0) = 0$. Applying Lemma 1.3.5, there exists a unitary transformation L_U of $\mathbb{C}^n$ such that $L_U = \varphi\circ\varphi_a$. Since φ_a is involutive, it turns out that $\varphi = L_U\varphi_a$. To prove the other equality, we proceed in the same way. □

To end this subsection, we prove a formula that will be used, for example, to establish some asymptotic estimates for certain important integrals operators on the ball and on the sphere.

Corollary 1.3.16. *Let $\varphi \in Aut(\mathbb{B}_n)$, we have*

$$J_{\mathbb{R}}\varphi(z) = \left(\frac{1-|a|^2}{|1-<z,a>|^2}\right)^{n+1},\ \forall z \in \mathbb{B}_n, \tag{1.3.11}$$

where $a = \varphi^{-1}(0)$.

Proof. There exists a unitary transformation L_U and an involutive automorphism φ_a such that

$$\varphi = L_U \varphi_a,$$

where $a = \varphi^{-1}(0)$. Hence,

$$J_{\mathbb{C}}\varphi(z) = U J_{\mathbb{C}}\varphi_a(z),$$

where U is a unitary matrix. Since $|det\ (U)| = 1$ and using Lemma 1.3.14, we obtain

$$\begin{aligned} det\,(J_{\mathbb{R}}\varphi(z)) &= |det\,(J_{\mathbb{C}}\varphi(z))\,|^2 \\ &= |det\ (U)||det J_{\mathbb{C}}\varphi_a(z)|^2 \\ &= |det J_{\mathbb{C}}\varphi_a(z)|^2 \\ &= det J_{\mathbb{R}}\varphi_a(z) \\ &= \left(\frac{1-|a|^2}{|1- < z, a > |^2}\right)^{n+1}. \end{aligned}$$

□

1.3.5 The Operator $\widetilde{\Delta}$

In Chapter 2, we characterise the weighted Bergman spaces in terms of various derivatives of a function. Hence, in this subsection, we talk about a fundamental operator that will be necessary. Let's explain how this subsection is organised: fix $z \in \mathbb{B}_n$ and let φ_z be an involutive automorphism, we introduce an operator which is defined in terms of the ordinary Laplacian and the previous automorphism φ_z. It is called the invariant Laplacian. After that, we demonstrate that this operator commutes with the automorphisms of $\mathbb{B}_n$. To conclude, for a twice differentiable function f on $\mathbb{B}_n$, we get a formula that describes the invariant Laplacian of f in terms of the ordinary partial derivatives. Of course, we start with the extension, to several variables, of the ordinary Laplacian on $\mathbb{C}^n$:

$$\Delta := \sum_{k=1}^{n}\left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2}\right) \overbrace{=}^{(1.1.2)} 4\sum_{k=1}^{n}\frac{\partial^2}{\partial z_k \partial \overline{z}_k}. \tag{1.3.12}$$

Definition 1.3.17 (The invariant Laplacian)**.** Given a twice differentiable function f on $\mathbb{B}_n$ and indicating with φ_z the involutive automorphism, that interchanges the points 0 and z, we define

$$(\widetilde{\Delta} f)(z) := \Delta(f \circ \varphi_z)(0), \;\; z \in \mathbb{B}_n. \tag{1.3.13}$$

In this proposition, we prove that this operator is invariant under the automorphisms of $\mathbb{B}_n$.

Proposition 1.3.18. *Let f be a twice differentiable function on $\mathbb{B}_n$, then*

$$\widetilde{\Delta}(f \circ \varphi) = (\widetilde{\Delta} f) \circ \varphi, \quad \forall\ \varphi \in Aut(\mathbb{B}_n). \tag{1.3.14}$$

Proof. Fix an element $z \in \mathbb{B}_n$, let $\varphi \in Aut(\mathbb{B}_n)$ and denote by $a = \varphi(z)$. Hence, the automorphism

$$L_U := \varphi_a \circ \varphi \circ \varphi_z;$$

is such that $L_U(0) = 0$. From Lemma 1.3.5, L_U is an unitary transformation. Writing

$L_U(z) = (L_U^1(z), ..., L_U^n(z))$, where $L_U^j : \mathbb{C}^n \longrightarrow \mathbb{C}$. Let $g \in C^2(\mathbb{C}^n)$, we have that

$$\Delta(g \circ U)(0) = 4\sum_{i=1}^{n}\left(\sum_{j,k=1}^{n}\left(\frac{\partial^2 g}{\partial z_k \partial \overline{z}_j} \circ L_U(0)\right)\frac{\partial L_U^j}{\partial z_i}(0)\frac{\partial L_U^k}{\partial \overline{z}_i}(0) + \sum_{j=1}^{n}\left(\frac{\partial g}{\partial \overline{z}_j}\right) \circ L_U(0)\underbrace{\frac{\partial^2 L_U^j}{\partial z_i \partial \overline{z}_i}(0)}_{=0}\right)$$

$$= 4\sum_{j,k=1}^{n}\left(\sum_{i=1}^{n}\left(\frac{\partial^2 g}{\partial z_k \partial \overline{z}_j} \circ L_U(0)\right)\frac{\partial L_U^j}{\partial z_i}(0)\frac{\partial L_U^k}{\partial \overline{z}_i}(0)\right)$$

$$= 4\sum_{j,k=1}^{n}\left(\frac{\partial^2 g}{\partial z_k \partial \overline{z}_j} \circ L_U(0)\right)\left(\sum_{i=1}^{n}\frac{\partial L_U^j}{\partial z_i}(0)\frac{\partial L_U^k}{\partial \overline{z}_i}(0)\right)$$

$$= 4\sum_{j,k=1}^{n}\left(\frac{\partial^2 g}{\partial z_k \partial \overline{z}_j} \circ L_U(0)\right)|det J_{\mathbb{C}} L_U(0)|^2$$

$$= 4\sum_{k=1}^{n}\left(\frac{\partial^2 g}{\partial z_k \partial \overline{z}_k} \circ L_U(0)\right).$$

That is,

$$\Delta(g \circ U)(0) = \Delta(g)(0).$$

As a consequence, we obtain

$$\begin{aligned}\widetilde{\Delta}(f \circ \varphi)(z) &= \Delta(f \circ \varphi \circ \varphi_z)(0)\\ &= \Delta(f \circ \varphi_a \circ L_U)(0)\\ &= \Delta(f \circ \varphi_a)(0)\\ &= \widetilde{\Delta} f(a)\\ &= (\widetilde{\Delta} f) \circ \varphi(z)\end{aligned}$$

that is

$$\widetilde{\Delta}(f \circ \varphi)(z) = (\widetilde{\Delta} f) \circ \varphi(z).$$

□

The invariant Laplacian admits a description using ordinary partial derivatives. This property is shown in the following proposition.

Proposition 1.3.19. *Suppose that f is a twice differentiable function in $\mathbb{B}_n$, then*

$$(\widetilde{\Delta} f)(z) = 4(1 - |z|^2)\sum_{i,j}^{n}(\delta_{i,j} - z_i \overline{z}_j)\frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}(z), \ \forall z \in \mathbb{B}_n, \quad (1.3.15)$$

where $\delta_{i,j}$ denotes the Kronecker delta.

Proof. Fix $z \in \mathbb{B}_n$, we write the involutive automorphism as follows

$$\varphi_z(w) = (\varphi_1(w), ..., \varphi_n(w)), w \in \mathbb{B}_n.$$

Applying the chain rule, we get

$$(\widetilde{\Delta} f)(z) = \Delta(f \circ \varphi_z)(0)$$

$$= 4 \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial \overline{x}_j}(x) \sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial w_k}(0) \overline{\frac{\partial \varphi_j}{\partial w_k}(0)}.$$

Denoting by $s_z = \sqrt{1 - |z|^2}$, the representation in power series of φ_z is

$$\varphi_z(w) = z - s_z w + \frac{s_z}{1 + s_z} < w, z > z + \ldots ,$$

where we omitted all the terms having $w-$degree 2 or higher. From the above representation, we find

$$\frac{\partial \varphi_i}{\partial z_k}(0) = -s_z \delta_{ik} + \frac{s_z}{1 + s_z} \overline{z}_k z_i \quad , \quad \overline{\frac{\partial \varphi_j}{\partial w_k}(0)} = -s_z \delta_{jk} + \frac{s_z}{1 + s_z} w_k \overline{z}_j.$$

Finally, after some calculations

$$\sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial w_k}(0) \overline{\frac{\partial \varphi_j}{\partial w_k}(0)} = \sum_{k=1}^{n} \left(-s_z \delta_{ik} + \frac{s_z}{1 + s_z} \overline{z}_k z_i \right) \left(-s_z \delta_{jk} + \frac{s_z}{1 + s_z} z_k \overline{z}_j \right)$$

$$= s_z^2 \sum_{k=1}^{n} \left(\delta_{ik} - \frac{\overline{z}_k z_i}{1 + s_z} \right) \left(\delta_{jk} - \frac{z_k \overline{z}_j}{1 + s_z} \right)$$

$$= s_z^2 \left\{ \sum_{k=1}^{n} \delta_{ik} \delta_{jk} - \sum_{k=1}^{n} \delta_{ik} \frac{z_k \overline{z}_j}{1 + s_z} - \sum_{k=1}^{n} \delta_{jk} \frac{z_i \overline{z}_k}{1 + s_z} + \frac{z_i \overline{z}_j}{(1 + s_z)^2} \sum_{k=1}^{n} |z_k|^2 \right\}$$

$$= s_z^2 \left\{ \delta_{ij} - z_i \overline{z}_j + z_i \overline{z}_j \left[1 - \frac{2}{1 + s_z} + \frac{|z|^2}{(1 + s_z)^2} \right] \right\}$$

$$= s_z^2 \left\{ \delta_{ij} - z_i \overline{z}_j + z_i \overline{z}_j \left[\frac{1 + s_z^2 + 2s_z - 2 - 2s_z + |z|^2}{(1 + s_z)^2} \right] \right\}$$

$$= s_z^2 \left\{ \delta_{ij} - z_i \overline{z}_j + z_i \overline{z}_j \left[\frac{|z|^2 - 1 + s_z^2}{(1 + s_z)^2} \right] \right\}$$

$$= s_z^2 \left\{ \delta_{ij} - z_i \overline{z}_j \right\},$$

where, in the last step, we used $s_z^2 = 1 - |z|^2$. Hence,

$$\sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial w_k}(0) \overline{\frac{\partial \varphi_j}{\partial w_k}(0)} = \delta_{ij} - z_i \overline{z}_j,$$

and we are done. □

1.4 Weighted L^p spaces

In this thesis, we will be interested in spaces of holomorphic functions, on the open unit ball, for which the $p-th$ power of the absolute value is Lebesgue integrable with respect to a weighted measure. The main purpose of this section is to provide a coherent exposition of the most importants objects concerning the weighted Lebesgue spaces which are necessary in this thesis.
The current section is organised as follows: we fix the weighted Lebesgue measure and collect some

of its main properties which will have many consequences in the holomorphic spaces studied in this dissertation.
Then, we present two crucial results regarding the behaviour of some integral operators defined on L^p: the first describes the asymptotic behaviour of a family of integral transforms, the second one is about the boundedness of integral operators having positive kernel.
This section ends with a change of variable formula that, as a consequence, defines an invariant measure under the automorphism group $Aut(\mathbb{B}_n)$.

1.4.1 The weighted measure dv_α

For sake of simplicity, we denote by dv the normalised standard Lebesgue volume measure on $\mathbb{B}_n$. In the following definition, we introduce the weighted normalised volume measure on $\mathbb{B}_n$.

Definition 1.4.1. If $\alpha > -1$, the weighted finite measure on $\mathbb{B}_n$ is defined as:

$$dv_\alpha(z) := \left\{ \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \right\} (1-|z|^2)^\alpha dv(z) \tag{1.4.1}$$

so that $v_\alpha(\mathbb{B}_n) = 1$ and where Γ denotes the gamma function. Furthermore, we denote by $L^p(\mathbb{B}_n, dv_\alpha)$, $0 \le p < \infty$, the Lebesgue space of measure equivalence classes of function such that

$$||f||_{p,\alpha} := \left(\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \right)^{1/p} < \infty. \tag{1.4.2}$$

We let $L^\infty(\mathbb{B}_n, dv_\alpha)$ denote the space of essentially bounded functions on $\mathbb{B}_n$. For $f \in L^\infty(\mathbb{B}_n, dv_\alpha)$, we define

$$||f||_{\infty,\alpha} = \operatorname{ess\,sup} \left\{ |f(z)| : z \in \mathbb{B}_n \right\}.$$

The space $L^\infty(\mathbb{B}_n, dv_\alpha)$ is a Banach space with the above norm.

Remark 1.4.2. If $1 \le p < \infty$, the space $L^p(\mathbb{B}_n, dv_\alpha)$ is a Banach space with the norm $|| \cdot ||_{p,\alpha}$. When $0 < p < 1$, $L^p(\mathbb{B}_n, dv_\alpha)$ is a complete metric space with the following distance:

$$\rho(f, g) = ||f - g||_{p,\alpha}^p.$$

In particular, $L^2(\mathbb{B}_n, dv_\alpha)$ is a Hilbert space whose inner product is denoted by $<,>_\alpha$.

Now we show a relevant property for this measure: the invariance under unitary transformation.

Proposition 1.4.3. *Let $\alpha > -1$, then the unitary invariance of dv_α holds. That is*

$$\int_{\mathbb{B}_n} f(L_U(z)) dv_\alpha(z) = \int_{\mathbb{B}_n} f(z) dv_\alpha(z), \ \forall \ f \in L^1(\mathbb{B}_n, dv_\alpha), \tag{1.4.3}$$

for every unitary transformation L_U.

Proof. Using the basic properties of the unitary transformation, putting $w := Uz$, we find:

$$\begin{aligned}
\int_{\mathbb{B}_n} f(L_U(z)) dv_\alpha(z) &= \int_{\mathbb{B}_n} f(Uz) \left\{ \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \right\} (1-|z|^2)^\alpha dv(z) \\
&= \int_{\mathbb{B}_n} f(w) \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} (1-|U^{-1}w|^2) |det J_{\mathbb{C}} L_U^{-1}(w)| dv(w) \\
&= \int_{\mathbb{B}_n} f(w) \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} (1-|w|^2) \underbrace{|det(U^{-1})|}_{=1} dv(w) \\
&= \int_{\mathbb{B}_n} f(w) dv_\alpha(w),
\end{aligned}$$

where $f \in L^1(\mathbb{B}_n, dv_\alpha)$. $\square$

1.4.2 L^p boundedness of a family of integral operators

Many operator-theoretic problems in the analysis of Bergman spaces involve estimating integral operators whose kernel is a function of the Bergman kernel. We present two estimates that we will use to prove the boundedness of some integral operator defined in the Bergman space.
The following theorem states asymptotic estimates, for a family of integral functions whose integration is performed over the open unit ball. We use the symbol $\backsim$ to indicate that two quantities have the same behaviour asymptotically. Thus, $A \backsim B$ means that A/B is bounded from above and below by two positive constants in the limit process in question. For what concerns the proof of this theorem, the interested reader can find it, for example, in the book of Zhu, Kehe, Spaces of Holomorphic Functions in the Unit Ball.

Theorem 1.4.4. *Let $c \in \mathbb{R}$ and $t > -1$, we introduce*

$$J_{c,t}(z) := \int_{\mathbb{B}_n} \frac{(1-|w|^2)^t}{|1-<z,w>|^{n+1+t+c}} dv(z), \;\; z \in \mathbb{B}_n.$$

Then, $J_{c,t}$ has the following asymptotic behaviour :
1) If $c < 0$, then $J_{c,t}$ is bounded in $\mathbb{B}_n$.
2) If $c = 0$, then

$$J_{c,t}(z) \backsim log\frac{1}{1-|z|^2}, \quad when \quad |z| \to 1^-.$$

3) If $c > 0$, then

$$J_{c,t}(z) \backsim \frac{1}{(1-|z|^2)^c}, \quad when \quad |z| \to 1^-.$$

The next theorem is an essential and general result concerning integral operators with non-negative kernel: Schur's test. This theorem will be used to describe the boundedness of a class of integral operators induced by Bergman type kernels on weighted Bergman spaces. The proof is a fairly simple application of Holder's inequality.

Theorem 1.4.5. *Let $(\mathcal{X}, d\mu_{\mathcal{X}})$, $(\mathcal{Y}, d\mu_{\mathcal{Y}})$ be measure spaces. Let T be the integral operator given by*

$$Tf(x) := \int_{\mathcal{Y}} K(x,y) f(y) \mu_{\mathcal{Y}}(y),$$

where K is a measurable positive kernel on $\mathcal{X} \times \mathcal{Y}$. Let $1 < p, q < +\infty$ be conjugate exponents. Suppose there exist positive functions $\phi : \mathcal{Y} \longrightarrow (0, +\infty)$, $\varphi : \mathcal{X} \longrightarrow (0, +\infty)$ such that

$$1) \int_{\mathcal{Y}} K(x,y)\phi^q(y) d\mu_{\mathcal{Y}}(y) \leq C\varphi(x)^q \qquad and \qquad 2) \int_{\mathcal{X}} K(x,y)\varphi^p(x) d\mu_{\mathcal{X}}(x) \leq C\phi(y)^p,$$

then $T : L^p(\mathcal{Y}) \longrightarrow L^p(\mathcal{X})$ is bounded.

Finally, we are going to need the following integral estimate.

Theorem 1.4.6. *Suppose a and α are two real parameters. Define two integral operators T and S by*

$$T\phi(z) := (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{\phi(w) dv_\alpha(w)}{(1-<z,w>)^{n+1+a+\alpha}}, \;\; \phi \in L^p(\mathbb{B}_n, dv_\alpha)$$

and

$$S\phi(z) := (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{\phi(w) dv_\alpha(w)}{|1-<z,w>|^{n+1+a+\alpha}}, \;\; \phi \in L^p(\mathbb{B}_n, dv_\alpha).$$

Then, for $-\infty < t < \infty$ and $1 \leq p < \infty$, the following conditions are equivalent:
a) T is bounded on $L^p(\mathbb{B}_n, dv_t)$.
b) S is bounded on $L^p(\mathbb{B}_n, dv_t)$.
c) $-pa < t+1 < p(\alpha+1)$.

1.4.3 Change of variables formula and Applications

During the study of the weighted Bergman space, we wish to obtain the boundedness of the point evaluations linear functionals. For this goal, we will need the following change of variable formula.

Proposition 1.4.7. *Let $\alpha > -1$ and $f \in L^1(\mathbb{B}_n, dv_\alpha)$, then*

$$\int_{\mathbb{B}_n} f \circ \varphi(z) dv_\alpha(z) = \int_{\mathbb{B}_n} f(z) \frac{(1-|a|^2)^{n+1+\alpha}}{|1- < z, a > |^{2(n+1+\alpha)}} dv_\alpha(z), \ \forall \ \varphi \in Aut(\mathbb{B}_n), \tag{1.4.4}$$

where $a = \varphi(0)$.

Proof. Every automorphism $\varphi \in Aut(\mathbb{B}_n)$ can be written as follows

$$\varphi = \varphi_a L_U,$$

where L_U is an unitary tranformation, φ_a is a involutive automorphism and $a = \varphi(0)$. Since dv_α is invariant under unitary transformations, assuming that $\varphi = \varphi_a$, we have that $\varphi^{-1} = \varphi$. Then, applying Lemma 1.3.14,

$$\int_{\mathbb{B}_n} f \circ \varphi(w) dv_\alpha(w) \overbrace{=}^{z=\varphi(w)} \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \int_{\mathbb{B}_n} f(z)(1-|z|^2)^\alpha \left(\frac{1-|a|^2}{|1- < z, a > |^2} \right)^{n+1} dv(z).$$

□

As a consequence, we have the following formulas. For their proof, the interested reader can find, for example, in the book of Kehe Zhu, Spaces of Holomorphic Functions in the Unit Ball.

Corollary 1.4.8. *If n and k are multi-indexes of nonnegative integers, then*

$$\int_{\mathbb{B}_n} z^m \overline{z^k} dv_\alpha(z) = \begin{cases} 0, & \text{if } m \neq k \\ \dfrac{m!\Gamma(n+\alpha+1)}{\Gamma(n+|m|+\alpha+1)}, & \text{if } m = k. \end{cases} \tag{1.4.5}$$

1.5 Differentiation

The purpose of this section is to illustrate the most important notions of differentiation on $\mathbb{B}_n$. Using these tools, we will characterise the weighted Bergman spaces in terms of various derivatives. We focus our attention on two different notions of differentiation. We start introducing the notion of radial derivative and, for a given function $f \in H(\mathbb{B}_n)$ that admits homogenous expansion in a neighborhood of the origin, we explicitely compute this type of derivative. Then, we prove a significant property: a holomorphic function can be obtained from its radial derivative. Finally, we define a family of invertible operators on the space $H(\mathbb{B}_n)$ and collect some fundamental properties.

1.5.1 Notions of differentiation

Definition 1.5.1 (Radial derivative)**.** For a holomorphic function f on $\mathbb{B}_n$, the radial derivative is

$$Rf(z) = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z). \tag{1.5.1}$$

The radial derivative will be used to produce equivalent norms on the weighted Bergman space.

Remark 1.5.2. In other words, for a fixed $z \in \mathbb{B}_n$, the radial derivative is a particular case of the directional derivative along the vector z. Hence, the radial derivative can be alternatively defined as

$$Rf(z) = \lim_{r \to 0} \frac{f(z+rz) - f(z)}{r}, \ r \in \mathbb{R}.$$

Proposition 1.5.3. *Let $f \in H(\mathbb{B}_n)$, suppose there exists a sequence of homogenous polynomials $(f_k(z))_{k\in\mathbb{N}}$, each one of them having degree k, and a polydisc $P(a,r)$, such that $\overline{P(a,r)} \subset \mathbb{B}_n$, so that*

$$f(z) = \sum_{k=0}^{+\infty} f_k(z) = \sum_{k=0}^{+\infty} \sum_{|i|=k} a_i z^i, \forall z \in P(a,r), \tag{1.5.2}$$

where the convergence is uniform on compact subsets of $P(a,r)$. Then, we obtain

$$Rf(z) = \sum_{k=1}^{+\infty} k f_k(z), \forall z \in P(a,r).$$

Proof.

$$Rf(z) = \sum_{j=0}^{+\infty} \sum_{|i|=j} \sum_{k=0}^{n} z_k a_{i_1,\dots,i_k} \frac{\partial}{\partial z_k} z_1^{i_1} \dots z_k^{i_k} \dots z_n^{i_n}$$

$$= \sum_{j=0}^{+\infty} \sum_{|i|=j} a_{i_1,\dots,i_k} z_1^{i_1} \dots z_k^{i_k} \dots z_n^{i_n} \overbrace{\sum_{k=0}^{n} i_k}^{=j}.$$

□

In the next proposition, our aim is to prove that we can recover a holomorphic function from its radial derivative.

Proposition 1.5.4. *Let $f \in H(\mathbb{B}_n)$. Then, the following formula holds*

$$\int_0^1 \frac{Rf(tz)}{t} dt = f(z) - f(0), \ \forall z \in \mathbb{B}_n. \tag{1.5.3}$$

Proof.

$$\begin{aligned} f(z) - f(0) &= \int_0^1 \frac{df}{dt}(tz)dt \\ &= \int_0^1 \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}(tz) z_k dt \\ &= \int_0^1 \frac{1}{t} \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}(tz)(tz_k) dt \\ &= \int_0^1 \frac{Rf(tz)}{t} dt \end{aligned}$$

□

1.5.2 The operator $R^{\alpha,t}$

An important tool in the study of holomorphic function spaces is the notion of fractional differential. There are numerous types of fractional differential, we introduce one that is intimately related to and interacts well with the Bergman kernel functions. More is true, the following family of operators yields an equivalent norm for both spaces of holomorphic functions studied in this thesis.

Definition 1.5.5. Let $\alpha, t \in \mathbb{R}$ such that neither $n+\alpha$ nor $n+\alpha+t$ is a negative integer, we introduce the operator $R^{\alpha,t}$ as follows

$$R^{\alpha,t}f(z) := \sum_{k=0}^{+\infty} \frac{\Gamma(n+1+\alpha)\Gamma(n+1+k+\alpha+t)}{\Gamma(n+1+\alpha+t)\Gamma(n+1+k+\alpha)} f_k(z), \ z \in P(0,r), \tag{1.5.4}$$

where $P(0,r)$ is a polydisc such that $\overline{P(0,r)} \subset \mathbb{B}_n$. Furthermore, the convergence is uniform on compact subsets of $P(0,r)$.

Proposition 1.5.6. *The operator $R^{\alpha,t}$ sends $H(\mathbb{B}_n)$ into itself. Moreover, $R^{\alpha,t}$ admits inverse operator $R_{\alpha,t}$ given by,*

$$R_{\alpha,t}f(z) := \sum_{k=0}^{+\infty} \frac{\Gamma(n+1+\alpha+t)\Gamma(n+1+k+\alpha)}{\Gamma(n+1+\alpha)\Gamma(n+1+k+\alpha+t)} f_k(z), \ z \in P(0,r), \tag{1.5.5}$$

and the convergence is uniform on compact subsets of $P(0,r)$.

Proof. Assume that the coordinates $z_1, ..., z_{j-1}, z_{j+1}, ..., z_n$, in (1.5.4), are given some fixed values $a_1, ..., a_{j-1}, a_{j+1}, ..., a_n$. Then, applying the ratio test, the power series is a convergent power series in z_j on $D(0, r_j)$. That is, we have proved that $R^{\alpha,t}$ is holomorphic in each variable separately. By Hartogs Theorem, we conclude that $R^{\alpha,t} \in H(\mathbb{B}_n)$. Furthermore, using the Identity Theorem, we deduce that $R^{\alpha,t}$ admits inverse operator. Hence,

$$\begin{aligned} R_{\alpha,t} \circ R^{\alpha,t} f(z) &= R_{\alpha,t}\left(\sum_{k=0}^{+\infty} \frac{\Gamma(n+1+\alpha)\Gamma(n+1+k+\alpha+t)}{\Gamma(n+1+\alpha+t)\Gamma(n+1+k+\alpha)} f_k(z)\right) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha)\Gamma(n+1+k+\alpha+t)}{\Gamma(n+1+\alpha+t)\Gamma(n+1+k+\alpha)} \frac{\Gamma(n+1+\alpha+t)\Gamma(n+1+k+\alpha)}{\Gamma(n+1+\alpha)\Gamma(n+1+k+\alpha+t)} f_k(z) \\ &= \sum_{k=0}^{\infty} f_k(z) \end{aligned}$$

□

Remark 1.5.7. In particular, considering $\alpha = -n$,

$$R^t f(z) := R^{-n,t} f(z) = \sum_{k=0}^{+\infty} \frac{\Gamma(1)\Gamma(1+k+t)}{\Gamma(1+t)\Gamma(1+k)} f_k(z) = \sum_{k=1}^{+\infty} k^t f_k(z).$$

Definition 1.5.8. The linear operator R^t is called fractional radial derivative.

Proposition 1.5.9. *The operator R^t is invertible with inverse given by*

$$R_t f(z) := R^{-t} f(z) = \sum_{k=1}^{+\infty} k^{-t} f_k(z). \tag{1.5.6}$$

Moreover, if we endow $H(\mathbb{B}_n)$ with the topology of uniform convergence on compact subsets, the operators R^t and R_t are continuous.

If $f(z) := \dfrac{1}{(1-<z,w>)^{n+1+\alpha}}$, then the operators $R^{\alpha,t}$ and $R_{\alpha,t}$ have the following form.

Proposition 1.5.10. *Let $\alpha, t \in \mathbb{R}$ so that neither $n+\alpha$ nor $n+\alpha+t$ is a negative integer. Then the operators $R^{\alpha,t}$ and $R_{\alpha,t}$ are the only continuous operators on $H(\mathbb{B}_n)$ that satisfy*

$$R^{\alpha,t}\left(\frac{1}{(1-<z,w>)^{n+1+\alpha}}\right) = \frac{1}{(1-<z,w>)^{n+1+\alpha+t}}, \ \forall\, w \in \mathbb{B}_n, \tag{1.5.7}$$

and

$$R_{\alpha,t}\left(\frac{1}{(1-<z,w>)^{n+1+\alpha+t}}\right) = \frac{1}{(1-<z,w>)^{n+1+\alpha}}, \ \forall w \ \in \mathbb{B}_n. \tag{1.5.8}$$

Proof. We observe that

$$\frac{1}{(1-<z,w>)^{n+1+\alpha}} = \sum_{k=0}^{\infty} \underbrace{\frac{\Gamma(n+1+k+\alpha)}{k!\Gamma(n+1+\alpha)} <z,w>^k}_{f_k(z)},$$

hence

$$R^{\alpha,t}\left(\frac{1}{(1-<z,w>)^{n+1+\alpha}}\right) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha)\Gamma(n+1+k+\alpha+t)}{\Gamma(n+1+\alpha+t)\Gamma(n+1+k+\alpha)} \frac{\Gamma(n+1+k+\alpha)}{k!\Gamma(n+1+\alpha)} <z,w>^k$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(n+1+k+\alpha+t)}{k!\Gamma(n+1+\alpha+t)} <z,w>^k$$

$$= \frac{1}{(1-<z,w>)^{n+1+\alpha+t}}.$$

Similarly, we prove (1.5.8.). □

We prove a further property that gives an alternative description of $R^{\alpha,t}$ in terms of polynomials and standard partial derivatives. This result will be used, in the next chapters, to show that the spaces of holomorphic functions of our interest can be decomposed into a series of very particular functions.

Proposition 1.5.11. *Assume N is a positive integer and α is a real number such that $n+\alpha$ is not negative integer. Then $R^{\alpha,N}$, considered as an operator which acts on $H(\mathbb{B}_n)$, is a linear partial differential operator whose order is N having polynomial coefficients, that is*

$$R^{\alpha,N} f(z) = \sum_{|m|\le N} p_m(z) \frac{\partial^m f}{\partial z^m}(z) \tag{1.5.9}$$

where p_m are polynomials.

Proof. Fix $w \in \mathbb{B}_n$, using the binomial formula,

$$\frac{1}{(1-<z,w>)^{n+1+\alpha+N}} = \frac{(1-<z,w>+<z,w>)^N}{(1-<z,w>)^{n+1+\alpha+N}}$$

$$= \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} \frac{<z,w>^k}{(1-<z,w>)^{n+1+\alpha+k}}.$$

Furthermore, for each k, using the multi-nomial formula, we can write

$$<z,w>^k = (z_1\overline{w}_1 + \ldots + z_n\overline{w}_n)^k$$

$$= \sum_{|m|=k} \frac{k!}{m!} z^m \overline{w}^m.$$

Then, there exists a family of constants $c_{m,k}$ such that

$$\frac{N!}{k!(N-k)!} \frac{<z,w>^k}{(1-<z,w>)^{n+1+\alpha+k}} = \sum_{|m|=k} c_{m,k} z^m \frac{\partial^m}{\partial z^m} \frac{1}{(1-<z,w>)^{n+1+\alpha}},$$

using Proposition 1.5.11, we obtain

$$R^{\alpha,N} \frac{1}{(1-<z,w>)^{n+1+\alpha}} = \sum_{k=0}^{N} \sum_{|m|=k} c_{m,k} z^m \frac{\partial^m}{\partial z^m} \frac{1}{(1-<z,w>)^{n+1+\alpha}}.$$

□

1.6 The Bergman Metric

In this section, we introduce and collect the basic properties of the Bergman metric. This metric derives from the Bergman kernel. The current section is organised as follows. First of all, we give the definition of the Bergman kernel for the open unit ball. Then, we give a detailed description of the Bergman matrix, so that we will formulate a rigorous definition of the Bergman metric. This section ends with the notion of the Bergman metric ball that will be used to decompose the open unit ball and compute the atomic decomposition of the holomorphic spaces of our interest.

Definition 1.6.1. For the open unit ball $\mathbb{B}_n$, we have that the Bergman kernel is given by

$$K(z,w) = \frac{1}{(1- < z, w >)^{n+1}}, \ z, w \in \mathbb{B}_n. \tag{1.6.1}$$

1.6.1 The Bergman matrix

In this paragraph, we discuss the Bergman matrix. We start giving the definition and, then, we prove that this matrix admits a description in terms of the involutive automorphism, the projections, P_z and Q_z, of $\mathbb{C}^n$. Then, we will have some crucial consequences such as the positivity and the invertibility of the Bergman matrix. More is true, we show that the invariance under automorphism holds for the Bergman matrix.

Definition 1.6.2 (Bergman matrix)**.** The Bergman matrix of $\mathbb{B}_n$ is the $n \times n$ complex matrix

$$B(z) := (b_{ij}(z))_{ij} := \frac{1}{n+1} \begin{pmatrix} \frac{\partial^2}{\partial \overline{z}_1 \partial z_1} logK(z,z) & \cdots & \frac{\partial^2}{\partial \overline{z}_1 \partial z_n} logK(z,z) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial \overline{z}_n \partial z_1} logK(z,z) & \cdots & \frac{\partial^2}{\partial \overline{z}_n \partial z_n} logK(z,z) \end{pmatrix}. \tag{1.6.2}$$

We briefly recall that denoting by

$$A(z) := (z_i \overline{z}_j)_{n \times n},$$

then, fixed $z \neq 0$, the orthogonal projection P_z from $\mathbb{C}^n$ onto the one-dimensional subspace $[z]$ generated by z can be written as

$$P_z = \frac{A(z)}{|z|^2}.$$

As well as, the orthogonal projection Q_z from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus [z]$ is written as

$$Q_z = I - \frac{A(z)}{|z|^2}.$$

After this short review, we can provide the following result.

Proposition 1.6.3. *For $z \in \mathbb{B}_n$, let $B(z)$ be the Bergman matrix. Then the following properties hold:*
a) $B(z) = [(1-|z|^2)I + A(z)]/(1-|z|^2)^2$.
b) $B(z)^{-1} = (1-|z|^2)[I - A(z)]$.
c) $B(z) = P_z/(1-|z|^2)^2 + Q_z/(1-|z|^2), \ z \neq 0$.
d) $det(B(z)) = K(z,z)$,
where φ_z denotes the involutive automorphism.

Proof. a) By the fact that

$$K(z,z) = \frac{1}{(1-|z|^2)^{n+1}},$$

it follows that

$$\log K(z,z) = -(n+1)\log(1-|z|^2),$$

hence

$$\frac{\partial}{\partial z_j} \log K(z,z) = (n+1)\frac{\overline{z}_j}{1-|z|^2}, \text{ for } j = 1, ..., n.$$

For what concerns the second mixed partial derivatives, we find:

$$\frac{\partial^2}{\partial \overline{z}_i \partial z_j} \log K(z,z) = (n+1)\frac{(1-|z|^2)\delta_{ij} + z_i\overline{z}_j}{(1-|z|^2)^2} \ , \text{ for } i,j = 1, ..., n.$$

Then, the Bergman matrix can be written as

$$B(z) = \frac{(1-|z|^2)I + A(z)}{(1-|z|^2)^2}.$$

b) Furthermore, since $A^2(z) = |z|^2 A(z)$, after some lines of calculations the following identity holds

$$(I - A(z))((1-|z|^2)I + A(z)) = (1-|z|^2)I,$$

so that

$$B(z)^{-1} = (1-|z|^2)(I - A(z)).$$

c) Using *a*)

$$B(z) = \frac{I}{(1-|z|^2)} - \frac{P_z}{(1-|z|^2)} + \frac{P_z}{(1-|z|^2)} + \frac{A(z)}{(1-|z|^2)^2}$$

$$= \frac{Q_z}{(1-|z|^2)} + \frac{P_z - \overbrace{|z|^2 P_z}^{=A(z)} + A(z)}{(1-|z|^2)^2}$$

$$= \frac{Q_z}{(1-|z|^2)} + \frac{P_z}{(1-|z|^2)^2}.$$

d) Since

$$J_{\mathbb{C}}\varphi_z(z) = -\left(\frac{P_z}{1-|z|^2} + \frac{Q_z}{\sqrt{1-|z|^2}}\right),$$

and

$$P_z^2 = P_z, \quad Q_z^2 = Q_z, \quad P_z Q_z = 0,$$

where φ_z is a involutive automorphism. We get

$$(J_{\mathbb{C}}\varphi_z(z))^2 = \frac{P_z}{(1-|z|^2)^2} + \frac{Q_z}{(1-|z|^2)}$$

$$= \frac{I - |z|^2 I + |z|^2 P_z}{(1-|z|^2)}$$

$$= \frac{1}{(1-|z|^2)^2}((1-|z|^2)I + |z|^2 P_z).$$

Recalling that $A(z) = |z|^2 P_z$, we obtain

$$B(z) = (J_{\mathbb{C}}\varphi_z(z))^2.$$

Since $J_{\mathbb{C}}\varphi_z(z)$ is self-adjoint and using Lemma 1.3.14, it turns out that

$$\begin{aligned}\det(B(z)) &= |\det(J_{\mathbb{C}}\varphi_z(z))|^2 \\ &= det J_{\mathbb{R}}\varphi_z(z)\left(\frac{1-|z|^2}{|1-<z,z>|^2}\right)^{n+1} \\ &= \left(\frac{1-|z|^2}{(1-|z|^2)^2}\right)^{n+1}.\end{aligned}$$

□

We immediately deduce the following corollary.

Corollary 1.6.4. *The Bergman matrix $B(z)$ is positive and invertible.*

In the next proposition, we prove that the Bergman matrix is invariant under automorphisms.

Proposition 1.6.5 (Invariance under automorphism)**.** *The Bergman matrix $B(z)$ satisfies*

$$B(z) = \overline{(J_{\mathbb{C}}\varphi(z))}B(\varphi(z))J_{\mathbb{C}}\varphi(z), \ \forall \ z \in \mathbb{B}_n, \forall \ \varphi \in Aut(\mathbb{B}_n). \tag{1.6.3}$$

Proof. Assume that $\varphi = \varphi_a$, where φ_a is an involutive automorphism, for some $a \in \mathbb{B}_n$. Hence, using (1.3.3) and (1.3.11), the Bergman kernel satisfies

$$K(z,z) = |det(J_{\mathbb{C}}\varphi(z))|^2 K(\varphi(z),\varphi(z)), \forall z \in \mathbb{B}_n.$$

We obtain

$$\log K(z,z) = \log|det(J_{\mathbb{C}}\varphi(z))|^2 + \log K(\varphi(z),\varphi(z)),$$

but since

$$\log|det(J_{\mathbb{C}}\varphi(z))|^2 = \log(det(J_{\mathbb{C}}\varphi(z))det(\overline{J_{\mathbb{C}}\varphi(z)})) = \log(det(J_{\mathbb{C}}\varphi(z))) + \log(det(\overline{J_{\mathbb{C}}\varphi(z)})).$$

Now

$$\begin{aligned}\frac{\partial^2}{\partial\overline{z}_i\partial z_j}\log|det(J_{\mathbb{C}}\varphi(z))|^2 &= (n+1)\frac{\partial}{\partial\overline{z}_i}\frac{\partial}{\partial z_j}\log\left(\frac{1-|a|^2}{|1-<z,a>|^2}\right) \\ &= (n+1)\frac{\partial}{\partial\overline{z}_i}\frac{\partial}{\partial z_j}\left\{\log\left(1-|a|^2\right) - \log\left(|1-<z,a>|^2\right)\right\} \\ &= -(n+1)\frac{\partial}{\partial\overline{z}_i}\frac{\partial}{\partial z_j}\left\{\log\left(|1-<z,a>|^2\right)\right\} \\ &= \overline{a}_j(n+1)\frac{\partial}{\partial\overline{z}_i}\frac{(1-\overline{<z,a>})}{|1-<z,a>|^2} \\ &= \overline{a}_j(n+1)\frac{\partial}{\partial\overline{z}_i}\frac{1}{1-<z,a>} \\ &= 0.\end{aligned}$$

Writing the automorphism as follows

$$\varphi(z) = (\varphi_1(z), ..., \varphi_n(z)), \ z \in \mathbb{B}_n,$$

applying the chain rule, we find that :

$$\frac{\partial}{\partial z_j}\log K(z,z)=\frac{\partial}{\partial z_j}\log|J_{\mathbb{C}}\varphi(z)|^2+\sum_{k=1}^{n}\frac{\partial}{\partial\varphi_k}\log K(\varphi(z),\varphi(z))\frac{\partial\varphi_k}{\partial z_j}.$$

Applying $\frac{\partial^2}{\partial\bar{z}_i}$ and the chain rule, we obtain

$$\frac{\partial^2}{\partial\bar{z}_i\partial z_j}\log K(z,z)=\sum_{k=1}^{n}\frac{\partial\varphi_k}{\partial z_j}\sum_{m=1}^{n}\frac{\partial^2}{\partial\varphi_k\partial\overline{\varphi}_m}\log K(\varphi(z),\varphi(z))\overline{\left(\frac{\partial\varphi_m}{\partial z_i}\right)},\forall\ i,j=1,...,n.$$

Finally, let $\varphi\in Aut(\mathbb{B}_n)$ be, then we can find an involutive automorphism φ_a and a unitary transformation L_U such that $\varphi=L_U\varphi_a$. Hence, arguing in the same way we complete the proof. □

1.6.2 The Bergman metric

For a rigorous definition of the Bergman metric, we need to introduce some tools. Fix $z,w\in\mathbb{B}_n$, given a smooth curve $\gamma:[0,1]\longrightarrow\mathbb{B}_n$, so that $\gamma(0)=z$ and $\gamma(1)=w$, we define

$$l(\gamma):=\int_0^1\left(\sum_{i,j=1}^{n}b_{ij}(\gamma(t))\gamma_i'(t)\overline{\gamma_j'(t)}\right)^{\frac{1}{2}}dt=\int_0^1<B(\gamma(t))\gamma'(t),\gamma'(t)>^{\frac{1}{2}}dt \qquad (1.6.4)$$

and call

$$C_{z,w}:=\{\ \gamma:[0,1]\longrightarrow\mathbb{B}_n\mid\gamma\text{ piecewise smooth curve s.t. }\gamma(0)=z,\quad\gamma(1)=w\ \}.$$

Remark 1.6.6. Using point $c)$ in Proposition 1.6.3, we deduce

$$<B(\gamma(t))\gamma'(t),\gamma'(t)>=\frac{|\gamma'(t)|^2-|\gamma'(t)|^2|\gamma(t)|^2+|<\gamma(t),\gamma'(t)>|^2}{(1-|\gamma(t)|^2)^2}.$$

Now, applying Cauchy-Schwarz, we get

$$\frac{|\gamma'(t)|^2-|\gamma'(t)|^2|\gamma(t)|^2+|<\gamma(t),\gamma'(t)>|^2}{(1-|\gamma(t)|^2)^2}\le\frac{|\gamma'(t)|^2}{(1-|\gamma(t)|^2)^2}.$$

Furthermore, we obtain an estimate from below as follows

$$\begin{aligned}\frac{|\gamma'(t)|^2-|\gamma'(t)|^2|\gamma(t)|^2+|<\gamma(t),\gamma'(t)>|^2}{(1-|\gamma(t)|^2)^2}&\ge\frac{|\gamma'(t)|^2-|\gamma'(t)|^2|\gamma(t)|^2}{(1-|\gamma(t)|^2)^2}\\&=\frac{|\gamma'(t)|^2(1-|\gamma(t)|^2)}{(1-|\gamma(t)|^2)^2}\\&=\frac{|\gamma'(t)|^2}{1-|\gamma(t)|^2}.\end{aligned}$$

In other words,

$$\int_0^1\frac{|\gamma'(t)|}{\sqrt{1-|\gamma(t)|^2}}dt\le l(\gamma)\le\int_0^1\frac{|\gamma'(t)|}{1-|\gamma(t)|^2}dt.$$

As a consequence, we prove that $l(\gamma)$ is bounded. Assume without loss of generality that $\gamma(0)=0$ and

$$Re(<\gamma(t),\gamma'(t)>)\ge0,\ \forall\ t\in[0,1],$$

then, integrating by parts, it turns out that

$$\int_0^1 \frac{|\gamma'(t)|dt}{1-|\gamma(t)|^2} \leq \left\{ \frac{t}{1-|\gamma(t)|^2}\bigg|_{t=0}^{t=1} + \int_0^1 t\frac{2Re(<\gamma(t),\gamma'(t)>)}{(1-|\gamma(t)|^2)^2}dt \right\} \sup_{t\in[0,1]} |\gamma'(t)|$$

$$\leq \left\{ \frac{1}{1-|\gamma(1)|^2} + \int_0^1 \frac{2Re(<\gamma(t),\gamma'(t)>)}{(1-|\gamma(t)|^2)^2}dt \right\} \sup_{t\in[0,1]} |\gamma'(t)|$$

$$= \left\{ \frac{|\gamma(1)|^2+1}{1-|\gamma(1)|^2} \right\} \sup_{t\in[0,1]} |\gamma'(t)|,$$

that is

$$l(\gamma) \leq \left\{ \frac{|\gamma(1)|^2+1}{1-|\gamma(1)|^2} \right\} \sup_{t\in[0,1]} |\gamma'(t)|.$$

Finally, since $\sqrt{1-|\gamma(t)|^2} \leq 1$, we find that

$$l(\gamma) \geq \int_0^1 |\gamma'(t)|dt.$$

Definition 1.6.7 (Bergman metric)**.** The Bergman metric is

$$\beta(z,w) := \inf_{\gamma\in C_{z,w}} l(\gamma). \tag{1.6.5}$$

Remark 1.6.8. It is clear that Remark 1.6.6 implies that $\beta : \mathbb{B}_n \times \mathbb{B}_n \longrightarrow [0,+\infty)$ is a finite metric.

We focus on a crucial property: the Bergman metric is invariant under automorphisms.

Proposition 1.6.9. *Let β be the Bergman metric, the following property holds:*

$$\beta(\varphi(z),\varphi(w)) = \beta(z,w), \forall\ z,w \in \mathbb{B}_n, \tag{1.6.6}$$

where $\varphi \in Aut(\mathbb{B}_n)$.

Proof. Using Proposition 1.6.5 we have that

$$\beta(\varphi(z),\varphi(w)) = \inf_{\varphi(\gamma)\in C_{z,w}} \int_0^1 < B(\varphi(\gamma(t)))J_{\mathbb{C}}\varphi(\gamma(t))\gamma'(t), J_{\mathbb{C}}\varphi(\gamma(t))\gamma'(t) >^{\frac{1}{2}} dt$$

$$= \inf_{\varphi(\gamma)\in C_{z,w}} \int_0^1 < J_{\mathbb{C}}\varphi(\gamma(t))^* B(\varphi(\gamma(t)))J_{\mathbb{C}}\varphi(\gamma(t))\gamma'(t), \gamma'(t) >^{\frac{1}{2}} dt$$

$$= \inf_{\gamma\in C_{z,w}} \int_0^1 < B(\gamma(t))\gamma'(t), \gamma'(t >^{\frac{1}{2}} dt = \beta(z,w),$$

□

To conclude this subsection, we provide a formula that describes the Bergman metric in terms of the involutive automorphism of $\mathbb{B}_n$.

Proposition 1.6.10. *Given φ_z an involutive automorphism that interchanges* 0 *and z, where $z \in \mathbb{B}_n$. Then*

$$\beta(z,w) = \frac{1}{2}log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|} \tag{1.6.7}$$

where $z,w \in \mathbb{B}_n$.

Proof. Since the Bergman metric is invariant under automorphism, we prove the result for $w = 0$. Fixed $z \in \mathbb{B}_n$ and let $\gamma : [0,1] \longrightarrow \mathbb{B}_n$ be a smooth curve such that

$$\gamma(0) = 0 \ \text{ and } \ \gamma(1) = z.$$

Moreover, we assume that $\gamma(t)$ is regular, that is

$$\gamma'(t) \neq 0, \quad \forall t \ \in [0,1].$$

Under these conditions, $\alpha(t) := |\gamma(t)|$ is smooth on $[0,1]$. Denoting by

$$\gamma(t) = (\gamma_1(t), ..., \gamma_n(t)),$$

we have that

$$\alpha'(t) = \frac{Re < \gamma'(t), \gamma(t) >}{|\gamma(t)|}.$$

Since $|\gamma(t)|^2 =< \gamma(t), \gamma(t) >= \alpha(t)^2$, we differentiate $\alpha(t)^2$ to obtain

$$\begin{aligned} 2\alpha(t)\alpha'(t) &= 2|\gamma(t)| \frac{Re < \gamma'(t), \gamma(t) >}{|\gamma(t)|} \\ &= 2Re < \gamma'(t), \gamma(t) > \\ &= 2Re < P_{\gamma(t)}\gamma'(t), \gamma(t) >, \end{aligned}$$

where $P_{\gamma(t)}$ denotes the orthogonal projection from $\mathbb{C}^n$ onto the one-dimensional subspace spanned by $\gamma(t)$. We apply Cauchy-Schwarz inequality to the previous result and it turns out that

$$\begin{aligned} |2\alpha(t)\alpha'(t)| &= |2Re < P_{\gamma(t)}\gamma'(t), \gamma(t) > | \\ &\leq 2| < P_{\gamma(t)}\gamma'(t), \gamma(t) > | \\ &\leq 2|P_{\gamma(t)}\gamma'(t)||\gamma(t)| \\ &= 2|P_{\gamma(t)}\gamma'(t)|\alpha(t), \end{aligned}$$

namely

$$|\alpha'(t)| \leq |P_{\gamma(t)}\gamma'(t)|, \quad t \in [0,1].$$

According to Remark 1.6.6.,

$$\begin{aligned} < B(\gamma(t))\gamma'(t), \gamma'(t) > &= \frac{|\gamma'(t)|^2}{(1 - |\gamma(t)|^2)^2} \\ &= \frac{|P_{\gamma(t)}\gamma'(t)|^2 + |Q_{\gamma(t)}\gamma'(t)|^2}{(1 - |\gamma(t)|^2)^2} \\ &\geq \frac{|P_{\gamma(t)}\gamma'(t)|^2}{(1 - |\gamma(t)|^2)^2}, \end{aligned}$$

Hence,

$$
\begin{aligned}
l(\gamma) &\geq \int_0^1 \frac{|P_{\gamma(t)}\gamma'(t)|}{1-|\gamma(t)|^2} \\
&\geq \left| \int_0^1 \frac{\alpha'(t)dt}{1-\alpha^2(t)} \right| \\
&\geq \int_0^1 \frac{\alpha'(t)dt}{1-\alpha^2(t)} \\
&\underbrace{=}_{s:=\alpha(t)} \int_0^{|z|} \frac{ds}{1-s^2} \\
&= \frac{1}{2} log \frac{1+|z|}{1-|z|}.
\end{aligned}
$$

In other words,

$$
l(\gamma) \geq \frac{1}{2} log \frac{1+|z|}{1-|z|}.
$$

Finally, considering the curve $\gamma(t) = tz, t \in [0,1]$, we verify that the equality holds. This means that

$$
\beta(0,z) = \frac{1}{2} log \frac{1+|z|}{1-|z|},
$$

and we are done. □

Remark 1.6.11. Recalling that the inverse hyperbolic tangent function, denoted by $artanh(x)$, is

$$
artanh(x) = \frac{1}{2} log \frac{1+x}{1-x},
$$

defined when $|x| < 1$. Hence, we easily deduce that equation 1.6.8 can be written as

$$
|\varphi_z(w)| = tanh(\beta(z,w)).
$$

1.6.3 The Bergman metric ball

After the introduction of the Bergman metric, we give the definition of Bergman metric ball and collect some properties. The Bergman metric ball will be used, for instance, to prove a local estimate of $|f(z)|$, from above, in terms of the norm of the Bergman weighted space.

Definition 1.6.12 (Bergman metric ball)**.** Fix $z \in \mathbb{B}_n$ and $r > 0$, the Bergman metric ball centered at z with radius r is

$$
D(z,r) = \{w \in \mathbb{B}_n \mid \beta(z,w) < r\}. \tag{1.6.8}
$$

Moreover, the volume of the Bergman metric ball is denoted by $v_B(D(z,r))$.

In the next Lemma, we explicitely calculate the volume of the Bergman metric ball.

Lemma 1.6.13. *Fix $z \in \mathbb{B}_n$ and $r > 0$, we have*

$$
v_B(D(z,r)) = \frac{R^{2n}(1-|z|^2)^{n+1}}{(1-R^2|z|^2)^{n+1}}, \tag{1.6.9}
$$

where $R = \tanh(r)$.

Proof. The invariance of the Bergman metric under automorphism of $\mathbb{B}_n$ says that

$$D(z,r) = \varphi_z(D(0,r)).$$

Furthermore,

$$v(\varphi_z(D(0,r))) = \int_{D(0,R)} \varphi_z(w)dv(w)$$

Hence, using the change of variables formula of Proposition 1.4.7, we find

$$\begin{aligned} v_B(D(z,r)) &= \int_{\varphi_z(D(0,r))} dv_B(w) \\ &= \int_{D(0,R)} \varphi_z(w)dv(w) \\ &= \int_{D(0,R)} \frac{(1-|z|^2)^{n+1}dv(w)}{|1-<z,w>|^{2(n+1)}} \\ &= (1-|z|^2)^{n+1} \int_{\mathbb{B}_n} \frac{R^{2n}dv(w)}{|1-<Rz,w>|^{2(n+1)}} \\ &= \frac{R^{2n}(1-|z|^2)^{n+1}}{(1-R^2|z|^2)^{n+1}}, \end{aligned}$$

where in the last equality we used the following identity

$$\int_{\mathbb{B}_n} \frac{(1-R^2|z|^2)^{n+1}}{|1-<Rz,w>|^{2(n+1)}}dv(w) = \frac{n!}{\pi^n}v(\mathbb{B}_n) = 1.$$

□

Corollary 1.6.14. *Let $r > 0$ be, there exists constants $c_r > 0$ and $C_r > 0$ such that*

$$c_r(1-|z|^2)^{n+1} \leq v_B(D(z,r)) \leq C_r(1-|z|^2)^{n+1}, \ \forall z \in \mathbb{B}_n. \tag{1.6.10}$$

In order to proceed, we recall that

$$v_\alpha(D(z,r)) := \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \int_{D(z,r)} (1-|z|^2)^\alpha dv(z), \ \alpha > -1.$$

Hence, for every α we have the following asymptotic estimate of $v_\alpha(D(z,r))$.

Lemma 1.6.15. *For every couple of real positive numbers r and α, there exist two positive constants C and c such that the following asymptotic estimate of v_α holds*

$$c(1-|z|^2)^{n+\alpha+1} \leq v_\alpha(D(z,r)) \leq C(1-|z|^2)^{n+1+\alpha}, \tag{1.6.11}$$

for all $z \in \mathbb{B}_n$.

Proof. Again, using Proposition 1.4.5, denoting by $R := tanh(r)$ and making some change of variables:

$$\begin{aligned} v_\alpha(D(z,r)) &= \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \int_{D(z,r)} (1-|z|^2)dv(z) \\ &= \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \int_{|w|<R} \frac{(1-|z|^2)^{n+1+\alpha}(1-|w|^2)^\alpha}{|1-<z,w>|^{2(n+1+\alpha)}}dv(w). \end{aligned}$$

Finally, observing that we can find two positive constants, denoted by c and C, such that

$$c \leq \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \frac{(1-|w|^2)^\alpha}{|1-<z,w>|^{2(n+1+\alpha)}} \leq C,$$

for all $z \in \mathbb{B}_n$ and $|w| < R$, we obtain the desired result. □

1.7 Subharmonic Functions

The aim of this section is to collect some results about subharmonic function. Many results of this subsection are obtained in the real variable case. Hence, the open unit ball $\mathbb{B}_n$ can be thought as in the real Euclidean space $\mathbb{R}^{2n}$.
For technical reason, it is convenient to include the definition of upper semicontinuous functions. Formally, such definition can be stated as follows.

Definition 1.7.1 (Upper semi-continuous function)**.** A function $f : \mathbb{B}_n \longrightarrow [-\infty, +\infty)$ is said to be upper semi-continuous if

$$\limsup_{z \to z_0} f(z) \leq f(z_0), \ \forall \ z_0 \in \mathbb{B}_n. \tag{1.7.1}$$

We introduce the notion of subharmonic function. We point out that the normalised surface measure on $\mathbb{S}_n$ is denoted by $d\sigma$.

Definition 1.7.2 (Subharmonic Function)**.** An upper semi-continous function $f : \mathbb{B}_n \longrightarrow [-\infty, +\infty)$ is said to be subharmonic if the following holds

$$f(a) \leq \int_{\mathbb{S}_n} f(a + r\zeta) d\sigma(\zeta), \ \forall a \in \mathbb{B}_n, r \in [0, 1 - |a|). \tag{1.7.2}$$

Remark 1.7.3. Intuitively, for a fixed point $a \in \mathbb{B}_n$, a subharmonic function f, on a, is no greater than the average of the values of f in a circle around a.

In the next theorem, we establish an equivalent criteria to recover subharmonic functions.

Theorem 1.7.4. *Given an upper semi-continuous function $f : \mathbb{B}_n \longrightarrow [-\infty, +\infty)$, the following conditions are equivalent:*
a) f is subharmonic in $\mathbb{B}_n$.
b) For every point a in $\mathbb{B}_n$, there exists $\epsilon \in (0, 1 - |a|)$ that satisfies

$$f(a) \leq \int_{\mathbb{S}_n} f(a + r\zeta) d\sigma(\zeta), \ \forall \ r \in [0, \epsilon). \tag{1.7.3}$$

We wish to understand how fast a function of the weighted Bergman space can grow near the boundary of $\mathbb{B}_n$. To achieve this goal, the next corollary will be crucial.

Corollary 1.7.5. 1) *If $\alpha > -1$ and given a subharmonic function f in $\mathbb{B}_n$, then*

$$f(a) \leq \int_{\mathbb{B}_n} f(a + rz) dv_\alpha(z), \ \forall a \in \mathbb{B}_n, \ r \in [0, 1 - |a|). \tag{1.7.4}$$

2) *Let $f \in H(\mathbb{B}_n)$ and $p \in \mathbb{R}^+$, then both $\log |f|$ and $|f|^p$ are subharmonic in $\mathbb{B}_n$.*

Proof. 1) The definition of subharmonic function means that

$$f(a) \leq \int_{\mathbb{S}_n} f(a + r\zeta) d\sigma(\zeta), a \in \mathbb{B}_n, r \in [0, 1 - |a|).$$

Then, integrating in polar coordinates, we obtain the desired result.
2) Fix $a \in \mathbb{B}_n$, if $f(a) = 0$, using the mean value property for harmonic functions, we have

$$\log |f(a)| \leq \int_{\mathbb{S}_n} \log |f(a + r\zeta)| d\sigma(\zeta).$$

As well as

$$|f(a)|^p \leq \int_{\mathbb{S}_n} |f(a + r\zeta)|^p d\sigma(\zeta), \ r \in [0, 1 - |a|).$$

If $f(a) \neq 0$, there exists a positive number $\epsilon < 1 - |a|$ such that

$$f(z) \neq 0, \quad \forall z \in U := \{z \in \mathbb{B}_n \mid |z - a| < \epsilon\}.$$

Since analytic branches of $\log f(z)$ and $f(z)^p$ can be defined on U and using the harmonicity of $\log|f(z)|$ on U, the next holds

$$\log|f(a)| = \int_{\mathbb{S}_n} \log|f(a + r\zeta)| d\sigma(\zeta), \; r \in [0, \epsilon).$$

Similarly, using the mean value property for holomorphic functions, we get

$$f(a)^p = \int_{\mathbb{S}_n} f(a + r\zeta)^p d\sigma(\zeta), \quad r \in [0, \epsilon),$$

and, considering the modulus on both sides of the equality, we easily find

$$|f(a)|^p \leq \int_{\mathbb{S}_n} |f(a + r\zeta)|^p d\sigma(\zeta), \quad r \in [0, \epsilon).$$

Finally, by point $b)$ of the previous Theorem, we conclude that both functions $\log|f(z)|$ and $|f(z)|^p$ are subharmonic in $\mathbb{B}_n$. □

1.8 Complex Interpolation of Banach Spaces

In this section, we talk about a fundamental tool in Analysis: Interpolation of Banach spaces. Roughly speaking, the Interpolation of Banach spaces consists on the construction of many Banach space that lie "in between" two other Banach spaces. There are two major methods for constructing interpolation of Banach spaces: the complex method and the real method. We discuss the complex method, because it proceeds by exploiting the powerful tools of Complex Analysis. We start extending the definition of holomorphic function with values in a Banach space. Then, we introduce the notion of complex interpolation. Besides, the Definition of compatible Banach spaces is also given. After that we state a theorem concerning the construction of complex interpolation spaces. We conclude this section giving an example, perhaps the most important, of complex interpolation spaces. We omit the proof of all these results because they are just tools that we will use in the description of the Function spaces studied in this manuscript. Furthermore, the proof of these results requires techniques which are much different from those in the rest of this thesis.
However, a standard reference for the theory of Complex Interpolation is J. Bergh and J. Lofstrom, Interpolation Spaces-An Introduction, Springer-Verlag, Berlin, 1976. Another possible reference can be, for example, J. Garnett, Bounded Analytic Functions, Academic Press, New York, 1982.

In the framework of Banach spaces, we meet at least two different definitions concerning holomorphic functions: strongly and weakly holomorphic functions. In this section we will essentially restrict ourselves to the first one and it will be called holomorphic function. However, these two definitions of holomorphy turn out to be equivalent.

Definition 1.8.1. Let X be a Banach space and $\Omega \subset \mathbb{C}$ an open subset. Let $f : \Omega \longrightarrow X$ be a function. f is called holomorphic in Ω if there exists

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \in X, \quad \forall z_0 \in \Omega.$$

In order to introduce the complex method of interpolation, we give the following definition.

Definition 1.8.2 (Compatible Banach spaces)**.** Given two Banach spaces X_0 and X_1, we say that they are compatible if there exists a Hausdorff topological linear space X such that both of them are continuously embedded in X.

Using the above notation, the embedding of X_0 and X_1 in X allows to consider the two linear subspaces $X_0 \cap X_1$ and $X_1 + X_2$ of X.

Proposition 1.8.3. *Equip $X_0 \cap X_1$ and $X_0 + X_1$ with the following norms:*

$$||x||_{X_0 \cap X_1} := \max(||x||_{X_0}, ||x||_{X_1}) \quad \textit{and} \quad ||x||_{X_0+X_1} := \inf\{||x_0||_{X_0} + ||x_1||_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}.$$

Hence, we get that $(X_0 \cap X_1, ||\cdot||_{X_0 \cap X_1})$ and $(X_1 + X_2, ||\cdot||_{X_0+X_1})$ are Banach spaces.

In this section, the open strip is $S := \{z \in \mathbb{C} \mid 0 < Re(z) < 1\}$.

Definition 1.8.4. Given a compatible pair of Banach spaces X_0 and X_1, let $\mathcal{F}(X_0, X_1)$ the space of all functions f from $\overline{S}$ into $X_0 + X_1$ with the following properties:
a) f is bounded and continuous on $\overline{S}$.
b) f is holomorphic in S.
c) $f(iy) \in X_0$, $\forall y \in \mathbb{R}$.
d) $f(1 + iy) \in X_1$, $\forall y \in \mathbb{R}$.

$\mathcal{F}(X_0, X_1)$ is clearly a vector space. We endow $\mathcal{F}(X_0, X_1)$ with the norm

$$||f||_{\mathcal{F}} := \max\left(\sup_{y\in\mathbb{R}} ||f(iy)||_{X_0}, \sup_{y\in\mathbb{R}} ||f(1 + iy)||_{X_1}\right).$$

Moreover, $\mathcal{F}$ becomes a Banach space with the above norm.

Definition 1.8.5. Given $0 \le \theta \le 1$, let X_θ be the space of vectors x in $X_0 + X_1$ such that $x = f(\theta)$ for some f in $\mathcal{F}(X_0, X_1)$. We norm X_θ with

$$||x||_{X_\theta} := \inf\{||f||_{\mathcal{F}} : x = f(\theta)\}.$$

We obtain that

Proposition 1.8.6. *$X_\theta := (X_\theta, ||\cdot||_{X_\theta})$ is a Banach space.*

Definition 1.8.7 (Complex Interpolation)**.** The space X_θ is called the complex interpolation space between X_0 and X_1.

Remark 1.8.8. In other words, Complex Interpolation studies the family of spaces X_θ that are intermediate spaces between X_0 and X_1 in the sense that

$$X_0 \cap X_1 \subset X_\theta \subset X_0 + X_1,$$

where the two inclusions maps are continuous. Moreover, to emphasise the fact that X_θ depends on both X_0 and X_1, we write

$$X_\theta = [X_0, X_1]_\theta.$$

The complex method of interpolation spaces is functorial in the sense of the following theorem.

Theorem 1.8.9. *Assume X_0 and X_1 are compatible, Y_0 and Y_1 are compatible. Fix $\theta \in (0, 1)$, suppose there exists a linear operator $T : X_0 + X_1 \longrightarrow Y_0 + Y_1$ that maps X_0 boundedly Y_0, with norm M_0, and X_1 boundedly into Y_1, with norm M_1. Under these conditions T maps $[X_0, X_1]_\theta$ boundedly into $[Y_0, Y_1]_\theta$,with norm at most $M_0^{1-\theta} M_1^\theta$.*

Finally, we give one of the most important examples of complex interpolation spaces. This interpolation concerns L^p spaces (over any measure space). We don't prove this result. However, the idea used in the proof follows the same lines as in the proof of the Riesz-Thorin theorem. Of course, interpolation can also be performed for many other normed vector spaces than the Lebesgue spaces, but we will just concentrate on Lebesgue spaces in this section to focus the discussion. That is, for example, determine the complex interpolation space of the holomorphic spaces studied later.

Theorem 1.8.10. *Let (X, μ) be a measure space and $1 \le p_0 < p_1 \le \infty$, then*

$$[L^{p_0}(X), L^{p_1}(X)]_\theta = L^p(X), \ 0 < \theta < 1, \tag{1.8.1}$$

with equals norms, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Chapter 2

Weighted Bergman spaces

In this chapter we study the weighted Bergman spaces, induced by radial weights. The current chapter is organised as follows. We start introducing the weighted Bergman spaces. Then, we provide an integral representation formula that, in the theory of Bergman spaces, will be proved to be very useful in many different situations. We will concentrate on the general aspects and basic properties of these spaces such as, for example, completeness, density of polynomials and invariance under automorphism. Hence, to prove these facts, we provide a pair of tools: the first one asserts that functions in a Bergman space cannot grow too rapidly near the boundary and the second one establishes a local estimate, from above, of the partial derivative in terms of a suitable constant and the norm of the function itself.
In the current chapter, after a briefly review of some definitions, notations, and some basic properties of Hilbert spaces with reproducing kernels, we discuss a concrete example: the space of all holomorphic functions on the ball which are square integrable with respect to the volume measure dv_α, for $\alpha > -1$.
In this chapter, we introduce one of the most important operators acting on holomorphic spaces: the Bergman projection, which is the integral operator induced by the Bergman kernel. After collect some basic properties, we will find necessary and sufficient conditions for which such operator is bounded. Then, we discuss a crucial consequence of the boundedness of the Bergman projections: the description of the dual space of the weighted Bergman space.
We present a new notion of derivative: invariant gradient and, so that, we will obtain a characterisation of the weighted Bergman space in terms of this type of derivative, the radial derivative and the holomorphic gradient respectively.
After that, we study the atomic decomposition of Bergman spaces: we introduce a family of functions that are called atoms, whose construction is based on some sharp estimates about Bergman metric and Bergman kernel functions in the unit ball $\mathbb{B}_n$, and show that every function in the Bergman space can be decomposed into a series of them.
Finally, we conclude this chapter with the complex interpolation of the weighted Bergman spaces.
For this chapter the main references are:
P. L. Duren, A. Schuster. Bergman Spaces. The American Mathematical Society, 2004.
K. Zhu. Spaces of Holomorphic Functions in the Unit Ball. Springer, 2005.

2.1 The Bergman Space A^p_α

The definition of Bergman space is given as follows.

Definition 2.1.1 (Weighted Bergman Space)**.** Given $\alpha > -1$ and $0 < p < +\infty$, the weighted Bergman space, on the open unit ball $\mathbb{B}_n$, is defined as

$$A^p_\alpha := L^p(\mathbb{B}_n, dv_\alpha) \cap H(\mathbb{B}_n). \tag{2.1.1}$$

In other words, the weighted Bergman space A^p_α consists of all holomorphic functions f on $\mathbb{B}_n$ for which

$$||f||_{p,\alpha} := \left\{ \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \right\}^{1/p} < +\infty.$$

When $1 \leq p < +\infty$, the space A^p_α, equipped with $||\cdot||_{p,\alpha}$, is a norm space. If $0 < p < 1$, the triangle inequality fails, but the inequality

$$||f+g||^p_{p,\alpha} \leq ||f||^p_{p,\alpha} + ||g||^p_{p,\alpha}$$

is often an adequate substitute. Hence, for $0 < p < 1$, the space A^p_α is a metric space where the metric is defined as

$$d(f,g) := ||f-g||^p_{p,\alpha}.$$

Finally, for $p = +\infty$, A^∞_α denotes the space of essentially bounded holomorphic functions in $\mathbb{B}_n$. This space is endowed with the following norm

$$||f||_\infty = \operatorname{ess\,sup}\{ \ |f(z)| : z \in \mathbb{B}_n\}.$$

However, regardless of what p is, we are going to call $||\cdot||_{p,\alpha}$ the norm of A^p_α.

Remark 2.1.2. The assumption that $\alpha > -1$ is crucial. In fact, the space $L^p(\mathbb{B}_n, dv_\alpha)$ does not contain any holomorphic function other than 0 when $\alpha \leq -1$. If $\alpha = 0$, we use A^p to denote the ordinary unweighted Bergman spaces. Moreover, since $H(\mathbb{B}_n)$ is a complex vector space, we deduce that A^p_α is a linear and convex subspace of $L^p(\mathbb{B}_n, dv_\alpha)$, for every $0 < p \leq \infty$.
By the definition, if $p < q$, we trivially notice that

$$A^q_\alpha \subset A^p_\alpha.$$

In the next Lemma, we establish an integral representation formula in $H(\mathbb{B}_n)$. In this thesis, such formula will be used in a large number of proofs. We remark that the only assumption that we use is the holomorphicity of f.

Lemma 2.1.3. *Let $\alpha > -1$ and $f \in H(\mathbb{B}_n)$, the following formula holds*

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w)dv_\alpha(w)}{(1-<z,w>)^{n+1+\alpha}}, \tag{2.1.2}$$

for all $z \in \mathbb{B}_n$.

Proof. Since $f \in H(\mathbb{B}_n)$, by the mean value property, we have that

$$f(0) = \int_{\mathbb{S}_n} f(r\xi)d\sigma(\xi), \quad 0 \leq r < 1.$$

Then, multiplying to both sides by $2nr^{2n-1}(1-r^2)^\alpha dr$, integrate in polar coordinates, it turns out that

$$\begin{aligned} f(0)\int_0^1 2nr^{2n-1}(1-r^2)^\alpha dr &= f(0)\frac{n!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \\ &= \int_0^1 2nr^{2n-1}(1-r^2)^\alpha dr \int_{\mathbb{S}_n} f(r\xi)d\sigma(\xi) \\ &= \int_{\mathbb{B}_n} f(w)(1-|w|^2)^\alpha dv(w) \end{aligned}$$

That is

$$f(0) = \int_{\mathbb{B}_n} f(w)dv_\alpha(w).$$

Hence, replacing f by $f \circ \varphi_z$, making an obvious change of variables, we obtain

$$f(z) = (1-|z|^2)^{n+1+\alpha} \int_{\mathbb{B}_n} \frac{f(w)dv_\alpha(w)}{(1-<w,z>)^{n+1+\alpha}(1-<z,w>)^{n+1+\alpha}}.$$

Finally, fixing $z \in \mathbb{B}_n$ and replacing f by the function $L(w) := (1-<w,z>)^{n+1+\alpha}f(w)$, we get the wished result. $\square$

In the next Theorem, we provide a pointwise estimate, for a function $f \in A^p_\alpha$, in terms of the L^p-norm. As a consequence, we will know more about how fast a function in A^p_α grows near the boundary of $\mathbb{B}_n$.

Theorem 2.1.4. *Let $0 < p < \infty$ and $\alpha > -1$, the following estimate holds*

$$|f(z)| \leq \frac{||f||_{p,\alpha}}{(1-|z|^2)^{(n+1+\alpha)/p}}, \ z \in \mathbb{B}_n, \tag{2.1.3}$$

where $f \in A^p_\alpha$.

Proof. We first assume that $z = 0$. The holomorphicity of f, on $\mathbb{B}_n$, implies that $|f|^p$ is subharmonic. Hence, using point 1) in Corollary 1.7.5 and Holder inequality, we find

$$|f(0)|^p \leq \int_{\mathbb{B}_n} |f(w)|^p dv_\alpha(w)$$

In general, for any $f \in H(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$, we introduce the holomorphic map

$$F(w) := f \circ \varphi_z(w) \frac{(1-|z|^2)^{(n+1+\alpha)/p}}{(1-<w,z>)^{2(n+1+\alpha)/p}}, \ w \in \mathbb{B}_n.$$

Applying Proposition 1.4.7 and putting $f(w) := f \circ \varphi_z(w)$, we have

$$\begin{aligned} ||F||^p_{p,\alpha} &= \int_{\mathbb{B}_n} |F(w)|^p dv_\alpha(w) \\ &= \int_{\mathbb{B}_n} |f \circ \varphi_z(w)|^p \frac{(1-|z|^2)^{(n+1+\alpha)}}{|1-<w,z>|^{2(n+1+\alpha)}} dv_\alpha(w) \\ &= \int_{\mathbb{B}_n} |f \circ \varphi_z \circ \varphi_z(w)|^p dv_\alpha(w) \\ &= ||f||^p_{p,\alpha} \end{aligned}$$

That is, the norm of F agrees with that of f. Moreover,

$$|f(z)(1-|z|^2)^{(n+1+\alpha)/p}| = |F(0)| \leq ||F||_{p,\alpha} = ||f||_{p,\alpha},$$

and we are done. $\square$

Basically, we proved that, for every function $f \in A^p_\alpha$, point-evaluations are bounded. Moreover, this fact will be treated during the description of the dual space of A^p_α.

Remark 2.1.5. A consequence of the previous estimate is that the convergence of a sequence of holomorphic functions, with respect to the L^p-norm, implies the uniform convergence on compact subsets and so, by Weirstrass theorem, the limit function is clearly holomorphic. As a prove, let $0 < \rho < 1$ and $f \in A^p_\alpha$, inequality (2.1.3) shows that

$$\sup_{|z|\leq\rho} |f(z)| \leq \frac{||f||_{p,\alpha}}{(1-\rho^2)^{(n+1+\alpha)/p}}.$$

Another important consequence of Theorem 2.1.4 should be noted. If a sequence of functions $(f_n)_{n\in\mathbb{N}} \in A^p_\alpha$ is bounded in norm, then it is locally bounded. So, by Montel's theorem, it constitutes a normal family; some subsequence converges locally uniformly on $\mathbb{B}_n$ to a function of A^p_α.

In order to prove that, when $1 \leq p < \infty$, A^p_α is a Banach space and, for $0 < p < 1$, a complete metric space we need the following important result. Actually, this is a sort of generalisation, on compact sets $\{z \in \mathbb{C}^n \mid |z| \leq r\}$ where $r \in (0,1)$, of Theorem 2.1.4. More is true, this result will be used, for example, to treat the Bergman space A^2_α, the description of the dual space of A^p_α and the characterisation in terms of derivatives of it.

Lemma 2.1.6. *Let $p > 0$, $\alpha > -1$, $0 < r < 1$ and $m = (m_1, ..., m_n)$ a multi-index of nonnegative integers. Then, there exists $C := C(z, r, n, m, \alpha) > 0$ such that*

$$\left|\frac{\partial^m f}{\partial z^m}(z)\right| \leq C||f||_{p,\alpha}, \tag{2.1.4}$$

for all $f \in A^p_\alpha$ and $|z| \leq r$.

Proof. We start assuming $m = 0$ and, in this case, we just apply Theorem 2.1.4 or Remark 2.1.5. After that, for $m = 1$, fix $\delta \in (r, 1)$, using Lemma 2.1.3, we get

$$f(\delta z) = \int_{\mathbb{B}_n} \frac{f(\delta w) dv_\alpha(w)}{(1- < z, w >)^{n+\alpha+1}}, \quad z \in \mathbb{B}_n.$$

Then, replacing δz by z, we find

$$f(z) = \int_{|w|<\delta} \frac{f(w) dv_\alpha(w)}{(1- < \frac{z}{\delta}, \frac{w}{\delta} >)^{n+\alpha+1}}, \quad |z| < \delta.$$

Since

$$\begin{aligned}
\left|\int_{|w|<\delta} \frac{\partial}{\partial z} \frac{f(w) dv_\alpha(w)}{(1- < z, w >)^{n+\alpha+1}}\right| &= \int_{|w|<\delta} \sum_{m=1}^{\infty} \frac{\Gamma(n + |m| + \alpha + 1)}{(m-1)!\Gamma(n + \alpha + 1)} \frac{|z|^{m-1}}{\delta^{m-1}} \frac{|\overline{w}|^m}{\delta^m} |f(w)| dv_\alpha(w) \\
&\underbrace{\leq}_{|w|<\delta} \int_{|w|<\delta} \sum_{m=1}^{\infty} \frac{\Gamma(n + |m| + \alpha + 1)}{(m-1)!\Gamma(n + \alpha + 1)} \frac{|z|^{m-1}}{\delta^{m-1}} |f(w)| dv_\alpha(w) \\
&= \frac{1}{(1 - |\frac{z}{\delta}|)^{2(n+\alpha+1)}} \int_{|w|<\delta} |f(w)| dv_\alpha(w) \\
&\leq \frac{\delta^{2(n+\alpha)+1}}{(\delta - |z|)^{2(n+\alpha+1)}} \sup \{|f(w)| : |w| \leq \delta\}.
\end{aligned}$$

This means that, for $|z| \leq \delta$, we have

$$\varphi(w) := \frac{\partial}{\partial z} \frac{f(w)}{(1- < z, w >)^{n+\alpha+1}} \in L^1(\mathbb{B}_n, dv_\alpha).$$

Hence, applying the dominated convergence theorem, differentiating under the integral sign and using

Theorem 2.1.4:

$$\left|\frac{\partial f}{\partial z}(z)\right| = \left|\frac{\partial}{\partial z}\int_{|w|<\delta}\frac{f(w)dv_\alpha(w)}{(1-<z,w>)^{n+\alpha+1}}\right|$$

$$= \left|\int_{|w|<\delta}\frac{\partial}{\partial z}\frac{f(w)dv_\alpha(w)}{(1-<z,w>)^{n+\alpha+1}}\right|$$

$$= \left|\int_{|w|<\delta}\frac{\partial}{\partial z}\sum_{m=0}^{\infty}\frac{\Gamma(n+|m|+\alpha+1)}{m!\Gamma(n+\alpha+1)}z^m\overline{w}^m f(w)dv_\alpha(w)\right|$$

$$< \int_{|w|<\delta}\sum_{m=1}^{\infty}\frac{\Gamma(n+|m|+\alpha+1)}{(m-1)!\Gamma(n+\alpha+1)}\frac{|z|^{m-1}}{\delta^{m-1}}|f(w)|dv_\alpha(w)$$

$$\leq \frac{\delta^{2(n+\alpha)+1}}{(\delta-|z|)^{2(n+\alpha+1)}}\sup\{|f(w)|:|w|\leq\delta\}$$

$$\underbrace{\leq}_{Theorem\ 2.1.4}\frac{\delta^{2(n+\alpha)+1}}{(\delta-|z|)^{2(n+\alpha+1)}}\frac{||f||_{p,\alpha}}{(1-|\delta|^2)^{(n+1+\alpha)/p}}$$

In other words, considering the limit when δ approachs to r, we have

$$\left|\frac{\partial f}{\partial z}(z)\right| \leq \frac{r^{2(n+\alpha)+1}}{(r-|z|)^{2(n+\alpha+1)}}\frac{||f||_{p,\alpha}}{(1-r^2)^{(n+1+\alpha)/p}}$$

Proceeding similarly, for any multiindex m, we easily find

$$\left|\frac{\partial^m f}{\partial z^m}(z)\right| \leq \frac{r^{(m+1)(n+\alpha)+1}}{(r-|z|)^{(m+1)(n+\alpha+1)}}\frac{||f||_{p,\alpha}}{(1-r^2)^{(n+1+\alpha)/p}},\ |z|\leq r,$$

and we are done. $\square$

As a consequence of all these tools, we have the following Corollary that contains the most important result of this section.

Corollary 2.1.7. *Let $p>0$ and $\alpha>-1$, the weighted Bergman space A^p_α is closed in $L^p(\mathbb{B}_n, dv_\alpha)$. Hence, for $1\leq p\leq\infty$, the weighted Bergman space A^p_α, with topology inherited from $L^p(\mathbb{B}_n, dv_\alpha)$, is a Banach space and is a complete metric space when $0<p<1$.*

Proof. Let $\{f_n(z)\}_{n\in\mathbb{N}}\in A^p_\alpha$, where $0<p<+\infty$, such that

$$\lim_{n\to\infty}||f_n-f||_{p,\alpha}=0,$$

for some $f\in L^p(\mathbb{B}_n, dv_\alpha)$. Since $\{f_n(z)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in A^p_α, by Theorem 2.1.4, we have that $\{f_n(z)\}_{n\in\mathbb{N}}$ is uniformly Cauchy on each set $\{z\in\mathbb{B}_n:|z|<r\}$, for $0<r<1$, and must converge to a holomorphic function in such set. By the arbitrariness of r, the sequence $\{f_n(z)\}_{n\in\mathbb{N}}$ converges to a holomorphic function $g(z)$ on $\mathbb{B}_n$. Finally, by the uniqueness of pointwise limits in Hausdorff spaces, we must have $f(z)=g(z)$ for almost all $z\in\mathbb{B}_n$. We've proved that $f\in H(\mathbb{B}_n)$ and, hence, $f\in A^p_\alpha$. Similarly, for $p=+\infty$, we prove that A^∞_α is closed in $L^\infty(\mathbb{B}_n, dv_\alpha)$ and hence is a Banach space. $\square$

In many applications, we need to approximate a general function in the Bergman space A^p_α by a sequence of nice functions. The following result gives two commonly used ways of doing this: fix $f\in A^p_\alpha$, we prove that f can be approximated in A^p_α norm by its dilations and polynomials.

Proposition 2.1.8. *Let $p>0$ and $\alpha>-1$. Then, polynomials are dense in A^p_α.*

Proof. For $0 < r < 1$, let $f \in A^p_\alpha$ and denoting by $f_r(z) := f(rz)$. Hence, we obtain that $\{f_r\}_{r\in(0,1)}$ are bounded and $\{f_r\}_{r\in(0,1)} \in A^p_\alpha$. Our aim is to show that

$$\lim_{r\to 1^-} ||f_r - f||_{p,\alpha} = 0.$$

We have

$$\begin{aligned} ||f_r - f||^p_{p,\alpha} &= \int_{|z|\le 1-\varepsilon} |f(z) - f(rz)|^p dv_\alpha(z) + \int_{1-\varepsilon<|z|<1} |f(z) - f(rz)|^p dv_\alpha(z) \\ &=: A + B. \end{aligned}$$

For every $0 < \varepsilon < 1$ fixed, since $\{f_r\}_{r\in(0,1)}$ converge uniformly to f on compact subsets, it follows that $A \to 0$ as $|z| \to 1^-$. Moreover,

$$\int_{1-\varepsilon<|z|<1} |f(rz)|^p dv_\alpha(z) \le \int_{1-\varepsilon<|z|<1} |f(z)|^p dv_\alpha(z).$$

Therefore, using the absolute continuity of the Lebesgue integral, we find

$$\begin{aligned} B &\le \int_{1-\varepsilon<|z|<1} |f(rz)|^p dv_\alpha(z) + \int_{1-\varepsilon<|z|<1} |f(z)|^p dv_\alpha(z) \\ &\le 2\int_{1-\varepsilon<|z|<1} |f(z)|^p dv_\alpha(z) \\ &\le \delta \end{aligned}$$

for ε small enough. By the arbitrariness of δ, we deduce

$$\lim_{r\to 1^-} ||f_r - f||_{p,\alpha} = 0.$$

Now, fixed $r \in (0, 1)$, we can approximate $\{f_r\}_{r\in(0,1)}$ uniformly by polynomials basically using the expansion in homogenous power series. That is, if $f(rz) = \sum_{k=0}^\infty a_k(zr)^k$, we define the sequence of polynomials as $f^N_r(z) := \sum_{k=0}^N a_k(rz)^k$. Then, an immediate application of the dominated convergence theorem implies that $\{f_r\}_{r\in(0,1)}$ can be approximated in the norm topology of A^p_α by polynomials. Finally, using the triangle inequality and the same notation used above

$$||f - f^N_r||_{p,\alpha} \le ||f - f_r||_{p,\alpha} + ||f_r - f^N_r||_{p,\alpha},$$

and we easily obtain the desired result. □

Remark 2.1.9. It is easy to see that the estimate given in Theorem 2.1.4 is optimal, namely, the exponent $(n + \alpha + 1)/p$ can't be improved. However, as a consequence of Proposition 2.1.6, using polynomials approximations and following the same lines as in the previous proof, we can show that

$$\lim_{|z|\to 1^-} (1 - |z|^2)^{(n+\alpha+1)/p} f(z) = 0, \tag{2.1.5}$$

whenever $f \in A^p_\alpha$, $p \in (0, +\infty)$ and $\alpha > -1$.

Proof. For a holomorphic polynomial $f(z) = \sum_{j=0}^N a_j z^j$ we have that

$$\begin{aligned} \lim_{|z|\to 1^-} (1 - |z|^2)^{(n+\alpha+1)/p} |f(z)| &= \lim_{|z|\to 1^-} (1 - |z|^2)^{(n+\alpha+1)/p} \left|\sum_{j=0}^N a_j z^j\right| \\ &\underbrace{\le}_{|z|<1} \lim_{|z|\to 1^-} (1 - |z|^2)^{(n+\alpha+1)/p} \underbrace{\sum_{j=0}^N |a_j|}_{<+\infty} = 0. \end{aligned}$$

Namely, in this case, (2.1.5) holds. Now, fix $f \in A^p_\alpha$ and $r \in (0,1)$, by the fact that $\{f_r\}_{r\in(0,1)}$ can be approximated uniformly by polynomials, we deduce

$$\lim_{|z|\to 1^-} (1-|z|^2)^{(n+\alpha+1)/p} f(rz) = 0.$$

Finally, for any $f \in A^p_\alpha$, since

$$\lim_{r\to 1^-} ||f - f_r||_{p,\alpha} = 0,$$

using the triangle inequality and Theorem 2.1.4, it turns out that

$$\lim_{|z|\to 1^-} (1-|z|^2)^{(n+\alpha+1)/p}|f(z)| \leq \lim_{|z|\to 1^-} (1-|z|^2)^{(n+\alpha+1)/p}|f(z)-f_r(z)| + \lim_{|z|\to 1^-} (1-|z|^2)^{(n+\alpha+1)/p}|f_r(z)|$$

$$\leq \lim_{|z|\to 1^-} ||f - f_r||_{p,\alpha} + 0 < \varepsilon,$$

for r large enough. This completes the proof. □

Proposition 2.1.8 states that given $f \in A^p_\alpha$, one can find a sequence of polynomials that approach f in norm. Actually, we can improve this result, for $p > 1$, by showing that f can be approximated by the most natural choice of polynomials: the partial sums of its homogenous expansion series. To this end, we will need the following instruments.
For any holomorphic function f, we define tha partial sum operator $S_N : H(\mathbb{B}_n) \longrightarrow H(\mathbb{B}_n)$ as follows

$$S_N f(z) = \sum_{n=0}^{N} a_n z^n,$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is the homogenous expansion of f.

Lemma 2.1.10. *Let $(X, ||\cdot||)$ be a Banach space of holomorphic functions on the open unit ball such that polynomials are dense. Then*

$$\lim_{N\to\infty} ||S_N f - f|| = 0, \forall f \in X \qquad \Longleftrightarrow \qquad \sup_{N\geq 1} ||S_N|| < \infty.$$

This Lemma is not proved. However, the interested reader can find all the details of this result on, for example, Bergman Spaces by Peter L. Duren and Alexander Schuster.
In the next theorem we study the approximation in norm by its Taylor polynomials. In fact, in the next theorem we show that this is possible if $1 < p < +\infty$.

Theorem 2.1.11. *Let $1 < p < \infty$. If $f \in A^p_\alpha$, then the partial sums of its Taylor series converge in norm to f.*

Proof. (Hint) We just observe that we can find a constant $C > 0$ such that

$$||S_N f||^p_{p,\alpha} \leq C||f||^p_{p,\alpha}, \; N \in \mathbb{N},$$

applying Lemma 2.1.10, the result follows easily. □

Remark 2.1.12. The fact that the above theorem fails when $p = 1$ can be shown, for $\alpha = 0$, by considering functions of the form $f(z) := \dfrac{(1-|a|^2)}{(1-<z,a>)^3}$, where $a \in \mathbb{B}_n$.

We end this section by proving that the map defined as $T_\varphi : f \longrightarrow f \circ \varphi$, where $\varphi \in Aut(\mathbb{B}_n)$, sends the Bergman space A^p_α into itself, for $0 < p < +\infty$.

Proposition 2.1.13. *Let $f \in A^p_\alpha$ and $\varphi \in Aut(\mathbb{B}_n)$, then*

$$f \circ \varphi \in A^p_\alpha,$$

where $\alpha > -1$ and $0 < p < +\infty$.

Proof. First of all, it is clear that $f \circ \varphi \in H(\mathbb{B}_n)$. Then, for $0 < p < \infty$, using the change of variables formula of Proposition 1.4.7 , we obtain

$$\begin{aligned}\int_{\mathbb{B}_n} |f \circ \varphi(z)|^p dv_\alpha(z) &= \int_{\mathbb{B}_n} |f(z)|^p \frac{(1-|a|^2)^{n+1+\alpha}}{|1-<z,a>|^{2(n+1+\alpha)}} dv_\alpha(z) \\ &\leq \int_{\mathbb{B}_n} |f(z)|^p \frac{(1-|a|^2)^{n+1+\alpha}}{(1-|a|)^{2(n+1+\alpha)}} dv_\alpha(z) \\ &= \frac{(1+|a|)^{n+1+\alpha}}{(1-|a|)^{n+1+\alpha}} \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z),\end{aligned}$$

where $a = \varphi(0)$. □

Remark 2.1.14. In Proposition 2.1.13, we had the linear operator

$$T(f) := f \circ \varphi,$$

and we proved that

$$||f \circ \varphi||_{p,\alpha} \leq \frac{(1+|a|)^{(n+1+\alpha)/p}}{(1-|a|)^{(n+1+\alpha)/p}} ||f||_{p,\alpha},$$

where $a = \varphi(0)$. Moreover, we easily notice that the following estimate from below holds:

$$||f \circ \varphi||_{p,\alpha} \geq \frac{(1-|a|)^{(n+1+\alpha)/p}}{(1+|a|)^{(n+1+\alpha)/p}} ||f||_{p,\alpha}$$

Denoting by $\mathcal{L}(A^p_\alpha, A^p_\alpha)$ the space of linear operators from A^p_α to itself and $||T||_{\mathcal{L}(A^p_\alpha, A^p_\alpha)} := \sup_{||f||_{p,\alpha}=1} ||f \circ \varphi||_{p,\alpha}$, we find that

$$\frac{(1-|a|)^{(n+1+\alpha)/p}}{(1+|a|)^{(n+1+\alpha)/p}} \leq ||T||_{\mathcal{L}(A^p_\alpha, A^p_\alpha)} \leq \frac{(1+|a|)^{(n+1+\alpha)/p}}{(1-|a|)^{(n+1+\alpha)/p}}.$$

Remark 2.1.15 (Uniformly convexity of A^p_α for $1 < p < \infty$)**.** It is known that the spaces $L^p(\mathbb{B}_n, dv_\alpha)$ are uniformly convex for $1 < p < \infty$, but not for $p = 1$ or $p = \infty$. Since every subspace of a uniformly convex space must have the same property, each of the Bergman spaces A^p_α, with $1 < p < \infty$, is uniformly convex.

2.1.1 The space A^2_α

In this section we concentrate on the case $p = 2$. We start recalling the most important facts about Hilbert spaces with reproducing kernel. In this short review, we suppose to deal with spaces of functions defined on $\mathbb{B}_n$ but, of course, this theory can be generalised on any domain of $\mathbb{C}^n$. We remark that all these results are provided without proofs. After such review, we will have the necessary tools to prove that A^2_α is a Hilbert space with reproducing kernel defining a canonical inner product on A^2_α and, so that, the associated reproducing kernel will be calculated in closed form. Hence, we will discuss about some properties of the kernel previously computed such as, for example, the invariance under $Aut(\mathbb{B}_n)$.

Let $\mathcal{H} := (\mathcal{H}, <,>_{\mathcal{H}})$ be a Hilbert space of functions, where $<,>_{\mathcal{H}}$ denotes the inner product, defined on a set $\mathbb{B}_n$ and suppose that the point evaluations are bounded linear functionals on $\mathcal{H}$, that is, for each $z \in \mathbb{B}_n$, there exists a constant $C_z > 0$ such that for all $\mathcal{H}$ we have

$$|f(z)| \leq C_z ||f||_{\mathcal{H}}. \tag{2.1.6}$$

Such inequality implies that the linear functional

$$L_z(f) := f(z), \quad f \in \mathcal{H},$$

is bounded. Applying Riesz-Fisher theorem, there exists $k_z \in \mathcal{H}$ such that, for all $f \in \mathcal{H}$ we have

$$< f, k_z >_{\mathcal{H}} = f(z).$$

Definition 2.1.16. We define a kernel function $K : \mathbb{B}_n \times \mathbb{B}_n \longrightarrow \mathbb{C}$ by setting

$$K(z, w) = k_z(w). \tag{2.1.7}$$

The kernel K is called the reproducing kernel for $\mathcal{H}$.

Proposition 2.1.17. *The kernel K satisfies the following properties:*
1) $f(z) =< f, K(\cdot, z) >_{\mathcal{H}}$, $\forall f \in \mathcal{H}$ and $z \in \mathbb{B}_n$.
2) $K(w, z) = \overline{K(z, w)}$, $\forall z, w \in \mathbb{B}_n$.

The following result establishes conditions that characterise the reproducing kernel of $\mathcal{H}$.

Lemma 2.1.18. *Let $H(z, w)$ be a function on $\mathbb{B}_n \times \mathbb{B}_n$ such that*
1) $H(\cdot, w) \in \mathcal{H}$, for all $w \in \mathbb{B}_n$ fixed;
2) $< f, H(\cdot, z) >_{\mathcal{H}} = f(z)$, for all $f \in \mathcal{H}$ and $z \in \mathbb{B}_n$.
Then, $H(z, w)$ coincides with the reproducing kernel $K(z, w)$ of $\mathcal{H}$.

It will also always be the case that, for each compact subset $E \subset \mathbb{B}_n$ there exists $C := C_E > 0$ such that for all $f \in \mathcal{H}$ we have

$$\sup_{z \in E} |f(z)| \leq C ||f||_{\mathcal{H}}. \tag{2.1.8}$$

If (2.1.8) holds, then clearly the convergence in $\mathcal{H}$ implies the uniform convergence on compact subsets of $\mathbb{B}_n$.
There is an intriguing formula connecting the Bergman kernel function with the notion of complete orthonormal system. We will use it to compute the Bergman kernel function for the unit ball.

Proposition 2.1.19. *Let $\mathcal{H}$ be a Hilbert space of holomorphic functions on $\mathbb{B}_n$ for which condition* (2.1.8) *holds. Let $\{\varphi_j\}$ be an orthonormal basis for $\mathcal{H}$. Then the series*

$$\sum_{j=1}^{+\infty} \varphi_j(z) \overline{\varphi_j(w)}$$

converges uniformly on compact subsets of $\mathbb{B}_n \times \mathbb{B}_n$ to the reproducing kernel $K(z, w)$ of $\mathcal{H}$.

An interesting consequence of this proposition is the following result that shows that the reproducing kernel K satisfies an extremal property. As a straightforward consequence of this result, we will calculate, on Section 2.3 of the current chapter, the operator norm of a linear functional defined on the space A^2_α: the point-evaluation linear functional.

Corollary 2.1.20. *Let $\mathcal{H}$ be a space of holomorphic functions on $\mathbb{B}_n$ for which condition* (2.1.8) *holds and let $K(z, w)$ be its reproducing kernel. Then*

$$K(z, z) = \sup_{f \in \mathcal{H}, ||f||_{\mathcal{H}} = 1} |f(z)|^2$$

After this introduction, we are ready to study an example of Hilbert space with reproducing kernel: the weighted Bergman space A^2_α.

Definition 2.1.21. We call the function

$$K^\alpha(z,w) := K^\alpha_w(z), \quad z, w \in \mathbb{B}_n \tag{2.1.9}$$

the reproducing kernel of A^2_α.

In the next theorem we prove the main result of this subsection.

Theorem 2.1.22. *Suppose $\alpha > -1$, the space A^2_α is a Hilbert space with reproducing kernel. Furthermore, A^2_α can be equipped with an inner product so that the associated reproducing kernel has the expression*

$$K^\alpha(z,w) = \frac{1}{(1- < z, w >)^{n+1+\alpha}}, \quad z, w \in \mathbb{B}_n. \tag{2.1.10}$$

Proof. For $p > 0$ and $\alpha > -1$, the Bergman spaces A^p_α are closed subspaces of $L^p(\mathbb{B}_n, dv_\alpha)$. Since $L^2(\mathbb{B}_n, dv_\alpha)$ is a Hilbert space, we obtain that A^2_α, equipped with the inner product $<,>_\alpha$, is a Hilbert space as well.
By Theorem 2.1.4, the point evaluations are bounded linear functionals on A^2_α. Hence, by the Riesz-Fisher Theorem, there exists a unique function $K^\alpha_w \in A^2_\alpha$ such that

$$f(w) =< f, K^\alpha_w >_\alpha = \int_{\mathbb{B}_n} f(z)\overline{K^\alpha_w(z)}dv_\alpha(z), \quad f \in A^2_\alpha, \tag{2.1.11}$$

that is called the reproducing formula for f in A^2_α.
In order to compute $K_\alpha(z,w)$, we proceed as follows. Applying Corollary 1.4.7 and Proposition 2.1.5, the following functions form an orthonormal basis:

$$e_m(z) = \sqrt{\frac{\Gamma(n + |m| + \alpha + 1)}{m!\Gamma(n + \alpha + 1)}} z^m, \tag{2.1.12}$$

where m runs over all n-tuples of nonnegative integers. Using the multinomial formula and Proposition 2.1.19, it turns out that

$$\begin{aligned} K_\alpha(z,w) &= \sum_m e_m(z)\overline{e_m(w)} \\ &= \sum_m \frac{\Gamma(n + |m| + \alpha + 1)}{m!\Gamma(n + \alpha + 1)} z^m \overline{w}^m \\ &= \frac{1}{(1- < z, w >)^{n+1+\alpha}}. \end{aligned}$$

Moreover,

$$\overline{K_\alpha(z,w)} = K_\alpha(w,z).$$

For any fixed $w \in \mathbb{B}_n$, we notice that

$$\phi(z) := \frac{1}{(1- < z, w >)^{n+1+\alpha}} \in H(\mathbb{B}_n),$$

and, since

$$\begin{aligned} \int_{\mathbb{B}_n} |K_\alpha(z,w)|^2 dv_\alpha(z) &= \sum_{m,j} e_m(w)\overline{e_j(w)} \underbrace{< e_m(z), e_j(z) >_\alpha}_{=\delta_{m,j}} \\ &= \sum_m |e_m(w)|^2 \\ &= \frac{1}{(1 - |w|)^{2(n+1+\alpha)}} < \infty. \end{aligned}$$

Hence,

$$K^{\alpha}(z,\cdot)\in A^2_{\alpha},$$

for $w\in\mathbb{B}_n$ fixed. Finally, from Lemma 2.1.3

$$f(z)=\int_{\mathbb{B}_n}\frac{f(w)}{(1-<z,w>)^{n+1+\alpha}}dv_{\alpha}(w),$$

in other words, K^{α} satisfies the reproducing property. $\square$

Remark 2.1.23 (Uniqueness of the reproducing kernel)**.** We should emphasise that among functions in A^2_{α}, the kernel $K_{\alpha}(z,w)$ is uniquely determined by its reproducing property. This is part of the Riesz representation theorem: if a function $l_z(w)=\overline{L(z,w)}$ belongs to A^2_{α} and also has the reproducing property, then $f(z)=<f,l_z>_{\alpha}$ and so $<f,k_z-l_z>_{\alpha}=0$ for every $f\in A^2_{\alpha}$. But this fact implies that $k_z-l_z=0$, so that $K_{\alpha}(z,w)=L(z,w)$ for all $z,w\in\mathbb{B}_n$.

In the next proposition we obtain a characterisation of A^2_{α} in terms of Taylor coefficients.

Proposition 2.1.24. *Let $f\in H(\mathbb{B}_n)$, assume that the homogenous expansion in power series of f is*

$$f(z)=\sum_{m=0}^{+\infty}a_m z^m, \forall z\in\mathbb{B}_n.$$

Then, $f\in A^2_{\alpha}$ if and only if the following condition is satisfied

$$\sum_{m\geq 0}\frac{m!\Gamma(n+\alpha+1)}{\Gamma(n+|m|+\alpha+1)}|a_m|^2<\infty. \tag{2.1.13}$$

Proof. After some computations, we find

$$\int_{\mathbb{B}_n}|f(z)|^2dv_{\alpha}(z)=\int_{\mathbb{B}_n}\sum_{k,m=0}^{\infty}a_k\overline{a}_m z^k\overline{z}^m dv_{\alpha}(z)$$

$$=\sum_{m=0}^{\infty}|a_m|^2\int_{\mathbb{B}_n}|z^m|^2dv_{\alpha}(z)$$

$$=\sum_{m\geq 0}\frac{m!\Gamma(n+\alpha+1)}{\Gamma(n+|m|+\alpha+1)}|a_m|^2.$$

$\square$

We conclude this section with one of the main features of the Bergman kernel: the invariance under the group $Aut(\mathbb{B}_n)$.

Proposition 2.1.25. *Let $K_{\alpha}(z,w)$ be the Bergman kernel. Then*

$$K_{\alpha}(z,w)=det(J_{\mathbb{C}}\varphi(z))K_{\alpha}(\varphi(z),\varphi(w))\overline{det(J_{\mathbb{C}}\varphi(w))}, \tag{2.1.14}$$

where $\varphi\in Aut(\mathbb{B}_n)$.

Proof. Defining $H(z,w) := det(J_{\mathbb{C}}\varphi(z))K_\alpha(\varphi(z),\varphi(w))\overline{det(J_{\mathbb{C}}\varphi(w))}$, we have

$$
\begin{aligned}
< f(w), H(w,z) >_\alpha &= \int_{\mathbb{B}_n} f(w)det(J_{\mathbb{C}}\varphi(w))K_\alpha(\varphi(w),\varphi(z))\overline{det(J_{\mathbb{C}}\varphi(z))}dv_\alpha(w) \\
&= \overline{det(J_{\mathbb{C}}\varphi(z))}\int_{\mathbb{B}_n}(f\circ\varphi^{-1})(\varphi(w))det(J_{\mathbb{C}}\varphi(w))K_\alpha(\varphi(w),\varphi(z))dv_\alpha(w) \\
&= \overline{det(J_{\mathbb{C}}\varphi(z))}\int_{\mathbb{B}_n}(f\circ\varphi^{-1})(\varphi(w))\frac{det(J_{\mathbb{C}}\varphi(w))}{|det(J_{\mathbb{C}}\varphi(w))|^2}K_\alpha(\varphi(w),\varphi(z))dv_\alpha(\varphi(w)) \\
&= \overline{det(J_{\mathbb{C}}\varphi(z))}\int_{\mathbb{B}_n}\left\{(f\circ\varphi^{-1})(\varphi(w))\frac{1}{\overline{det(J_{\mathbb{C}}\varphi(w))}}\right\}K_\alpha(\varphi(w),\varphi(z))dv_\alpha(\varphi(w)) \\
&= \overline{det(J_{\mathbb{C}}\varphi(z))}f(z)\frac{1}{\overline{det(J_{\mathbb{C}}\varphi(z))}} \\
&= f(z).
\end{aligned}
$$

Finally, using Lemma 2.1.17, we obtain the desired formula. □

2.2 Bergman Type Projections

We verified that A^p_α is a closed supspace of $L^p(\mathbb{B}_n, dv_\alpha)$. This implies, in particular, that there exists an orthogonal projection P_α from $L^2(\mathbb{B}_n, dv_\alpha)$ to A^2_α, which is called the Bergman projection. The boundedness of the Bergman projection on $L^2(\mathbb{B}_n, dv_\alpha)$ is trivial from the general theory of Hilbert spaces, but its boundedness on $L^p(\mathbb{B}_n, dv_\alpha)$, for $p \neq 2$, is not obvious at all. Furthermore, the Bergman projection is a central object in the study of holomorphic function spaces. In fact, for example, the boundedness of the Bergman projection on $L^p(\mathbb{B}_n, dv_\alpha)$ immediately gives the duality between the Bergman spaces. Hence, understanding its behaviour and estimating its size is therefore of vital importance on several occasions. We start giving the definition of Bergman projection.

Definition 2.2.1. Let $\alpha > -1$, we introduce

$$P_\alpha(f)(z) := \int_{\mathbb{B}_n} f(w)K^\alpha(z,w)dv_\alpha(w), \quad f \in L^2(\mathbb{B}_n, dv_\alpha). \tag{2.2.1}$$

Remark 2.2.2. First of all, we prove that $P_\alpha(f)(z)$ is well defined. Using the uniform convergence on compact subsets of K_α, it turns out that

$$
\begin{aligned}
|P_\alpha(f)(z)| &\leq \int_{\mathbb{B}_n}|f(w)|\sum_{k=0}^{\infty}\frac{\Gamma(n+k+\alpha+1)}{k!\Gamma(n+\alpha+1)}|< z,w >|^k dv_\alpha(w) \\
&\underbrace{<}_{|w|<1}\int_{\mathbb{B}_n}|f(w)|\sum_{k=0}^{\infty}\frac{\Gamma(n+k+\alpha+1)}{k!\Gamma(n+\alpha+1)}|z|^k dv_\alpha(w) \\
&= \frac{1}{(1-|z|)^{n+1+\alpha}}\int_{\mathbb{B}_n}|f(w)|dv_\alpha(w)
\end{aligned}
$$

From Holder inequality, we deduce

$$|P_\alpha(f)(z)| < \frac{1}{(1-|z|)^{n+1+\alpha}}||f||_{p,\alpha}, \ z \in \mathbb{B}_n, \ 1 \leq p < +\infty.$$

For $p = \infty$, we have

$$
\begin{aligned}
|P_\alpha(f)(z)| &\le \int_{\mathbb{B}_n} |f(w)||K^\alpha(z,w)|dv_\alpha(w) \\
&\le ||f||_{\infty,\alpha} \int_{\mathbb{B}_n} |K^\alpha(z,w)|dv_\alpha(w) \\
&\underbrace{\le}_{Holder} ||f||_{\infty,\alpha} \left(\int_{\mathbb{B}_n} |K^\alpha(z,w)|^2 dv_\alpha(w) \right)^{1/2} \\
&= ||f||_{\infty,\alpha} \frac{1}{(1-|z|^2)^{(n+1+\alpha)/2}}, \ z \in \mathbb{B}_n.
\end{aligned}
$$

This means that, although the Bergman projection of P_α is originally defined on $L^2(\mathbb{B}_n, dv_\alpha)$, we clearly extends the domain of P_α to $L^1(\mathbb{B}_n, dv_\alpha)$. In other words, since $L^p(\mathbb{B}_n, dv_\alpha) \subseteq L^1(\mathbb{B}_n, dv_\alpha)$ for $1 \le p \le \infty$, we can apply P_α to a function in $L^p(\mathbb{B}_n, dv_\alpha)$ whenever $1 \le p \le \infty$. Moreover, these inequalities suggest how the behaviour of the Bergman projection is near the boundary of $\mathbb{B}_n$. In fact, for $1 \le p < \infty$, we prove that

$$
\lim_{|z| \to 1^-} P_\alpha(f)(z)(1-|z|)^{n+1+\alpha} = 0, \ f \in L^p(\mathbb{B}_n, dv_\alpha)
$$

Let $f(z) := I_{\mathbb{B}_n}(z)$ the indicator function be, then

$$
\begin{aligned}
\lim_{|z| \to 1^-} P_\alpha(f)(z)(1-|z|)^{n+1+\alpha} &= \lim_{|z| \to 1^-} (1-|z|)^{n+1+\alpha} \\
&= 0.
\end{aligned}
$$

Hence, for any simple function we easily obtain the same result. Finally, fix $f \in L^p(\mathbb{B}_n, dv_\alpha)$ and let a sequence of simple functions $f_n(z)$ be such that

$$
\lim_{n \to +\infty} ||f_n - f||_{p,\alpha} = 0,
$$

then

$$
\begin{aligned}
\lim_{|z| \to 1^-} |P_\alpha(f)(z)|(1-|z|)^{n+1+\alpha} &\le \lim_{|z| \to 1^-} |P_\alpha(f - f_n)(z)|(1-|z|)^{n+1+\alpha} + \lim_{|z| \to 1^-} |P_\alpha(f_n)(z)|(1-|z|)^{n+1+\alpha} \\
&\le \lim_{|z| \to 1^-} ||f - f_n||_{p,\alpha} \\
&< \varepsilon,
\end{aligned}
$$

for n large enough. Similarly, for $p = +\infty$, we show that

$$
\lim_{|z| \to 1^-} P_\alpha(f)(z)(1-|z|^2)^{(n+1+\alpha)/2} = 0.
$$

Remark 2.2.3. For $0 < p < 1$, in general, the Bergman projection is not well-defined. As a prove, we consider the following counterexample. For sake of simplicity, we suppose that $\alpha = 0$ and $z = 0$. Let the following function be

$$
f(w) := \frac{1}{1-|w|^2} \in L^p(\mathbb{B}_n, dv) \setminus L^1(\mathbb{B}_n, dv), \ p \in (0,1).
$$

Hence, under these conditions and using integration in polar coordinates, we get that the Bergman projection is

$$
P_0(f)(0) = \int_{\mathbb{B}_n} \frac{1}{1-|w|^2} dv(w) = \infty.
$$

A further property of the Bergman projection is given by

Proposition 2.2.4. *Let $\alpha > -1$, we have that*

$$P_\alpha(f)(z) \in H(\mathbb{B}_n),$$

where $f \in L^p(\mathbb{B}_n, dv_\alpha)$ and $1 \leq p \leq \infty$. Moreover, P_α sends the weighted Bergman space A^p_α to itself.

Proof.

$$\begin{aligned}
P_\alpha(f)(z) &= \int_{\mathbb{B}_n} f(w) K^\alpha(z,w) dv_\alpha(w) \\
&= \int_{\mathbb{B}_n} f(w) \sum_m^\infty \frac{\Gamma(n+|m|+\alpha+1)}{m!\Gamma(n+\alpha+1)} z^m \overline{w}^m dv_\alpha(w) \\
&= \sum_m^\infty \sqrt{\frac{\Gamma(n+|m|+\alpha+1)}{m!\Gamma(n+\alpha+1)}} z^m \underbrace{\int_{\mathbb{B}_n} f(w) \sqrt{\frac{\Gamma(n+|m|+\alpha+1)}{m!\Gamma(n+\alpha+1)}} \overline{w}^m dv_\alpha(w)}_{:=p_m} \\
&= \sum_m^\infty p_m e_m(z).
\end{aligned}$$

The fact that $P_\alpha : A^p_\alpha \longrightarrow A^p_\alpha$, for $0 < p < \infty$, follows easily from Lemma 2.1.3. This completes the proof. □

In the next Lemma, we prove that the operator P_α maps $L^2(\mathbb{B}_n, dv_\alpha)$ boundedly onto the Bergman space A^2_α.

Lemma 2.2.5. *Let $\alpha > -1$, the Bergman projection is the Hilbert space orthogonal projection of $L^2_\alpha(\mathbb{B}_n, dv_\alpha)$ onto its closed subspace A^2_α.*

Proof. Since $P_\alpha f(z) \in H(\mathbb{B}_n)$, applying Lemma 2.1.3 , we get

$$P_\alpha f(z) =< P_\alpha f, K^\alpha_z >_\alpha .$$

By the other hand, using Fubini Theorem

$$\begin{aligned}
< P_\alpha f, K^\alpha_z >_\alpha &= \int_{\mathbb{B}_n} P_\alpha f(w) K^\alpha(z,w) dv_\alpha(w) \\
&= \int_{\mathbb{B}_n} \left\{ \int_{\mathbb{B}_n} f(x) K^\alpha(w,x) dv_\alpha(x) \right\} K^\alpha(z,w) dv_\alpha(w) \\
&= \int_{\mathbb{B}_n} f(x) \left\{ \int_{\mathbb{B}_n} K^\alpha(z,w) K^\alpha(w,x) dv_\alpha(w) \right\} dv_\alpha(x) \\
&= \int_{\mathbb{B}_n} f(x) K^\alpha(z,x) dv_\alpha(x) \\
&=< f, K^\alpha_z >_\alpha,
\end{aligned}$$

that is,

$$< P_\alpha f, K^\alpha_z >_\alpha =< f, K^\alpha_z >_\alpha,$$

where $f \in L^2_\alpha(\mathbb{B}_n, dv_\alpha)$ and $z \in \mathbb{B}_n$. In other words, we used the reproducing property of K^α to $P_\alpha f$. Hence, we deduce that $(f - Pf)$ is orthogonal to K^α_z. Furthermore, the last equality also implies that

the Bergman projection must be given by integration against the Bergman kernel.
Now, $P_\alpha f(z)$ is well defined for all $f \in L^2_\alpha(\mathbb{B}_n, dv_\alpha)$. This fact was proved on Remark 2.2.2.. However, since $K^\alpha \in A^2_\alpha \subset L^2(\mathbb{B}_n, dv_\alpha)$, using Cauchy-Schwarz we notice that

$$\left| \int_{\mathbb{B}_n} f(w) K^\alpha(z,w) dv_\alpha(w) \right| \leq ||K^\alpha||_{2,\alpha} ||f||_{2,\alpha} < \infty.$$

Finally, using the notation of Proposition 2.2.4 and Bessel's inequality, we have

$$\begin{aligned}\int_{\mathbb{B}_n} |P_\alpha f(z)|^2 dv_\alpha(z) &= \sum_{m,j} p_m \overline{p_j} < e_m(z), e_j(z) >_\alpha \\ &= \sum_m |p_m|^2 \\ &\leq ||f||^2_{2,\alpha},\end{aligned}$$

we conclude that $P_\alpha f(z) \in A^2_\alpha$. □

Remark 2.2.6. Alternatively, we can show that P_α is the orthogonal projection proving that $P_\alpha \circ P_\alpha = P_\alpha$ and $< P_\alpha f, g >_\alpha = < f, P_\alpha g >_\alpha$ for $f, g \in L^2_\alpha(\mathbb{B}_n, dv_\alpha)$. The former identity follows by the reproducing property of the Bergman kernel when acting on functions in A^2_α, while the latter one follows from the hermitian symmetry property. Hence, we have

$$\begin{aligned}< P_\alpha f, g >_\alpha &= < \sum_m < f, e_m >_\alpha e_m, g >_\alpha \\ &= \sum_m < f, e_m >_\alpha < e_m, g >_\alpha \\ &= < f, \sum_m < g, e_m >_\alpha e_m >_\alpha \\ &= < f, P_\alpha g >_\alpha .\end{aligned}$$

That is, we used the characterisation of orthogonal projections.

Corollary 2.2.7. *Let $f \in L^2(\mathbb{B}_n, dv_\alpha)$ and $\varphi \in Aut(\mathbb{B}_n)$. Then*

$$P_\alpha(f \circ \varphi)(z) \in A^2_\alpha.$$

Proof. Under these conditions, it turns out that $(f \circ \varphi)(z) \in L^2(\mathbb{B}_n, dv_\alpha)$. Hence, applying the previous lemma, the wished result is obtained. □

Remark 2.2.8. In Lemma 2.2.5 we showed that

$$||P_\alpha(f)||_{2,\alpha} \leq ||f||_{2,\alpha},$$

this fact implies that

$$||P_\alpha||_{\mathcal{L}(L^2(\mathbb{B}_n, dv_\alpha), A^2_\alpha)} \leq 1.$$

Actually, the exact operator norm is one. This can be proved as follows

$$\begin{aligned}||P_\alpha(f)||_{2,\alpha} &= ||P_\alpha \circ P_\alpha(f)||_{2,\alpha} \\ &= ||P_\alpha(P_\alpha(f))||_{2,\alpha} \\ &\leq ||P_\alpha||_{\mathcal{L}(L^2(\mathbb{B}_n, dv_\alpha), A^2_\alpha)} ||P_\alpha(f)||_{2,\alpha}\end{aligned}$$

that is

$$1 \leq ||P_\alpha||_{\mathcal{L}(L^2(\mathbb{B}_n, dv_\alpha), A^2_\alpha)}.$$

The fact that $||P_\alpha||_{\mathcal{L}(L^2(\mathbb{B}_n,dv_\alpha),A^2_\alpha)} = 1$ will be crucial when we discuss the dual space of A^2_α.
Furthermore, let $\varphi \in Aut(\mathbb{B}_n)$, then we have that

$$\begin{aligned} ||P_\alpha(f \circ \varphi)||_{2,\alpha} &\leq ||f \circ \varphi||_{2,\alpha} \\ &\leq \frac{(1+|a|)^{(n+1+\alpha)/2}}{(1-|a|)^{(n+1+\alpha)/2}} ||f||_{2,\alpha}, \end{aligned}$$

that is

$$||P_\alpha(f \circ \varphi)||_{2,\alpha} \leq \frac{(1+|a|)^{(n+1+\alpha)/2}}{(1-|a|)^{(n+1+\alpha)/2}} ||f||_{2,\alpha},$$

where $a = \varphi(0)$.
Finally, we prove that the operator P_α is not injective. It suffices to prove that $Ker(P_\alpha) \neq \{0\}$. Let the function $f(z) := \overline{z}^k$ be, where $k \in \mathbb{N}^n \setminus \{0\}$. Then

$$\begin{aligned} P_\alpha(\overline{z}^k)(z) &= \sum_m^\infty z^m \frac{\Gamma(n+|m|+\alpha+1)}{m!\Gamma(n+\alpha+1)} \int_{\mathbb{B}_n} \overline{w}^{m+k} dv_\alpha(w) \\ &= \sum_m^\infty z^m \frac{\Gamma(n+|m|+\alpha+1)}{m!\Gamma(n+\alpha+1)} \int_0^1 r^k (1-r^2)^\alpha dr \underbrace{\int_{\mathbb{S}_n} \xi^{m+k} d\sigma(\xi)}_{=0,\ m+k\geq 1} \\ &= 0. \end{aligned}$$

We remark that we have applied a more general result concerning formulas for images of special classes of functions under the Bergman projection. The interested reader can find further details on the book Bergman spaces by Peter Duren and Alexander Schuster, Chapter 2 Lemma 6.

In Chapter 1, we presented some estimates concerning the boundedness of a family of integral operators. These tools are going to be used to understand how the operator P_α acts on other $L^p(\mathbb{B}_n, dv_\alpha)$ spaces, for $p \neq 2$.
As a special case of Theorem 1.4.6, considering the operator T and putting $a = 0$, in the following result are given sufficient and necessary conditions for the Bergman projection to be bounded.

Theorem 2.2.9. *Let $-1 < \gamma < \infty$, $-1 < \alpha < \infty$ and $1 \leq p < \infty$. Then*

$$P_\gamma : L^p(\mathbb{B}_n, dv_\alpha) \longrightarrow A^p_\alpha \text{ is bounded} \quad \Longleftrightarrow \quad p(\gamma+1) > \alpha+1. \tag{2.2.2}$$

First, considering $\gamma = \alpha$ and, secondly, $p = 1$, we find the following corollary.

Corollary 2.2.10. *P_γ is a bounded projection from $L^p(\mathbb{B}_n, dv_\alpha)$ onto A^p_α if and only if $p > 1$, and P_α is a bounded projection from $L^1(\mathbb{B}_n, dv_\alpha)$ onto A^1_α if and only if $\gamma > \alpha$.*

Remark 2.2.11. Note that, for $p \geq 1$ and $p(\gamma+1) > (\alpha+1)$, A^p_α can be thought as the quotient space induced by the Bergman type projection

$$P_\gamma : L^p(\mathbb{B}_n, dv_\alpha) \longrightarrow A^p_\alpha. \tag{2.2.3}$$

By the open mapping theorem the quotient norm on A^p_α is equivalent to the $||\cdot||_{p,\alpha}$ norm on A^p_α because the operator in (2.2.3) is bounded and onto.

Remark 2.2.12 (Unboundedness of the Bergman projection)**.** Whithout loss of generality we assume $\alpha = \gamma = 0$ and $p = 1$, we prove that the Bergman projection is not a bounded operator from $L^1(\mathbb{B}_n, dv)$ to $L^1(\mathbb{B}_n, dv)$. If $P_0 : L^1(\mathbb{B}_n, dv) \longrightarrow L^1(\mathbb{B}_n, dv)$ was bounded, $P_0 : L^\infty(\mathbb{B}_n, dv) \longrightarrow L^\infty(\mathbb{B}_n, dv)$ would also be bounded. But for fixed $\xi \in \mathbb{B}_n$, define the function

$$g_\xi(z) = (1 - \overline{z}\xi)^2 |1 - \overline{z}\xi|^{-2}.$$

Then $g_\xi \in L^\infty(\mathbb{B}_n, dv)$ and $||g_\xi||_\infty = 1$, while

$$(P_0 g_\xi)(z) = \int_{\mathbb{B}_n} |1 - \overline{\xi} z|^{-2} dv(z) = 2 \int_0^1 \frac{r}{1 - r^2 |\xi|^2} dr$$

is not bounded for $\xi \in \mathbb{B}_n$. This shows that P_0 does not map $L^\infty(\mathbb{B}_n, dv)$ boundedly to $L^\infty(\mathbb{B}_n, dv)$, and, hence, that the Bergman projection is not a bounded operator from $L^1(\mathbb{B}_n, dv)$ to $L^1(\mathbb{B}_n, dv)$.

2.3 Duality

The aim of this section is to identify the dual space of A^p_α, when $1 < p < +\infty$. We will see that the boundedness of the Bergman projection plays a fundamental role in this representation. For $0 < p \leq 1$, the dual space structure of A^p_α is more delicated and involves the Bloch spaces, which are introduced in the next chapter. In order to proceed, we begin with reviewing some notions from Functional Analysis.

Suppose $(X, || \cdot ||_X)$ is a normed vector space over the field $\mathbb{C}$, a linear functional

$$f : X \longrightarrow \mathbb{C}$$

is said to be bounded if there exists a positive constant C such that

$$|f(x)| \leq C ||x||_X \tag{2.3.1}$$

for all $x \in X$.
The dual space of X, denoted by X^*, is the vector space of all bounded linear functionals on X. Namely,

$$X^* := \{\ f : X \longrightarrow \mathbb{C} \mid f \text{ bounded linear functional}\} .$$

For any bounded linear functional f on X, we use $||f||_{X^*}$ to denote the smallest constant C satisfying (2.3.1), that is

$$||f||_{X^*} = \inf \{C \geq 0 : |f(x)| \leq C ||x||_X\} ,$$

or equivalently

$$||f||_{X^*} = \sup \{|f(x)| \mid ||x||_X = 1\} .$$

Then, it is easy to check that X^* becomes a Banach space with this norm.
The bidual space, denoted by X^{**}, consists of all continuous linear functionals $h : X^* \longrightarrow \mathbb{C}$. This space is endowed with the following norm:

$$||h||_{X^{**}} = \sup \{|h(f)| \mid ||f||_{X^*} = 1\} .$$

Each element $x \in X$ generates a scalar function $J(x) : X^* \longrightarrow \mathbb{C}$ by the formula:

$$J(x)(f) = f(x),\ \ f \in X^*,$$

and $J(x)$ is a continuous linear functional on X^*, that is, $J(x) \in X^{**}$. One obtains in this way a linear map

$$J : X \longrightarrow X^{**}$$

called evaluation map. By the Hahn-Banach theorem, J is injective and preserves norms:

$$||J(x)||_{X^{**}} = ||x||_X,$$

that is, J maps X isometrically onto its image $J(X)$ in X^{**}. Furthermore, the image $J(X)$ is closed in X^{**}. The space X is called reflexive if it satisfies the following equivalent conditions:

1) the evaluation map $J : X \longrightarrow X^{**}$ is surjective,

2) the evaluation map $J : X \longrightarrow X^{**}$ is an isometric isomorphism of normed spaces,

3) the evaluation $J : X \longrightarrow X^{**}$ is an isomorphism of normed spaces.

Moreover, every closed linear subspace of a reflexive space is reflexive and, if X is Banach space, the following are equivalent:

1) The space X is reflexive.

2) (James' theorem) Every continuous linear functional on X attains its norm, that is, there exists an element x of unit norm such that $|f(x)| = ||f||_{X^*}$.

Let $\alpha > -1$ and $\beta > -1$, it is a well-known fact that, for $1 < p < \infty$, the dual space of $L^p(\mathbb{B}_n, dv_\alpha)$ is isometrically isomorphic to $L^q(\mathbb{B}_n, dv_\beta)$, where

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad \gamma = \frac{\alpha}{p} + \frac{\beta}{q},$$

and the pairing is given by

$$F_g(f) = \int_{\mathbb{B}_n} f(z)\overline{g(z)}dv_\gamma(z), \quad f \in L^p(\mathbb{B}_n, dv_\alpha),\ g \in L^q(\mathbb{B}_n, dv_\beta).$$

Essentially the above representation holds for functionals $(A^p_\alpha)^*$. Namely, each such functional is uniquely represented in similar way by a function $g \in A^q_\beta$, but there is an important difference: for $p \neq 2$, the induced isomorphism between $(A^p_\alpha)^*$ and A^q_β is no longer an isometry, although the norms of F_g and g are equivalent in the sense that they are bounded by constant multiples of each other.

Besides, recalling that, by Theorem 2.1.4, the point evaluation, at any $z \in \mathbb{B}_n$, is a bounded linear functional on A^p_α and, hence, $(A^p_\alpha)^*$ is a nontrivial Banach space for all $p > 0$ and all $\alpha > -1$. Now, we can prove the main goal of this section.

Theorem 2.3.1. *Let $\alpha > -1$, $\beta > -1$, $1 < p, q < \infty$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \textit{and} \quad \gamma = \frac{\alpha}{p} + \frac{\beta}{q}.$$

Then, the following identification holds

$$(A^p_\alpha)^* = A^q_\beta \tag{2.3.2}$$

under the integral pairing

$$F_g(f) = < f, g >_\gamma = \int_{\mathbb{B}_n} f(z)\overline{g(z)}dv_\gamma(z), \tag{2.3.3}$$

where $f \in A^p_\alpha$, $g \in A^q_\beta$. Furthermore, the identification of (2.3.2) *holds so that the norms of F_g and g are equivalent. In other words,*

$$C_1||F_g||_{(A^p_\alpha)^*} \leq ||g||_{q,\beta} \leq C_2||F_g||_{(A^p_\alpha)^*},$$

for some positive constants C_1 and C_2.

Proof. First of all, we show that the weighted measure $dv_\gamma(z)$ is well-defined:

$$\gamma = \frac{\alpha}{p} + \frac{\beta}{q}$$

$$> -\frac{1}{p} - \frac{1}{q}$$

$$= -1.$$

We start proving $A^q_\beta \subseteq (A^p_\alpha)^*$. Let $g \in A^q_\beta$ and $f \in A^p_\alpha$, we define the linear functional

$$F_g(f) := < f, g >_\gamma = \int_{\mathbb{B}_n} (1-|z|^2)^{\alpha/p} f(z) \overline{(1-|z|^2)^{\beta/q} g(z)} dv(z)$$

hence, applying Holder's inequality, we find

$$|F_g(f)| \leq ||f||_{p,\alpha} ||g||_{q,\beta},$$

that is

$$||F_g||_{(A^p_\alpha)^*} \leq ||g||_{q,\beta},$$

in other words, $F_g \in (A^p_\alpha)^*$. To show that g is unique, assuming without loss of generality that

$$g(z) = \sum_k a_k z^k, \ \forall z \in \mathbb{B}_n,$$

then

$$F_g(z^m) = \sum_k \overline{a}_k \int_{\mathbb{B}_n} z^m \overline{z}^k dv_\gamma(z)$$

$$= \overline{a}_m \int_{\mathbb{B}_n} |z^m|^2 dv_\gamma(z)$$

$$= \overline{a}_m \frac{m! \Gamma(n+\gamma+1)}{\Gamma(n+|m|+\gamma+1)}.$$

Thus, if $F_{g_1}(z^m) = F_{g_2}(z^m)$ for all m, since polynomials are dense on every Bergman space, then, we deduce $g_1 = g_2$.
Conversely, for what concerns $(A^p_\alpha)^* \subseteq A^q_\beta$, let $F \in (A^p_\alpha)^*$, by the Hahn-Banach theorem, F extends to a bounded linear functional on $L^p(\mathbb{B}_n, dv_\alpha)$ with the same norm. So that, by the duality of L^p spaces, there exists $h \in L^q(\mathbb{B}_n, dv_\alpha)$ such that

$$F(f) = \int_{\mathbb{B}_n} f(z) \overline{h(z)} dv_\alpha(z),$$

where $f \in L^p(\mathbb{B}_n, dv_\alpha)$. Let

$$H(z) := \frac{c_\alpha}{c_\gamma}(1-|z|^2)^{(\alpha-\beta)/q} h(z), \quad z \in \mathbb{B}_n.$$

Then,

$$\int_{\mathbb{B}_n} |H(z)|^q dv_\beta(z) = \frac{c_\alpha^q}{c_\gamma^q} \int_{\mathbb{B}_n} (1-|z|^2)^{\alpha-\beta} |h(z)|^q dv_\beta(z)$$

$$= \frac{c_\alpha^q}{c_\gamma^q} \int_{\mathbb{B}_n} |h(z)|^q dv_\alpha(z) < \infty,$$

that is, $H \in L^q(\mathbb{B}_n, dv_\beta)$ and

$$\int_{\mathbb{B}_n} f(z)\overline{H(z)}dv_\gamma(z)\frac{c_\alpha}{c_\gamma} = \int_{\mathbb{B}_n} f(z)(1-|z|^2)^{(\alpha-\beta)/q}\overline{h(z)}(1-|z|^2)^{\frac{\alpha}{p}+\frac{\beta}{q}}dv(z)$$

$$= \int_{\mathbb{B}_n} f(z)\overline{h(z)}dv_\alpha(z)$$

$$= F(f), \quad f \in A^p_\alpha.$$

Moreover, since the following conditions hold

$$\alpha > -1 \iff q(\gamma+1) > \beta+1 \quad \text{and} \quad \beta > -1 \iff p(\gamma+1) > \alpha+1.$$

Hence, as a consequence of Theorem 2.2.6, both projections $P_\gamma : L^p(\mathbb{B}_n, dv_\alpha) \longrightarrow A^p_\alpha$ and $P_\gamma : L^q(\mathbb{B}_n, dv_\beta) \longrightarrow A^q_\beta$ are bounded. Finally, denoting by $g := P_\gamma(H)$, we have that $g \in A^q_\beta$ and

$$F(f) =< f, H >_\gamma$$

$$=< P_\gamma(f), H >_\gamma$$

$$=< f, P_\gamma(H) >_\gamma$$

$$=< f, g >_\gamma,$$

that is

$$F(f) =< f, g >_\gamma, \ \forall f \in A^p_\alpha.$$

Now, since $||F||_{(A^q_\beta)^*} = ||H||_{q,\beta}$, we get

$$||g||_{q,\beta} = ||P_\gamma(H)||_{q,\beta}$$

$$\leq ||P_\gamma||_{\mathcal{L}(L^q(\mathbb{B}_n, dv_\beta), A^q_\beta)}||H||_{q,\beta}$$

$$= ||P_\gamma||_{\mathcal{L}(L^q(\mathbb{B}_n, dv_\beta), A^q_\beta)}||F||_{(A^q_\beta)^*},$$

namely, the above identification holds with equivalent norms.
Finally, if $p = q = 2$, from Remark 2.2.8, we know that

$$||P_\gamma||_{\mathcal{L}(L^2(\mathbb{B}_n, dv_\beta), A^2_\beta)} = 1,$$

hence, in such case, we obtain

$$||g||_{q,\beta} \leq ||F||_{(A^q_\beta)^*} \leq ||g||_{q,\beta},$$

that is,

$$||g||_{q,\beta} = ||F||_{(A^q_\beta)^*}.$$

In other words,

$$(A^2_\alpha)^* = A^2_\beta,$$

is an isometric isomorphism. This completes the proof. □

Remark 2.3.2. It is evident from the previous proof that the representation of the dual space of A^p_α depends heavily on the boundedness of the Bergman projection. The proof does not extend to A^1_α because, if $p = 1$, the Bergman projection is bounded if and only if we have that both $\gamma > \alpha$ and $\gamma > \beta$ hold. But, under such conditions, we would have that

$$\gamma > \alpha \Rightarrow \beta > \alpha$$

and

$$\gamma > \beta \Rightarrow \alpha > \beta,$$

which is clearly a contradiction.

In particular, choosing $\alpha = \beta$, we find

Corollary 2.3.3. *Let $\alpha > -1$, $1 < p, q < \infty$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, the following identification holds with equivalent norms

$$(A^p_\alpha)^* = A^q_\alpha$$

under the integral pairing

$$< f, g >_\alpha, \quad f \in A^p_\alpha, g \in A^q_\alpha.$$

Moreover,

$$(A^p_\alpha)^{**} = A^p_\alpha. \tag{2.3.4}$$

That is, for $1 < p < \infty$, the Bergman spaces A^p_α are reflexive.

Remark 2.3.4. Actually, (2.3.4) can be alternatively proved as follows. For $1 < p < \infty$, the space $L^p(\mathbb{B}_n, dv_\alpha)$ is reflexive. Since $A^p_\alpha \subset L^p(\mathbb{B}_n, dv_\alpha)$ is a closed subspace, we deduce that the weighted Bergman spaces A^p_α are reflexive, for $1 < p < \infty$. Hence, as a consequence, by James Theorem every linear functional of $(A^p_\alpha)^*$ attains its norm.

Remark 2.3.5 (Point evaluation Linear Functional)**.** From Theorem 2.1.4, for any $f \in A^p_\alpha$ and $0 < p < \infty$, we had that

$$|f(z)| \leq \frac{||f||_{p,\alpha}}{(1 - |z|^2)^{(n+1+\alpha)/p}}.$$

Hence, the linear functional given by the point evaluation, denoted by $L_z(f) := f(z)$, satisfies

$$||L_z||_{(A^p_\alpha)^*} \leq \frac{1}{(1 - |z|^2)^{(n+1+\alpha)/p}}, \quad 0 < p < +\infty.$$

For $p = 2$ we are able to exactly calculate this norm. In fact, applying the extremal property of Corollary 2.1.20, it turns out that

$$||L_z||_{(A^2_\alpha)^*} = \frac{1}{(1 - |z|^2)^{(n+1+\alpha)/2}}.$$

Finally, we can find the element where L_z attains its norm as follows: let a point $w \in \mathbb{B}_n$, the function

$$f(z) := a \frac{K^\alpha(z, w)}{\sqrt{K^\alpha(w, w)}}, \quad |a| = 1,$$

is such that

$$\begin{aligned} ||f||_{2,\alpha} &= \frac{1}{\sqrt{K^\alpha(w, w)}} ||K^\alpha(z, w)||_{2,\alpha} \\ &= \frac{\sqrt{K^\alpha(w, w)}}{\sqrt{K^\alpha(w, w)}} \\ &= 1. \end{aligned}$$

Hence, choosing $w = z$, we find

$$|f(z)| = \frac{K^\alpha(z,z)}{\sqrt{K^\alpha(z,z)}}$$

$$= \sqrt{K^\alpha(z,z)}$$

$$= \frac{1}{(1-|z|^2)^{(n+1+\alpha)/2}}.$$

We give a further example of linear functional on $L^p(\mathbb{B}_n, dv_\alpha)$, for $1 \le p \le \infty$, and the behaviour of its norm operator: the Bergman projection P_α.

Example 2.3.6. In Remark 2.2.2, we proved that

$$|P_\alpha(f)(z)| < \frac{||f||_{p,\alpha}}{(1-|z|)^{n+1+\alpha}}, z \in \mathbb{B}_n,\ 1 \le p < \infty.$$

Hence, fixed $z \in \mathbb{B}_n$, the linear operator

$$P_\alpha(\cdot)(z) : f \in L^p(\mathbb{B}_n, dv_\alpha) \longrightarrow P_\alpha(f)(z) \in \mathbb{C}$$

satisfies

$$||P_\alpha||_{(L^p(\mathbb{B}_n,dv_\alpha))^*} < \frac{1}{(1-|z|)^{n+1+\alpha}}, z \in \mathbb{B}_n,\ 1 \le p < \infty.$$

Finally, for $p = \infty$, we have

$$||P_\alpha||_{(L^\infty(\mathbb{B}_n,dv_\alpha))^*} < \frac{1}{(1-|z|^2)^{(n+1+\alpha)/2}}.$$

Remark 2.3.7 (Uniformly convexity and Strictly convexity)**.** For $0 < p \le 1$, assume that A^p_α is not reflexive. This fact will be proved in the next chapter. Hence, from Milman-Pettis theorem, the space A^p_α is not uniformly convex. However A^p_α, for $0 < p \le 1$ is strictly convex. This last assertion can be verified as follows. Suppose $f, g \in A^1_\alpha$, so that $||f||_{1,\alpha} = ||g||_{1,\alpha} = 1$, and $||f+g||_{1,\alpha} = 1$. Then

$$||f+g||_{1,\alpha} = ||f||_{1,\alpha} + ||g||_{1,\alpha},$$

this is possible only if $f(z) = \lambda g(z)$, where $\lambda(z) > 0$ at every point z where $f(z)g(z) \neq 0$. But $\lambda = f/g$ is a meromorphic function, so the requirement that $\lambda(z) > 0$ forces it to be constant. Thus $f(z) = cg(z)$ for some constant $c > 0$. But now it follows $c = 1$, since $||f||_{1,\alpha} = ||g||_{1,\alpha} = 1$. Thus $f = g$, which proves that A^1_α is strictly convex. Finally, using the same approach, we can also show that the Bergman space A^p_α, for $0 < p < 1$, are strictly convex.

2.4 Characterisation in terms of derivatives

We begin the current section recalling the notion of holomorphic gradient and introduce the invariant gradient. After that, we obtain a result that compares the various derivatives that we use for a holomorphic function and, as a consequence, show that the holomorphic gradient admits an estimate from above and below in terms of the invariant gradient and the radial derivative respectively. The invariance, under the action of $Aut(\mathbb{B}_n)$, of the invariant gradient is also proved. Then, we provide an embedding lemma for Bergman spaces. Finally, the main goal of this section will be proved: we establish various characterisations of A^p_α in terms of higher order derivatives and, hence, obtain some straightforward consequence as well.

In this section, and throughout this thesis, we will write

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), ..., \frac{\partial f}{\partial z_n}(z)\right) \tag{2.4.1}$$

and call $|\nabla f(z)|$ the holomorphic gradient of f at z. Similarly,

$$\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0), \tag{2.4.2}$$

where φ_z is the involutive automorphism. $|\widetilde{\nabla} f(z)|$ will be called the invariant gradient of f at z.

The following formula provides the bridge between the objects mentioned above and the radial derivative.

Lemma 2.4.1. *Let $f \in H(\mathbb{B}_n)$, then the following formula holds*

$$|\widetilde{\nabla} f(z)|^2 = (1-|z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2), \quad \forall z \in \mathbb{B}_n \tag{2.4.3}$$

Proof. Writing

$$f(z) = u(x_1, y_1, ..., x_n, y_n) + iv(x_1, y_1, ..., x_n, y_n), \text{ where } u, v : \mathbb{B}_n \longrightarrow \mathbb{R},$$

so that we get

$$|f|^2 = u(x_1, y_1, ..., x_n, y_n)^2 + v(x_1, y_1, ..., x_n, y_n)^2.$$

After some computations, we find

$$\partial_{x_j}|f|^2 = 2u(x,y)\partial_{x_j}u(x,y) + 2v(x,y)\partial_{x_j}v(x,y), \quad j = 1, ..., n,$$

as well as

$$\partial_{y_j}|f|^2 = 2u(x,y)\partial_{y_j}u(x,y) + 2v(x,y)\partial_{y_j}v(x,y), \quad j = 1, ..., n.$$

Hence,

$$\partial_{x_jx_j}|f|^2 = 2\left(\partial_{x_j}u(x,y)\right)^2 + 2u(x,y)\partial_{x_jx_j}u(x,y) + 2\left(\partial_{x_j}v(x,y)\right)^2 + 2v(x,y)\partial_{x_jx_j}v(x,y), \quad j = 1, ..., n,$$

$$\partial_{y_jy_j}|f|^2 = 2\left(\partial_{y_j}u(x,y)\right)^2 + 2u(x,y)\partial_{y_jy_j}u(x,y) + 2\left(\partial_{y_j}v(x,y)\right)^2 + 2v(x,y)\partial_{y_jy_j}v(x,y), \quad j = 1, ..., n.$$

Since, as a consequence of the Cauchy-Riemann equations in several variables, u and v are harmonic functions, we deduce that

$$\begin{aligned}
\Delta(|f|^2)(0) &= \sum_{j=1}^{n}\left(\partial_{x_jx_j}|f|^2 + \partial_{y_jy_j}|f|^2\right) \\
&= 2u(0,0)\underbrace{\Delta u(0,0)}_{=0} + 2v(0,0)\underbrace{\Delta v(0,0)}_{=0} + 2\underbrace{|\nabla u|^2}_{=|\nabla f(0)|^2} + 2\underbrace{|\nabla v|^2}_{=|\nabla f(0)|^2} \\
&= 4|\nabla f(0)|^2,
\end{aligned}$$

that is

$$\Delta(|f|^2)(0) = 4|\nabla f(0)|^2.$$

Moreover, using the definition of invariant Laplacian and (2.4.2), the above equality implies

$$\begin{aligned}
\widetilde{\Delta}(|f|^2)(0) &= \Delta(|f|^2 \circ \varphi_0)(0) \\
&= \Delta(|f|^2)(0) \\
&= 4|\nabla f(0)|^2 \\
&= 4|\widetilde{\nabla} f(0)|^2,
\end{aligned}$$

by the other hand,

$$\begin{aligned}
4|\widetilde{\nabla} f(z)|^2 &= 4|\widetilde{\nabla}(f\circ\varphi_z)(0)|^2 \\
&= \widetilde{\Delta}(|f\circ\varphi_z|^2)(0) \\
&= \widetilde{\Delta}(|f|^2)(z),
\end{aligned}$$

that is

$$4|\widetilde{\nabla} f(z)|^2 = \widetilde{\Delta}(|f|^2)(z). \tag{2.4.4}$$

Since, using the conjugation properties of Wirtinger operators, we have

$$\begin{aligned}
|Rf(z)|^2 &= \left|\sum_{i=1}^{n} z_k \frac{\partial f}{\partial z_i}(z)\right|^2 \\
&=< \sum_{i=1}^{n} z_k \frac{\partial f}{\partial z_i}(z), \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) > \\
&= \sum_{j,i=1}^{n} < z_i \frac{\partial f}{\partial z_i}(z), z_j \frac{\partial f}{\partial z_j}(z) > \\
&= \sum_{j,i=1}^{n} z_i \frac{\partial f}{\partial z_i}(z) \overline{z}_j \overline{\frac{\partial f}{\partial z_j}}(z) \\
&= \sum_{j,i=1}^{n} z_i \frac{\partial f}{\partial z_i}(z) \overline{z}_j \frac{\partial \overline{f}}{\partial \overline{z}_j}(z) \\
&= \sum_{j,i=1}^{n} z_i \overline{z}_j \frac{\partial |f|^2}{\partial z_i \partial \overline{z}_j}(z),
\end{aligned}$$

that is

$$|Rf(z)|^2 = \sum_{j,i=1}^{n} z_i \overline{z}_j \frac{\partial |f|^2}{\partial z_i \partial \overline{z}_j}(z).$$

To conclude, we use Proposition 1.3.19. as follows

$$\begin{aligned}
\widetilde{\Delta}(|f|^2)(z) &= 4(1-|z|^2)\sum_{i,j=1}^{n}(\delta_{i,j} - z_i\overline{z}_j)\frac{\partial^2 |f|^2}{\partial z_i \partial \overline{z}_j}(z) \\
&= 4(1-|z|^2)\left(\sum_{i,j=1}^{n}\delta_{i,j}\frac{\partial^2 |f|^2}{\partial z_i \partial \overline{z}_j}(z) - \sum_{i,j=1}^{n} z_i\overline{z}_j\frac{\partial^2 |f|^2}{\partial z_i \partial \overline{z}_j}(z)\right) \\
&= 4(1-|z|^2)\left(\sum_{j=1}^{n}\frac{\partial^2 |f|^2}{\partial z_j \partial \overline{z}_j}(z) - |Rf(z)|^2\right) \\
&= 4(1-|z|^2)\left(|\nabla f(z)|^2 - |Rf(z)|^2\right),
\end{aligned}$$

namely,

$$\widetilde{\Delta}(|f|^2)(z) = 4(1-|z|^2)\left(|\nabla f(z)|^2 - |Rf(z)|^2\right).$$

Finally, we substitute this last equality in (2.4.4) and the wished result is obtained. □

Remark 2.4.2. In the former proof we used the following property

$$\frac{\partial f}{\partial z_i}(z)\frac{\partial \overline{f}}{\partial \overline{z}_j}(z) = \frac{\partial^2 |f|^2}{\partial z_i \partial \overline{z}_j}(z),$$

where f is any holomorphic function. As a prove, using Cauchy-Riemann equations (that is (1.2.1)), the conjugation properties of Wirtinger operators and the product rule, we find

$$\begin{aligned}
\frac{\partial^2 |f|^2}{\partial z_i \partial \overline{z}_j}(z) &= \frac{\partial}{\partial \overline{z}_j}\frac{\partial (f\overline{f})}{\partial z_i}(z) \\
&= \frac{\partial}{\partial \overline{z}_j}\left\{ f(z)\frac{\partial \overline{f}}{\partial z_i}(z) + \overline{f(z)}\frac{\partial f}{\partial z_i}(z)\right\} \\
&= \frac{\partial}{\partial \overline{z}_j}\left\{ \overline{f(z)}\frac{\partial f}{\partial z_i}(z)\right\} \\
&= \frac{\partial \overline{f}}{\partial \overline{z}_j}(z)\frac{\partial f}{\partial z_i}(z) + \overline{f(z)}\frac{\partial}{\partial z_i}\frac{\partial f}{\partial \overline{z}_j}(z) \\
&= \frac{\partial \overline{f}}{\partial \overline{z}_j}(z)\frac{\partial f}{\partial z_i}(z).
\end{aligned}$$

By the fact that the invariant Laplacian is invariant under the action of the automorphism group, we have the following important corollary.

Corollary 2.4.3. $|\widetilde{\nabla} f|$ *is Möbius invariant, that is*

$$|\widetilde{\nabla}(f\circ\varphi)(z)| = |(\widetilde{\nabla} f)\circ\varphi(z)|, \quad \forall f \in H(\mathbb{B}_n), \tag{2.4.5}$$

where $\varphi \in Aut(\mathbb{B}_n)$.

Proof. Since

$$4|\widetilde{\nabla} f(z)|^2 = \widetilde{\Delta}(|f|^2)(z),$$

this fact implies

$$\begin{aligned}
4|(\widetilde{\nabla} f)\circ\varphi(z)|^2 &= \widetilde{\Delta}(|f|^2)(\varphi(z)) \\
&= \widetilde{\Delta}(|f\circ\varphi|^2)(z) \\
&= 4|\widetilde{\nabla}(f\circ\varphi)(z)|^2,
\end{aligned}$$

and this completes our proof. □

A fundamental tool, that follows from the previous lemma, will be the following chain of inequalities.

Lemma 2.4.4. *Let* $f \in H(\mathbb{B}_n)$, *then*

$$(1-|z|^2)|Rf(z)| \leq (1-|z|^2)|\nabla f(z)| \leq |\widetilde{\nabla} f(z)|, \ \forall\ z \in \mathbb{B}_n \tag{2.4.6}$$

Proof.

$$|Rf(z)| \doteq \left|\sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}\right|$$

$$\underbrace{\leq}_{Cauchy-Schwarz} \left\{\sum_{k=1}^{n} |z_k|^2\right\}^{1/2} \left\{\sum_{k=1}^{n} \left|\frac{\partial f}{\partial z_k}\right|^2\right\}^{1/2}$$

$$= |z||\nabla f(z)| \leq |\nabla f(z)|,$$

that is,

$$|Rf(z)| \leq |\nabla f(z)|,$$

and, multiplying both sides by $(1-|z|^2)$, we clearly get the first inequality. For the second inequality, using Lemma 2.4.1 and $|Rf(z)| \leq |z||\nabla f(z)|$, it turns out that

$$\begin{aligned}
|\widetilde{\nabla} f(z)| &= \sqrt{(1-|z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2)} \\
&\geq \sqrt{(1-|z|^2)(|\nabla f(z)|^2 - |z|^2|\nabla f(z)|^2)} \\
&= (1-|z|^2)|\nabla f(z)|.
\end{aligned}$$

□

The next lemma is a key estimate for the study of Bergman spaces, when $0 < p \leq 1$. It will be needed several times later in this thesis.

Lemma 2.4.5. *Given* $0 < p \leq 1$ *and* $\alpha > -1$, *then*

$$\int_{\mathbb{B}_n} |f(z)|(1-|z|^2)^{(n+1+\alpha)/p-(n+1)} dv(z) \leq \frac{n!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} ||f||_{p,\alpha}, \quad \forall\ f \in A^p_\alpha. \tag{2.4.7}$$

Proof. Writing

$$|f(z)| = |f(z)|^p |f(z)|^{1-p},$$

denoting by $c_{\alpha,n} := \dfrac{n!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}$ and applying Theorem 2.1.4 to $|f(z)|^{1-p}$, we get

$$\begin{aligned}
\int_{\mathbb{B}_n} |f(z)|(1-|z|^2)^{(n+1+\alpha)/p-(n+1)} dv(z) &= \int_{\mathbb{B}_n} |f(z)|^p |f(z)|^{1-p} (1-|z|^2)^{(n+1+\alpha)/p-(n+1)} dv(z) \\
&\leq \int_{\mathbb{B}_n} |f(z)|^p \left(\frac{||f||_{p,\alpha}}{(1-|z|^2)^{(n+1+\alpha)/p}}\right)^{1-p} (1-|z|^2)^{(n+1+\alpha)/p-(n+1)} dv(z) \\
&= c_{\alpha,n} ||f||^{1-p}_{p,\alpha} \int_{\mathbb{B}_n} |f(z)|^p \underbrace{\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} (1-|z|^2)^\alpha dv(z)}_{dv_\alpha(z)} \\
&= c_{\alpha,n} ||f||_{p,\alpha},
\end{aligned}$$

and we are done. □

Remark 2.4.6. As a further consequence of Theorem 2.1.4, we notice that the exponent $\beta := \dfrac{n+1+\alpha}{p} - (n+1)$ is the best possible one. Moreover, the previous result should be considered as an embedding of the Bergman space A^p_β into A^1_α, for $0 < p \leq 1$.

Remark 2.4.7. Since, for $0 < p < 1$,

$$\gamma := \frac{n+1+\alpha}{p} - (n+1) > -1,$$

defining the linear functional $\Phi : A^p_\alpha \longrightarrow \mathbb{C}$ as follows

$$\Phi(f) := \int_{\mathbb{B}_n} f(z) dv_\gamma(z).$$

Hence, as a consequence of the previous Lemma, we find

$$||\Phi||_{(A^p_\alpha)^*} \leq 1.$$

Moreover, (2.4.7) can be written as follows

$$||f||_{1,\gamma} \leq ||f||_{p,\alpha}, \quad 0 < p \leq 1,$$

that is, the continuous embedding of A^p_β into A^1_α.

If $n = 1$, we clearly have

$$Rf(z) = zf'(z), \quad |\nabla f(z)| = |f'(z)|, \quad |\widetilde{\nabla} f(z)| = (1-|z|^2)|f'(z)|,$$

and notice that the functions

$$(1-|z|^2)|Rf(z)|, \quad (1-|z|^2)|\nabla f(z)|, \quad |\widetilde{\nabla} f(z)|$$

have exactly the same boundary behaviour on the unit disk. In higher dimensions, the three functions above no longer have the same boundary behaviour. However, in the next theorem we show that when integrated against the weighted volume measures dv_α, these differential-based functions not only exhibit the same behaviour, they also behave the same as the original function $f(z)$. Besides, this result will be used to illustrate the complex interpolation of the Bloch space.

Theorem 2.4.8. *Let $\alpha > -1$, $p > 0$ and $f \in H(\mathbb{B}_n)$, then the following are equivalent:*
a) $f \in A^p_\alpha$
b) $|\widetilde{\nabla} f(z)| \in L^p(\mathbb{B}_n, dv_\alpha)$
c) $(1-|z|^2)|\nabla f(z)| \in L^p(\mathbb{B}_n, dv_\alpha)$
d) $(1-|z|^2)|Rf(z)| \in L^p(\mathbb{B}_n, dv_\alpha)$.

Moreover, the quantities

$$\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p dv_\alpha(z)\ , \quad \int_{\mathbb{B}_n} (1-|z|^2)|\nabla f(z)|^p dv_\alpha(z)\ , \quad \int_{\mathbb{B}_n} (1-|z|^2)|Rf(z)|^p dv_\alpha(z),$$

are all comparable to

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z),$$

whenever $f \in H(\mathbb{B}_n)$.

Proof. We start proving that *b)* implies *c)*. From the inequality

$$(1-|z|^2)|\nabla f(z)| \leq |\widetilde{\nabla} f(z)|$$

of (2.4.6), multiplying both sides by $(1-|z|^2)^\alpha$ and then integrating, we easily obtain

$$||(1-|z|^2)\nabla f||_{p,\alpha} \leq ||\widetilde{\nabla} f||_{p,\alpha}.$$

$c) \Rightarrow d)$, from the inequality

$$(1-|z|^2)|Rf(z)| \leq (1-|z|^2)|\nabla f(z)|$$

of (2.4.6), and, proceeding similarly as in the previous case, we get the desired result.
Now, our aim is to prove that $a)$ implies $b)$. Defining $g(w) := f \circ \varphi_z(w)$ and choosing $\beta > \alpha$, where φ_z is the involutive automorphism, from Lemma 2.1.6 there exists a constant $C_1 := C_1(z) > 0$ such that

$$|\nabla g(0)|^p \leq C_1 \int_{\mathbb{B}_n} |g(w)|^p dv_\beta(w)$$

Hence, using the change of variables formula of Proposition 1.4.7, our result is

$$|\widetilde{\nabla} f(z)|^p \leq C_1 (1-|z|^2)^{n+1+\beta} \int_{\mathbb{B}_n} \frac{|f(w)|^p dv_\beta(w)}{|1-<z,w>|^{2(n+1+\beta)}},$$

then, integrating both sides respect to $dv_\alpha(z)$ and using Fubini's theorem, we find that

$$\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p dv_\alpha(z) \leq C_1 \int_{\mathbb{B}_n} (1-|z|^2)^{n+1+\beta} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(w)|^p dv_\beta(w)}{|1-<z,w>|^{2(n+1+\beta)}}$$

$$\underbrace{=}_{Fubini} C_1 \int_{\mathbb{B}_n} |f(w)|^p dv_\beta(w) \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\beta}}{|1-<z,w>|^{2(n+1+\beta)}} dv_\alpha(z)$$

$$= C_1 \int_{\mathbb{B}_n} |f(w)|^p dv_\beta(w) \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\beta+\alpha}}{|1-<z,w>|^{n+1+(n+1+\beta+\alpha)+\beta-\alpha}} dv(z).$$

Denoting by $t := n+1+\alpha$ and $c := \beta - \alpha > 0$ and applying Theorem 1.4.4, we find

$$\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p dv_\alpha(z) \leq C_1 \int_{\mathbb{B}_n} |f(w)|^p dv_\beta(w) \int_{\mathbb{B}_n} \frac{(1-|z|^2)^t}{|1-<z,w>|^{n+1+t+c}} dv(z)$$

$$\leq C_2 \int_{\mathbb{B}_n} |f(w)|^p dv_\beta(w)(1-|w|^2)^{-(\beta-\alpha)}$$

$$= C_2 \int_{\mathbb{B}_n} |f(w)|^p (1-|w|^2)^{-(\beta-\alpha)}(1-|w|^2)^\beta dv(w)$$

$$= C_2 \int_{\mathbb{B}_n} |f(w)|^p (1-|w|^2)^\alpha dv(w)$$

$$= C_2 \int_{\mathbb{B}_n} |f(w)|^p dv_\alpha(w),$$

that is

$$\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p dv_\alpha(z) \leq C_2 \int_{\mathbb{B}_n} |f(w)|^p dv_\alpha(w).$$

Finally, we substitute $f - f(0)$ instead of f:

$$\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p dv_\alpha(z) \leq C_2 \int_{\mathbb{B}_n} |f(z)-f(0)|^p dv_\alpha(z),$$

and we obtain the desired result.
We wish to prove that $d)$ implies $a)$. We apply Theorem 2.1.4, choosing $\beta > 0$ large enough, to obtain

$$Rf(z) = \int_{\mathbb{B}_n} \frac{Rf(w) dv_\beta(w)}{(1-<z,w>)^{n+1+\beta}}, \quad z \in \mathbb{B}_n. \tag{2.4.8}$$

We observe that

$$Rf(0) = \sum_{k=1}^{n} \frac{\partial f}{\partial z_i}(0)0$$

$$= 0.$$

Hence, (2.4.8) can be written as

$$Rf(z) = \int_{\mathbb{B}_n} Rf(w) \left(\frac{1}{(1- < z, w >)^{n+1+\beta}} - 1 \right) dv_\beta(w)$$

More is true, using formula (1.5.3), we also have

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t} dt$$

$$= \int_{\mathbb{B}_n} Rf(w) \int_0^1 \left(\frac{1}{(1- < tz, w >)^{n+1+\beta}} - 1 \right) dv_\beta(w) \frac{dt}{t},$$

that is,

$$f(z) - f(0) = \int_{\mathbb{B}_n} Rf(w) \int_0^1 \left(\frac{1}{(1- < tz, w >)^{n+1+\beta}} - 1 \right) dv_\beta(w) \frac{dt}{t}, \tag{2.4.9}$$

If we define the kernel

$$H(z, w) = \int_0^1 \left(\frac{1}{(1 - t < z, w >)^{n+1+\beta}} - 1 \right) \frac{dt}{t},$$

then, (2.4.9) becomes

$$f(z) - f(0) = \int_{\mathbb{B}_n} Rf(w) H(z, w) dv_\beta(w).$$

We proceed estimating the kernel $H(z, w)$ as follows

$$|H(z, w)| = \left| \int_0^1 \left(\frac{1 - (1 - t < z, w >)^{n+\beta+1}}{(1 - t < z, w >)^{n+\beta+1}} \right) \frac{dt}{t} \right|$$

$$\leq C \left| \int_0^1 \frac{< z, w > dt}{(1 - t < z, w >)^{n+\beta+1}} \right|$$

$$= C \left| -(1 - t < z, w >)^{-(n+\beta)} |_{t=0}^{t=1} \right|$$

$$\leq \frac{\tilde{C}}{|1- < z, w > |^{n+\beta}},$$

that is

$$|H(z, w)| \leq \frac{\tilde{C}}{|1- < z, w > |^{n+\beta}}, \forall z, w \in \mathbb{B}_n.$$

Then, we have that

$$|f(z) - f(0)| \leq \tilde{C} \int_{\mathbb{B}_n} \left| \frac{Rf(w)}{(1- < z, w >)^{n+\beta}} \right| (1 - |w|^2)^\beta dv(w). \tag{2.4.10}$$

We use this inequality as follows: we consider two cases, the first is regarding $1 \leq p < \infty$. Chosen β large enough so that

$$0 < \alpha + 1 < p\beta.$$

Then, as a consequence of Theorem 1.4.6, where we put $\alpha := \beta - 1$, $a = 0$, and $\phi(w) := |Rf(w)|(1-|w|^2)$, we find

$$\int_{\mathbb{B}_n} |f(z)-f(0)|^p dv_\alpha(z) \leq \tilde{C} \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \left\{ |Rf(w)|(1-|w|^2) \right\} \frac{(1-|w|^2)^{\beta-1}}{|1-<z,w>|^{n+\beta}} dv(w) \right)^p dv_\alpha(z),$$

that is, $f \in L^p(\mathbb{B}_n, dv_\alpha)$.
We prove the case $0 < p < 1$. We choose β large enough so that

$$\beta = \frac{n+1+\beta'}{p} - (n+1)$$

where $\beta' > \alpha + p > -1$. Hence, using Lemma 2.4.5 and the inequality (2.4.10), where we put $\alpha := \beta'$, we have that

$$|f(z)-f(0)|^p \leq \int_{\mathbb{B}_n} \left| \frac{Rf(w)}{(1-<z,w>)^{n+\beta}} \right| (1-|w|^2)^{\frac{n+1+\beta'}{p}-(n+1)} dv(w)$$

$$\underbrace{\leq}_{Lemma\ 2.4.5} \left\| \frac{Rf(w)}{(1-<z,w>)^{n+\beta}} \right\|_{p,\beta'}.$$

Afterthat, applying Fubini and Theorem 1.4.4 part c), where we put $t = \alpha$, $c := np + p\beta - n - 1 - \alpha > 0$, it turns out that

$$\int_{\mathbb{B}_n} |f(z)-f(0)|^p dv_\alpha(z) \leq \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \left| \frac{Rf(w)}{|1-<z,w>|^{n+\beta}} \right|^p dv_{\beta'}(w)$$

$$= \int_{\mathbb{B}_n} |Rf(w)|^p dv_{\beta'}(w) \int_{\mathbb{B}_n} \frac{(1-|z|^2)^\alpha}{|1-<z,w>|^{pn+p\beta}} dv(z)$$

$$= \int_{\mathbb{B}_n} |Rf(w)|^p dv_{\beta'}(w) \int_{\mathbb{B}_n} \frac{(1-|z|^2)^\alpha}{|1-<z,w>|^{n+1+\alpha+(pn+p\beta-n-1-\alpha)}} dv(z)$$

$$\underbrace{\leq}_{Theorem\ 1.4.4} C_J \int_{\mathbb{B}_n} |Rf(w)|^p dv(w)(1-|w|^2)^{\alpha+p} dv(w)$$

that is

$$\int_{\mathbb{B}_n} |f(z)-f(0)|^p dv_\alpha(z) \leq C_J \int_{\mathbb{B}_n} \left((1-|z|^2)|Rf(z)| \right)^p dv_\alpha(z).$$

Finally, it's easy to notice that this proof produces equivalent norms on A^p_α in terms of radial derivative, the holomorphic gradient and the invariant gradient of f. □

In the following corollary we provide a criterion, in terms of partial derivatives, to determine functions of the Bergman spaces.

Corollary 2.4.9. *Let $\alpha > -1$, $p > 0$, $N \in \mathbb{N}$ and $f \in H(\mathbb{B}_n)$. Then*

$$f \in A^p_\alpha \iff (1-|z^2|)^N \frac{\partial^m f}{\partial z^m}(z) \in L^p(\mathbb{B}_n, dv_\alpha), \ |m| = N \tag{2.4.11}$$

Proof. We start proving ($\Longrightarrow$) by induction. For $N = 1$, we just apply the equivalence a) and c) of the last theorem. For $N = 2$, let $f \in A^p_\alpha$, using the same equivalence, it turns out that

$$\frac{\partial f}{\partial z_i} \in A^p_{\alpha+p}, \ i = 1, ..., n.$$

Hence, we apply the implication $a) \Longrightarrow c)$ to the function $\dfrac{\partial f}{\partial z_i}$, we obtain that

$$(1-|z|^2)\frac{\partial^2 f}{\partial z_i \partial z_j}(z) \in L^p(\mathbb{B}_n, dv_{\alpha+p}), \quad i,j = 1,...,n.$$

To prove the general case, that is for any arbitrary $N \in \mathbb{N}$, we basically use the same idea.
($\Longleftarrow$) In this case, we just repeat reversing the same argument. $\square$

A further consequence is given by the following corollary.

Corollary 2.4.10. *Suppose $\alpha > 1$, $p > 0$ and assume that $f \in A^p_\alpha$. Then the integral*

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z)$$

is comparable to

$$\sum_{|m|\leq N} \left|\frac{\partial^m f}{\partial z^m}(0)\right| + \sum_{|m|=N} \int_{\mathbb{B}_n} \left|(1-|z|^2)^N \frac{\partial^m f}{\partial z^m}(z)\right| dv_\alpha(z),$$

whenever $f \in H(\mathbb{B}_n)$.

2.4.1 Characterisation in terms of $R^{\alpha,t}$

The aim of this subsection is to obtain a characterisation of the Bergman space A^p_α, for every $0 < p < \infty$, in terms of the operator $R^{\alpha,t}$. We start with the following corollary, that follows from the integral representation formula of (2.1.2)

Corollary 2.4.11. *Assume that $\alpha > -1$, $t > 0$ and $f \in H(\mathbb{B}_n)$. If neither $n+\alpha$ nor $n+\alpha+t$ is not a negative integer, then*

$$R^{\alpha,t} f(z) = \lim_{r\to 1^-} \int_{\mathbb{B}_n} \frac{f(rw)dv_\alpha(w)}{(1-<z,w>)^{n+1+\alpha+t}}, \tag{2.4.12}$$

and

$$R_{\alpha,t} f(z) = \lim_{r\to 1^-} \int_{\mathbb{B}_n} \frac{f(rw)dv_{\alpha+t}(w)}{(1-<z,w>)^{n+1+\alpha}}. \tag{2.4.13}$$

In particular, the limits above always exist.

Proof. Fixed $r \in (0,1)$, let the dilation function $f_r(z) = f(rz)$ be. Applying Lemma 2.1.3, we have

$$f_r(z) = \int_{\mathbb{B}_n} \frac{f_r(w)dv_\alpha(w)}{(1-<z,w>)^{n+1+\alpha}}, \quad z \in \mathbb{B}_n.$$

Now, since

$$\lim_{r\to 1^-} \frac{f_r(w)}{(1-<z,w>)^{n+1+\alpha+t}} = \frac{f(w)}{(1-<z,w>)^{n+1+\alpha+t}}, \ \forall\, w \in \mathbb{B}_n,$$

and

$$\begin{aligned}
\int_{\mathbb{B}_n} \frac{f(w)dv_\alpha(w)}{(1-<z,w>)^{n+1+\alpha+t}} &= \int_{\mathbb{B}_n} f(w) R^{\alpha,t}\left(\frac{1}{(1-<z,w>)^{n+1+\alpha}}\right) dv_\alpha(w) \\
&= R^{\alpha,t} \int_{\mathbb{B}_n} \frac{f(w)dv_\alpha(w)}{(1-<z,w>)^{n+1+\alpha}} \\
&= R^{\alpha,t} f(z).
\end{aligned}$$

Hence, we deduce that

$$\lim_{r\to 1^-} \int_{\mathbb{B}_n} \frac{f_r(w)dv_\alpha(w)}{(1-<z,w>)^{n+1+\alpha+t}} = R^{\alpha,t} f(z),$$

this proves (2.4.12). To show (2.4.13), we proceed similarly. $\square$

Remark 2.4.12. Moreover, we've proved the following identity

$$\int_{\mathbb{B}_n} \frac{R^{\alpha,t} f(w) dv_\alpha(w)}{(1- < z, w >)^{n+1+\alpha}} = \int_{\mathbb{B}_n} f(w) R^{\alpha,t} \left(\frac{1}{(1- < z, w >)^{n+1+\alpha}} \right) dv_\alpha(w),$$

where $f \in H(\mathbb{B}_n)$.

In order to prove the main goal of this subsection, we need the following lemma concerning the image of the Bergman kernel under the action of the operator $R^{s,t}$ and its inverse operator $R_{s,t}$.

Lemma 2.4.13. *Suppose neither $n + s$ nor $n + s + t$ is a negative integer. If $\beta = s + N$ for some positive integer N, then there exists a one-variable polynomial $h(< z, w >)$ of degree N such that*

$$R^{s,t} \frac{1}{(1- < z, w >)^{n+1+\beta}} = \frac{h(< z, w >)}{(1- < z, w >)^{n+1+\beta+t}}. \tag{2.4.14}$$

Furthermore, there exists a polynomial $P(z, w)$ such that

$$R_{s,t} \frac{1}{(1- < z, w >)^{n+1+\beta+t}} = \frac{P(z, w)}{(1- < z, w >)^{n+1+\beta}}. \tag{2.4.15}$$

Proof. Using the definition of $R^{s,t}$ for $\beta = s + 1$, there exists a positive constant C such that

$$\begin{aligned}
R^{s,t} \frac{1}{(1- < z, w >)^{n+1+\beta}} &= C \sum_{k=0}^{\infty} \frac{k + n + 1 + s}{k!} \Gamma(n + 1 + k + s + t) < z, w >^k \\
&= C \sum_{k=1}^{\infty} \frac{\Gamma(n + 1 + k + s + t)}{(k - 1)!} < z, w >^k + \\
&+ C \sum_{k=0}^{\infty} \frac{n + 1 + s}{k!} \Gamma(n + 1 + k + s + t) < z, w >^k \\
&= C < z, w > \sum_{k=0}^{\infty} \frac{\Gamma(n + 1 + k + s + t)}{k!} < z, w >^k + \\
&+ C \sum_{k=0}^{\infty} \frac{n + 1 + s}{k!} \Gamma(n + 1 + k + s + t) < z, w >^k \\
&= \frac{C\Gamma(n + 1 + \beta + t)}{(1- < z, w >)^{n+1+\beta+t}} + C \frac{(n + \beta)\Gamma(n + \beta + t)}{(1- < z, w >)^{n+\beta+t}} \\
&= C \frac{\Gamma(n + 1 + \beta + t) + (n + \beta)\Gamma(n + \beta + t)(1- < z, w >)}{(1- < z, w >)^{n+\beta+t+1}}.
\end{aligned}$$

Namely, for $\beta = s + 1$, (2.4.14) holds. In general, if $\beta = s + N$, there exists a positive constant C_N such that

$$R^{s,t} \frac{1}{(1- < z, w >)^{n+1+\beta}} = C_N \sum_{k=0}^{\infty} \frac{p(k)\Gamma(n + 1 + k + s + t)}{k!} < z, w >^k,$$

where $p(k)$ is a polynomial having degree N that can be written as a linear combination of

$$1, \quad k, \quad k(k - 1), \quad \ldots \ , k(k - 1), \ldots , (k - N + 1).$$

Hence, to prove (2.4.14), we proceed exactly as in the case $N = 1$.
To show (2.4.15), using Proposition 1.5.10, Proposition 1.5.11 and the fact that $R_{s,t}$ and $R^{s+t,N}$ commute, it turns out that

$$\begin{aligned} R_{s,t}\frac{1}{(1-<z,w>)^{n+1+\beta+t}} &= R_{s,t}R^{s+t,N}\frac{1}{(1-<z,w>)^{n+1+s+t}} \\ &= R^{s+t,N}R_{s,t}\frac{1}{(1-<z,w>)^{n+1+s+t}} \\ &= R^{s+t,N}\frac{1}{(1-<z,w>)^{n+1+s}} \\ &= \sum_{|m|\le N} p_m(z)\frac{\partial}{\partial z^m}\frac{1}{(1-<z,w>)^{n+1+s}} \end{aligned}$$

where $p_m(z)$ is a polynomial. This completes our proof. □

Remark 2.4.14. As a consequence of the previous lemma, we can compare the behaviour of $R^{s,t}$ and $R^{\beta,t}$. In fact, there exist a family of constants $\{C_k\}_k$ such that

$$h(<z,w>) = \sum_{k=0}^{N} C_k(1-<z,w>)^k.$$

Then, using Proposition 1.5.10, we find

$$\begin{aligned} R^{s,t}\frac{1}{(1-<z,w>)^{n+1+\beta}} &= \sum_{k=0}^{N} C_k\frac{1}{(1-<z,w>)^{n+1+\beta+t-k}} \\ &= \sum_{k=0}^{N} C_k R_{\beta+t-k,k}R^{\beta,t}\frac{1}{(1-<z,w>)^{n+1+\beta}}. \end{aligned}$$

Finally, we differentiate respect to $\overline{w}$ to obtain

$$R^{s,t} = C_0R^{\beta,t} + \sum_{k=1}^{N} C_k R_{\beta+t-k,k}R^{\beta,t},$$

where we easily notice that $C_0 \neq 0$. For any $f \in H(\mathbb{B}_n)$, the function $R_{\beta+t-k,k}R^{\beta,t}f$ is a $k-$th integral of $R^{\beta,t}f$, for every $1 \le k \le N$, and, hence, is more regular than $R^{\beta,t}f$. In other words, the behaviour of $R^{s,t}f$ and $R^{\beta,t}f$ are often the same. Finally, we estimate $h(<z,w>)$ as follows. Using the above notatio, Cauchy-Schwartz inequality, we get

$$|h(<z,w>)| \le \sum_{k=0}^{N} |C_k|2^k$$

and, from (2.4.14) and putting $C := \sum_{k=0}^{N} |a_k|2^k$, we easily deduce that

$$\begin{aligned} \left|R^{s,t}\frac{1}{(1-<z,w>)^{n+1+\beta}}\right| &= \left|\frac{h(<z,w>)}{(1-<z,w>)^{n+1+\beta+t}}\right| \\ &\le \left|\frac{h(<z,w>)}{(1-<z,w>)^{n+1+\beta+t}}\right| \\ &\le \frac{C}{|1-<z,w>|^{n+1+\beta+t}} \end{aligned}$$

We are ready to prove the main goal of this subsection.

Theorem 2.4.15. *Assume $\alpha > -1$, $p > 0$ and $t > 0$. If neither $n+s$ nor $n+s+t$ is a negative integer, then there exists positive constant c and C such that*

$$c\int_{\mathbb{B}_n}|f(z)|^p dv_\alpha(z) \leq \int_{\mathbb{B}_n}|(1-|z|^2)^t R^{s,t}f(z)|^p dv_\alpha(z) \leq C\int_{\mathbb{B}_n}|f(z)|^p dv_\alpha(z), \tag{2.4.16}$$

where $f \in H(\mathbb{B}_n)$.

Proof. Let $\beta = s+N$, where N is a positive integer large enough so that $\beta > -1$. Then, we use the integral representation formula of Lemma 2.1.3, we can write

$$f(z) = \int_{\mathbb{B}_n}\frac{f(w)dv_\beta(w)}{(1-<z,w>)^{n+1+\beta}},\ z \in \mathbb{B}_n.$$

Applying the operator $R^{s,t}$ to f, using the previous formula, Remark 2.4.14 and Lemma 2.4.11, we can find a positive constant C_1 such that

$$\begin{aligned}|R^{s,t}f(z)| &\leq \left|R^{s,t}\int_{\mathbb{B}_n}\frac{f(w)dv_\beta(w)}{(1-<z,w>)^{n+1+\beta}}\right| \\ &\leq \left|\int_{\mathbb{B}_n}R^{s,t}\frac{f(w)dv_\beta(w)}{(1-<z,w>)^{n+1+\beta}}\right| \\ &= \left|\int_{\mathbb{B}_n}\frac{h(<z,w>)f(w)dv_\beta(w)}{(1-<z,w>)^{n+1+\beta+t}}\right| \\ &\leq C_1\int_{\mathbb{B}_n}\frac{|f(w)|dv_\beta(w)}{|1-<z,w>|^{n+1+\beta+t}}.\end{aligned}$$

We rewrite

$$|R^{s,t}f(z)| \leq C_1\int_{\mathbb{B}_n}\frac{|f(w)|dv_\beta(w)}{|1-<z,w>|^{n+1+\beta+t}}, \tag{2.4.17}$$

that implies

$$|R^{s,t}f(z)|(1-|z|^2)^t \leq C_1(1-|z|^2)^t\int_{\mathbb{B}_n}\frac{|f(w)|dv_\beta(w)}{|1-<z,w>|^{n+1+\beta+t}},\ z \in \mathbb{B}_n.$$

If $p \geq 1$ and N is large enough so that

$$\alpha + 1 < p(\beta+1),$$

so that, from Theorem 1.4.6, there exists $C_2 > 0$, independent of f, such that

$$\int_{\mathbb{B}_n}|(1-|z|^2)^t R^{s,t}f(z)|^p dv_\alpha(z) \leq C_2\int_{\mathbb{B}_n}|f(z)|^p dv_\alpha(z).$$

If $0 < p < 1$, we write

$$\beta = \frac{n+1+\alpha'}{p} - (n+1),$$

where N is assumed large enough such that $\alpha' > \alpha$. Using Lemma 2.4.5, (2.4.15) and, again, (2.4.17), there exists a constant $C_3 > 0$ such that

$$\begin{aligned}|R^{s,t}f(z)|^p &\leq C_1^p\left\{\int_{\mathbb{B}_n}\frac{|f(w)|dv_\beta(w)}{|1-<z,w>|^{n+1+\beta+t}}\right\}^p \\ &\leq C_3C_1^p\int_{\mathbb{B}_n}\left|\frac{f(w)}{(1-<z,w>)^{n+1+\beta+t}}\right|^p dv_{\alpha'}(w) \\ &= C_3C_1^p\int_{\mathbb{B}_n}\frac{|f(w)|^p}{|1-<z,w>|^{n+1+\alpha'+tp}}dv_{\alpha'}(w)\end{aligned}$$

After that, multiplyng from both sides by $(1-|z|^2)^{tp}dv_\alpha(z)$, integrating and, hence, from Fubini's Theorem and Theorem 1.4.4, there exists a positive constant C_4, independent of f, such that

$$\begin{aligned}\int_{\mathbb{B}_n}(1-|z|^2)^{tp}|R^{s,t}f(z)|^p dv_\alpha(z) &\leq C_3C_1^p\int_{\mathbb{B}_n}(1-|z|^2)^{tp}dv_\alpha(z)\int_{\mathbb{B}_n}\frac{|f(w)|^p}{|1-<z,w>|^{n+1+\alpha'+tp}}dv_{\alpha'}(w)\\ &\leq C_3C_1^p\int_{\mathbb{B}_n}|f(w)|^p dv_{\alpha'}(w)\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{tp}}{|1-<z,w>|^{n+1+\alpha'+tp}}dv_\alpha(z)\\ &\leq C_3C_1^p\int_{\mathbb{B}_n}|f(w)|^p dv_{\alpha'}(w)\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{tp+\alpha}}{|1-<z,w>|^{n+1+\alpha'-\alpha+(tp+\alpha)}}dv(z)\\ &\leq \tilde{C}C_3C_1^p\int_{\mathbb{B}_n}|f(w)|^p(1-|w|^2)^{\alpha-\alpha'}dv_{\alpha'}(w)\\ &= \tilde{C}C_3C_1^p\int_{\mathbb{B}_n}|f(w)|^p dv_\alpha(w).\end{aligned}$$

Next, assuming that the function $(1-|z|^2)^tR^{s,t}f(z)\in L^p(\mathbb{B}_n)$ and from Remark 2.4.12, the function

$$g(z)=\frac{c_{\beta+t}}{c_\beta}(1-|z|^2)^tR^{\beta,t}f(z)\in L^p(\mathbb{B}_n,dv_\alpha).$$

Furthermore, using Corollary 2.4.11, Fubini's theorem and Lemma 2.1.3, it turns out that

$$\begin{aligned}P_\beta(g)(z) &= \int_{\mathbb{B}_n}\frac{g(w)}{(1-<z,w>)^{n+1+\beta}}(1-|w|^2)^\beta dv_\beta(w)\\ &= \int_{\mathbb{B}_n}\frac{c_{\beta+t}}{c_\beta}(1-|w|^2)^tR^{\beta,t}f(w)\frac{1}{(1-<z,w>)^{n+1+\beta}}dv_\beta(w)\\ &= \int_{\mathbb{B}_n}\frac{c_{\beta+t}}{c_\beta}(1-|w|^2)^t\left\{\lim_{r\to1^-}\int_{\mathbb{B}_n}\frac{f(rx)dv_\beta(x)}{(1-<w,x>)^{n+1+\beta+t}}\right\}\frac{1}{(1-<z,w>)^{n+1+\beta}}dv_\beta(w)\\ &= \lim_{r\to1^-}\int_{\mathbb{B}_n}f(rx)dv_\beta(x)\int_{\mathbb{B}_n}\frac{dv_{\beta+t}(w)}{(1-<w,x>)^{n+1+\beta+t}(1-<z,w>)^{n+1+\beta}}\\ &= \lim_{r\to1^-}\int_{\mathbb{B}_n}\frac{f(rx)dv_\beta(x)}{(1-<z,x>)^{n+1+\beta}}\\ &= f(z).\end{aligned}$$

If $1\leq p<\infty$, from Theorem 2.2.9, we deduce that $f\in A^p_\alpha$. For $0<p<1$, writing $f=P_\beta g$ as

$$f(z)=C_5\int_{\mathbb{B}_n}\frac{R^{\beta,t}f(w)}{(1-<z,w>)^{n+1+\beta}}(1-|w|^2)^{\beta+t}dv_\beta(w),\quad z\in\mathbb{B}_n,$$

where C_5 is a suitable positive constant. We also assume that N is large enough so that

$$\beta+t=\frac{n+1+\alpha'}{p}-(n+1)$$

for some $\alpha'>1$. Applying Lemma 2.4.5, there exists a positive constant C_6 such that

$$|f(z)|^p\leq C_6\int_{\mathbb{B}_n}\frac{|R^{\beta,t}f(w)|^p}{|1-<z,w>|^{(n+1+\beta)p}}dv_{\alpha'}(w),$$

for all $z \in \mathbb{B}_n$. Now, since

$$(n+1+\beta)p = n+1+\alpha' - pt$$
$$= n+1+\alpha+(\alpha' - pt - \alpha),$$

again, assuming that N is large enough so that

$$\alpha' - pt - \alpha > 0,$$

and, finally, using Theorem 1.4.4 and Fubini's theorem, there exists a positive constant C_7 such that

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \le C_7 \int_{\mathbb{B}_n} (1-|w|^2)^{pt} |R^{s,t} f(w)|^p dv_\alpha(w).$$

We proved (2.4.16). □

To conclude this subsection, we give the following trivial corollary.

Corollary 2.4.16. *Let $p > 0$, $\alpha > -1$, $t > 0$ and assume that neither $n+s$ nor $n+s+t$ is a negative integer. Let $f \in H(\mathbb{B}_n)$, then the following are equivalent*
1) $f \in A^p_\alpha$.
2) $(1-|z|^2)^t R^{s,t} f(z) \in L^p(\mathbb{B}_n, dv_\alpha)$

2.5 Atomic Decomposition

The purpose of this section is to present a proof of the atomic decomposition for weighted Bergman spaces. That is, for any $f \in A^p_\alpha$, we show that f can be described in terms of a linear combination of a particular family of functions, called atoms, that are defined using the reproducing kernel of the space A^2_α. In some sense, they play the role of an orthonormal basis for this space, although we will see that they are not mutually orthogonal. Besides, as a consequence of this process, we will deduce a fundamental topological property of the Bergman space: the lack of local convexity, for $0 < p < 1$.

First of all, denoting by $\beta(\cdot,\cdot)$ the Bergman metric, we recall that for $r > 0$ and $z \in \mathbb{B}_n$ the set

$$D(z,r) = \{w \in \mathbb{B}_n : \beta(z,w) < r\} \tag{2.5.1}$$

is the Bergman metric ball at z.
In order to give a constructive proof of the atomic decomposition for Bergman spaces, we need a pair of estimates. This is the content of the next lemma.

Lemma 2.5.1. *For every $r > 0$ there exists a positive constant C_r such that*

$$C_r^{-1} \le \frac{1-|a|^2}{1-|z|^2} \le C_r \tag{2.5.2}$$

and

$$C_r^{-1} \le \frac{1-|a|^2}{|1- < a,z > |} \le C_r \tag{2.5.3}$$

for all $a, z \in \mathbb{B}_n$ such that $\beta(a,z) < r$. Furthermore, if r is bounded above, the constant C_r can be chosen to be indipendent of r.

Proof. Under the above conditions, writing $z = \varphi_a(w)$ for some $w \in \mathbb{B}_n$ with $\beta(0,w) < r$, applying Lemma 1.3.9 follows that

$$1-|z|^2 = \frac{(1-|w|^2)(1-|a|^2)}{|1- < a,w > |^2}.$$

We notice that $D(0,r)$ is actually a Euclidean ball centered at the origin and having radius less than 1. This fact allows us to find a positive constant C such that

$$C^{-1} \leq \frac{|1-<a,w>|^2}{1-|w|^2} \leq C, \ \forall a \in \mathbb{B}_n, \forall w \in D(0,r),$$

so

$$C^{-1} \leq \frac{1-|a|^2}{1-|z|^2} \leq C$$

for all a and z with $\beta(a,z) < r$. (2.5.2) is proved.
In order to prove (2.5.3), we proceed as follows. Using the involution property of Proposition 1.3.11, we know that $z = \varphi_a(w)$ if and only if $w = \varphi_a(z)$ and, hence, applying Lemma 1.3.9, we get

$$1-|w|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-<a,z>|^2}.$$

To conclude, using the boundedness of $1-|w|^2$, from above and below, and since $1-|a|^2$ is comparable to $1-|z|^2$, the result follows easily. $\square$

As a consequence of Lemma 1.6.15 and Lemma 2.5.1 we have the following corollary.

Corollary 2.5.2. *Let $\alpha \in \mathbb{R}$, $r_1 > 0$, $r_2 > 0$, and $r_3 > 0$. Then there exists a constant $C > 0$ such that*

$$C^{-1} \leq \frac{v_\alpha(D(z,r_1))}{v_\alpha(D(w,r_2))} \leq C \tag{2.5.4}$$

for all $z, w \in \mathbb{B}_n$ with $\beta(z,w) \leq r_3$.

Many techniques in Mathematical Analysis involve covering lemmas, namely, ways to decompose the underlying domain into special nice pieces. In the next Lemma, we provide an example of this technique.

Lemma 2.5.3. *Let $R > 0$ and $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that every Bergman metric ball of radius r, where $r \leq R$, can be covered by N Bergman metric balls having radius r/M.*

Proof. Let a Bergman metric ball $D(a,r)$ be, where $0 < r \leq R$. Put $\delta = r/M$ and let $\{D(a_k, \delta/2)\}_{k=1}^N$ be a finite covering of $D(a,r)$, where each $a_k \in D(a,r)$. Our aim is to obtain an other covering of $D(a,r)$: put $a_1' := a_1$ and let a_2' be the first of $\{a_2, a_3, ...\}$ such that $\beta(a_2', a_1') \geq \delta/2$. Then, we choose the first term a_3', after a_2', whose Bergman distance is at least $\delta/2$ from both a_1' and a_2'. The covering obtained at the end of this process, denoted by $\{D(a_k', \delta)\}$, satisfies

$$\beta(a_i', a_j') \geq \delta/2, \quad i \neq j.$$

By the fact that the family of sets $\{D(a_k', r/4M)\}$ are disjoint and contained in $D(a, r + r/(4M))$, the following inequality holds

$$\sum_k v\left(D\left(a_k', \frac{r}{4M}\right)\right) \leq v\left(D\left(a, r+\frac{r}{4M}\right)\right)$$

Using the previous corollary, there exists a constant $C := C(R,M) > 0$, that is independent of r, such that

$$v\left(D\left(a, r+\frac{r}{4M}\right)\right) \leq Cv\left(D\left(a_k', \frac{r}{4M}\right)\right)$$

for every k. Finally, putting $N = [C]+1$, the wished result follows easily. $\square$

We now present a useful decomposition of the open unit ball into Bergman metric balls.

Theorem 2.5.4. *There exists $N \in \mathbb{N}$ such that for any $0 < r \leq 1$ we can find a sequence $\{a_k\} \in \mathbb{B}_n$ such that the folllowing properties hold:*

1) $\mathbb{B}_n = \bigcup_k D(a_k, r)$.
2) *The sets $D(a_k, r/4)$ are mutually disjoint.*
3) *Each point $z \in \mathbb{B}_n$ belongs to at most N of the sets $D(a_k, 4r)$.*

Proof. 1) Fix $r \in (0, 1]$, following the same lines as in the first part of the previous proof, we can find a sequence $\{a_k\}_k \in \mathbb{B}_n$ such that

$$\mathbb{B}_n = \bigcup_k D(a_k, r)$$

and that $\beta(a_i, a_j) \geq r/2$ for all $i \neq j$.
2) This property follows as a fairly application of the triangle inequality.
3) As a consequence of the previous Lemma, every $D(a_k, 4r)$ can be covered by N Bergman metric balls having radius $r/4$, where N is independent of r. Hence, assuming by contradiction

$$z \in \bigcap_{i=1}^{N+1} D(a_{k_i}, 4r)$$

then $a_{k_i} \in D(z, 4r)$ for $1 \leq i \leq N+1$. Let $D(z_i, r/4)$, $1 \leq i \leq N$, be a cover of $D(z, 4r)$. Then at least one of $D(z_i, r/4)$ must contain two of a_{k_j}, $1 \leq j \leq N+1$. Applying the traingle inequality, such points must have Bergman distance less than $r/2$, which contradicts the second assumption made on $\{a_k\}$ during the previous paragraph. □

Remark 2.5.5. We notice that we could have proved the result for any fixed radius greater than $r/4$.

Using the above notation, we give the following definition.

Definition 2.5.6. We will call r the separation constant for the sequence $\{a_k\}$. Moreover, the sequence of complex numbers $\{a_k\}$ is called an $r-$lattice in the Bergman metric.

For the sake of convenience, we collect some elementary facts, on the Bergman metric, about holomorphic functions in the unit ball $\mathbb{B}_n$ and the automorphism group $Aut(\mathbb{B}_n)$. In the following Lemma, for any $f \in A^p_\alpha$, we prove that the point-evaluation, for a fixed $z \in \mathbb{B}_n$, can be estimated from above by $v_\alpha(D(z, r))$ and the integral of $|f|^p$, respect to dv_α, performed on $D(z, r)$ for some $r > 0$.

Lemma 2.5.7. *Assume $r > 0$, $p > 0$ and $\alpha > -1$. Then there exists a constant $C > 0$ such that*

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p dv_\alpha(w)$$

for all $f \in H(\mathbb{B}_n)$ and all $z \in \mathbb{B}_n$.

Proof. From Proposition 1.6.10, $D(0, r)$ is a Euclidean ball centered at the origin with Euclidean radius $R = tanh(r)$. Then, using the subharmonicity of $|f|^p$ and Corollary 1.7.5, it turns out that

$$|f(0)|^p \leq \frac{1}{v_\alpha(D(0,r))} \int_{D(0,r)} |f(w)|^p dv_\alpha(w)$$

where $f \in H(\mathbb{B}_n)$. We substitute f by $f \circ \varphi_z$ and use the change of variables formula of Proposition 1.4.7, we get

$$|f(z)|^p \leq \frac{1}{v_\alpha(D(0,r))} \int_{D(z,r)} |f(w)|^p \frac{(1-|z|^2)^{n+1+\alpha}}{|1-<z,w>|^{2(n+1+\alpha)}} dv_\alpha(w)$$

Finally, applying Lemma 2.5.1, the result is obtained. □

The following result is an identity concerning the involutive automorphism. Actually, this is a generalisation of Lemma 1.3.9 and is not proved.

Lemma 2.5.8. *Suppose $a \in \mathbb{B}_n$ and let φ_a the involution automorphism be. Then*

$$1- < \varphi_a(z), \varphi_a(w) >= \frac{(1- < a, a >)(1- < z, w >)}{(1- < z, a >)(1- < a, w >)}, \tag{2.5.5}$$

for all $z, w \in \overline{\mathbb{B}}_n$.

Fix $, u, v \in \mathbb{B}_n$ and $R > 0$ such that $u, v \in D(0, R)$. In the next lemma we provide an estimate from below of the distance, under the Bergman metric, between u and v in terms of the hermitian inner product of $\mathbb{C}^n$ involving z, u and v.

Lemma 2.5.9. *Let $R > 0$ and $b \in \mathbb{R}$. Then, for all $z, u,$ and $v \in \mathbb{B}_n$ so that $\beta(u, v) \leq R$, there exists constant $C > 0$ such that*

$$\left| \frac{(1- < z, u >)^b}{(1- < z, v >)^b} - 1 \right| \leq C\beta(u, v), \tag{2.5.6}$$

where $\beta(u, v)$ is the Bergman metric.

Proof. Since $\beta(u, v) \leq R$, denoting by $r := tanh(R) \in (0, 1)$, we can write $v = \varphi_u(w)$ where $|w| \leq r$. Let $z' = \varphi_u(z)$. Hence, as a consequence of Lemma 2.5.1, it turns out that

$$1- < z, u >= \frac{1 - |u|^2}{1- < z', u >},$$

and

$$1- < z, v >= \frac{(1 - |u|^2)(1- < z', w >)}{(1- < z', u >)(1- < u, w >)}.$$

Then, using these two equalities, we find

$$\frac{(1- < z, u >)^b}{(1- < z, v >)^b} = \frac{(1- < u, w >)^b}{(1- < z', w >)^b}$$

and hence

$$\frac{(1- < z, u >)^b}{(1- < z, v >)^b} - 1 = \frac{(1- < u, w >)^b - (1- < z', w >)^b}{(1- < z', w >)^b}.$$

By the fact that $|z'| < 1$ and $|w| < r$, we get

$$1 - r < |1- < z', w > | < 2.$$

Furthermore, because of $| < u, w > | < r$ and $| < z', w > | < r$, there exists $C_1 := C_1(r, b) > 0$ such that

$$\begin{aligned} |(1- < u, w >)^b - (1- < z', w >)^b| &\leq C_1| < z', w > - < u, w > | \\ &\leq 2C_1|w|. \end{aligned}$$

Finally, since the Bergman metric is equivalent to the Euclidean metric on the relatively compact set $|w| < r$, there exists $C_2 > 0$ such that

$$\begin{aligned} |w| &\leq C_2\beta(0, w) \\ &= C_2\beta(0, \varphi_u(v)) \\ &= C_2\beta(u, v) \end{aligned}$$

wher $v = \varphi_u(w)$ and $\beta(u, v) < R$. $\square$

Remark 2.5.10. Putting $b = 1$ in the previous Lemma, applying the triangle inequality and, then, repeating the same argument with $b = -1$, we find that for every $R > 0$ there exists a constant $C > 0$ that satisfy

$$C^{-1} \leq \frac{|1- < z, u > |}{|1- < z, v > |} \leq C \tag{2.5.7}$$

for every u,v and $z \in \mathbb{B}_n$, such that $\beta(u, v) \leq R$.

Lemma 2.5.11. *For every $k \geq 1$ there exists a Borel set D_k such that the conditions hold*

1) $D(a_k, r/4) \subset D_k \subset D(a_k, r)$, $\forall k \geq 1$.
2) $D_k \cap D_j = \emptyset$, *for* $k \neq j$.

3) $\mathbb{B}_n = \bigcup_k D_k$

Proof. Let $k \geq 1$, we define the family of sets

$$E_k = D(a_k, r) - \bigcup_{j \neq k} D(a_j, r/4)$$

so that E_k satisfies

$$D(a_k, r/4) \subset E_k \subset D(a_k, r)$$

Furthermore, $\{E_k\}_{k \in \mathbb{N}}$ is a covering for $\mathbb{B}_n$. To prove this fact, we notice that if $z \in \mathbb{B}_n$, then $z \in D(a_k, r)$ for some k. If $z \in D(a_j, r/4)$ for some $j \neq k$, then $z \in E_j$; otherwise $z \in E_k$.
We construct the Borel sets as follows: let $D_1 = E_1$ and inductively define

$$D_{k+1} = E_{k+1} - \bigcup_{j=1}^{k} D_j, \quad k \geq 1$$

so that $\{D_k\}_k$ is a disjoint cover of $\mathbb{B}_n$. As a prove, if $z \in \mathbb{B}_n$, then $z \in E_k$ for some k. If $k = 1$, then $z \in D_1$. If $k > 1$, then either we have $z \in D_i$ for some $1 \leq i < k$, or we have $z \in D_k$.
For each $k \geq 1$ we have

$$D_k \subset E_k \subset D(a_k, r).$$

We prove that $D(a_k, r/4) \subset D_k$. The case $k = 1$ follows easily from $D_1 = E_1$. For $k \geq 1$ we proceed as follows: fix $z \in D(a_{k+1}, r/4) \subset E_{k+1}$. Then $z \notin E_i$ for any $1 \leq i \leq k$, which implies that $z \notin D_i$ for any $1 \leq i \leq k$. This shows that

$$z \in E_{k+1} - \bigcup_{i=1}^{k} D_i = D_{k+1},$$

and we are done. □

We wish to improve the partition of the sets $\{D_k\}_k$ in Lemma 2.5.11. To this end, we partition the set D_1 and use automorphisms to carry the partition to $\{D_k\}_{k>1}$. In order to proceed, we pick η a positive radius such that the quotient η/r is small. Then, fix $\{z_k\}_{k=1,\ldots,J} \in D(0, r)$. These points depend on η, in the sense that $\{D(z_j, \eta)\}$ cover $D(0, r)$ and that $\{D(z_j, \eta/4)\}$ are disjoint. After that, following the same lines as in the former proof, we can enlarge every set $D(z_j, \eta/4) \cap D(0, r)$ to a Borel set E_j so that $E_j \subset D(z_j, \eta)$ and

$$D(0, r) = \bigcup_{j=1}^{J} E_j$$

is a disjoint union. The automorphisms are used as follows: for $k \geq 1$ and $1 \leq j \leq J$, we define $a_{kj} = \varphi_{a_k}(z_j)$ and

$$D_{kj} = D_k \cap \varphi_{a_k}(E_j)$$

where $a_{kj} \in D(a_k, r)$ for all $k \geq 1$ and $1 \leq j \leq J$. By the fact that

$$D_k = \bigcup_{j=1}^{J} D_{kj}$$

is a disjoint union for every k, we get a disjoint decomposition of $\mathbb{B}_n$

$$\mathbb{B}_n = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{J} D_{kj}.$$

We introduce the following two operators on $L^1(\mathbb{B}_n, dv_\beta)$ and $H(\mathbb{B}_n)$ respectively.

Definition 2.5.12. Fix a real parameter $b > n$ and define $\beta = b - (n+1)$. We define

$$Tf(z) = \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{b-(n+1)}}{|1- < z, w > |^b} f(w) dv(w), \quad f \in L^1(\mathbb{B}_n, dv_\beta) \tag{2.5.8}$$

and

$$Sf(z) = \sum_{k=1}^{\infty} \sum_{j=1}^{J} \frac{v_\beta(D_{kj}) f(a_{kj})}{(1- < z, a_{kj} >)^b} \tag{2.5.9}$$

where $f \in H(\mathbb{B}_n)$.

Remark 2.5.13. We remark that the operator T depends on the parameter b. Moreover, the operator S depends on both the parameter b and the partition $\{D_{kj}\}$ and hence, as a consequence, depends on the separation constant r and η.

The following Lemma plays a fundamental role in the atomic decomposition of Bergman spaces.

Lemma 2.5.14. *Let any $p > 0$, $\alpha > -1$, there exists a constant $C > 0$, independent of r and η, such that*

$$|f(z) - Sf(z)| \leq C\sigma \sum_{k=1}^{\infty} \frac{(1-|a_k|^2)^{(pb-n-1-\alpha)/p}}{|1- < z, a_k > |^b} \left[\int_{D(a_k, 2r)} |f(w)|^p dv_\alpha(w) \right]^{1/p} \tag{2.5.10}$$

for all $r \leq 1$, $z \in \mathbb{B}_n$, $f \in H(\mathbb{B}_n)$ and where

$$\sigma = \eta + \frac{tanh(\eta)}{(tanh(r))^{1-2n(1-1/p)}} \tag{2.5.11}$$

Proof. Using Lemma 2.1.3 and since $\beta > -1$, we can write

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w) dv_\beta(w)}{(1- < z, w >)^b}, z \in \mathbb{B}_n.$$

$\{D_{kj}\}$ is a partition of $\mathbb{B}_n$, this fact implies

$$v_\beta(\mathbb{B}_n) = \sum_{k=1}^{\infty} \sum_{j=1}^{J} \int_{D_{kj}} dv_\beta(w)$$

and, hence, we can write

$$f(z) - Sf(z) = \sum_{k=1}^{\infty} \sum_{j=1}^{J} \int_{D_{kj}} \left[\frac{f(w)}{(1- < z, w >)^b} - \frac{f(a_{kj})}{(1- < z, a_{kj} >)^b} \right] dv_\beta(w).$$

We apply the triangle inequality to obtain that

$$|f(z) - Sf(z)| \leq I(z) + H(z),$$

where

$$I(z) = \sum_{k=1}^{\infty}\sum_{j=1}^{J} \frac{1}{|1-<z,a_{kj}>|^b} \int_{D_{kj}} |f(w) - f(a_{kj})| dv_\beta(w),$$

and

$$H(z) = \sum_{k=1}^{\infty}\sum_{j=1}^{J} \frac{1}{|1-<z,a_{kj}>|^b} \int_{D_{kj}} \left| \frac{(1-<z,w>)^b}{(1-<z,a_{kj}>)^b} - 1 \right| |f(w)| dv_\beta(w)$$

For sake of simplicity, we will denote

$$I_{kj} = \int_{D_{kj}} |f(w) - f(a_{kj})| \, dv_\beta(w),$$

and, using the change of variables formula of Proposition 1.4.7, we get

$$I_{kj} = (1-|a_{kj}|^2)^{n+1+\beta} \int_{E_{kj}} \frac{|f \circ \varphi_{a_{kj}}(w) - f \circ \varphi_{a_{kj}}(0)|}{|1-<w,a_{kj}>|^{2(n+1+\beta)}} dv_\beta(w)$$

where

$$\begin{aligned} E_{kj} &= \varphi_{a_{kj}}(D_{kj}) \\ &\subset \varphi_{a_{kj}} \circ \varphi_{a_k}(D(z_j,\eta)) \\ &= \varphi_{a_{kj}}(D(a_{kj},\eta)) \\ &= D(0,\eta) \end{aligned}$$

We observe that, for $w \in E_{kj}$, the quantities $(1-|w|^2)^\beta$ and $|1-<w,a_{kj}>|$ are both bounded from below and from above. Furthermore, since $a_{kj} \in D(a_k, r)$, we deduce that the quantities $1-|a_{kj}|^2$ and $1-|a_k|^2$ are comparable. Hence, there exists a constant $C > 0$, independent of r and η, such that I_{kj} can be estimated as follows

$$I_{kj} \le C(1-|a_k|)^{n+1+\beta} \int_{E_{kj}} |f \circ \varphi_{a_{kj}}(w) - f \circ \varphi_{a_{kj}}(0)| dv(w).$$

We put $r' = tanh(r)$, $\eta' = tanh(\eta)$, and $R = \eta'/r'$. Moreover, since η is much smaller that r, we can assume $R \le 1/2$.
For any $h \in H(\mathbb{B}_n)$, using Lemma 2.1.6, there exists a constant $C > 0$, independent of r, η, k and j, such that

$$|\nabla h(z)| \le C \left(\int_{\mathbb{B}_n} |h(w)|^p dv(w) \right)^{1/p}, \; |z| \le R.$$

In particular, defining $h(z) = g(r'z)$, where

$$g(z) = f \circ \varphi_{a_{kj}}(z), \; z \in \mathbb{B}_n.$$

After the most obvious change of variables, we get

$$r'|\nabla g(r'z)| \le C \left(\frac{1}{(r')^{2n}} \int_{D(0,r)} |g(w)|^p dv(w) \right)^{1/p}, \; z \in \mathbb{B}_n,$$

we can rewrite the above as

$$|\nabla g(z)| \le C \frac{1}{(r')^{1+(2n/p)}} \left(\int_{D(0,r)} |g(w)|^p dv(w) \right)^{1/p}, \forall z \in D(0,\eta). \tag{2.5.12}$$

Now, for any $w \in E_{kj} \subset D(0,\eta)$, the following identity holds

$$g(w) - g(0) = \int_0^1 \left(\sum_{i=1}^n w_i \frac{\partial g}{\partial w_i}(tw) \right) dt$$

and clearly implies that

$$|g(w) - g(0)| \leq \eta' \sup \{ |\nabla g(u)| : u \in D(0,\eta) \} .$$

Hence, as a consequence, I_{kj} can be estimated

$$I_{kj} \leq C\eta' (1 - |a_k|)^{n+1+\beta} v(E_{kj}) \sup \{ |\nabla g(u)| : u \in D(0,\eta) \} .$$

We combine the above with (2.5.12) to obtain

$$I_{kj} \leq \frac{C\eta'}{(r')^{1+(2n/p)}} (1 - |a_k|^2)^{n+1+\beta} v(E_{kj}) \left(\int_{D(0,r)} |g(w)|^p dv(w) \right)^{1/p} .$$

But, by a change of variables, we notice that

$$\int_{D(0,r)} |g(w)|^p dv(w) = \int_{D(a_{kj},r)} |f(w)|^p \frac{(1 - |a_{kj}|^2)^{n+1} dv(w)}{|1- < w, a_{kj} > |^{2(n+1)}} .$$

Since Lemma 2.5.9 says that the quantities $1 - |a_{kj}|^2$ and $|1- < w, a_{kj} > |$ are both comparable to $1 - |a_k|^2$, where $w \in D(a_{kj}, r)$, and, by the fact that $D(a_{kj}, r) \subset D(a_k, 2r)$, we have that

$$\int_{D(0,r)} |g(w)|^p dv(w) \leq \frac{C}{(1 - |a_k|^2)^{n+1}} \int_{D(a_k,2r)} |f(w)|^p dv(w).$$

More is true, since $1 - |a_k|^2$ is comparable to $1 - |w|^2$ for $w \in D(a_k, 2r)$, we find

$$\int_{D(0,r)} |g(w)|^p dv(w) \leq \frac{C}{(1 - |a_k|^2)^{n+1+\alpha}} \int_{D(a_k,2r)} |f(w)|^p dv_\alpha(w).$$

The above inequality combined with the estimate of the previous paragraph gives

$$I_{kj} \leq \frac{C\eta'}{(r')^{1+(2n/p)}} (1 - |a_k|^2)^{(pb-n-1-\alpha)/p} v(E_{kj}) \left(\int_{D(a_k,2r)} |f|^p dv_\alpha \right)^{1/p} .$$

Since,

$$\sum_{j=1}^J v(E_{kj}) \leq Jv(D(0,\eta)) = J(\eta')^{2n}$$

and

$$\begin{aligned} v(D(0,r)) &= \sum_{j=1}^J v(E_j) \\ &\geq \sum_{j=1}^J v(D(z_j, \eta/4)) \\ &\geq CJ(\eta')^{2n} \end{aligned}$$

and the last equality follows from Lemma 1.6.13, we get

$$\sum_{j=1}^J v(E_{kj}) \leq Cv(D(0,r)) = C(r')^{2n}.$$

Combining this with the estimate in the previous paragraph, we obtain

$$\sum_{j=1}^{J} I_{kj} \leq \frac{C\eta'}{(r')^{1-2n+(2n/p)}}(1-|a_k|^2)^{(pb-n-1-\alpha)/p}\left(\int_{D(a_k,2r)}|f|^p dv_\alpha\right)^{1/p}.$$

From Lemma 2.5.9, for each $k \geq 1$ and $1 \leq j \leq J$, we have that $|1- < z, a_{kj} > |^b$ is comparable to $|1- < z, a_k > |^b$. Hence

$$I(z) \leq \frac{C\eta'}{(r')^{1-2n+(2n/p)}}\sum_{k=1}^{\infty}\frac{(1-|a_k|^2)^{(pb-n-1-\alpha)/p}}{|1- < z, a_k > |^b}\left[\int_{D(a_k,2r)}|f|^p dv_\alpha\right]^{1/p}.$$

In order to estimate $H(z)$, for sake of simplicity, we let

$$H_{kj} = \int_{D_{kj}}\left|\frac{(1- < z, a_{kj} >)^b}{(1- < z, w >)^b} - 1\right||f(w)|dv_\beta(w)$$

where $k \geq 1$ and $1 \leq j \leq J$. Applying Lemma 2.5.9 and (2.5.2) the following estimate

$$H_{kj} \leq C\eta(1-|a_k|^2)^\beta\int_{D_{kj}}|f(w)|dv(w).$$

Afterthat, for every $w \in D_{kj}$, using Lemma 2.5.9 we find

$$|f(w)| \leq C\left(\frac{1}{(1-|a_k|^2)^{n+1+\alpha}}\int_{D(a_k,2r)}|f(w)|^p dv_\alpha(w)\right)^{1/p}.$$

Hence, we can estimate H_{kj} :

$$H_{kj} \leq C\eta(1-|a_k|^2)^{\beta-(n+1+\alpha)/p}v(D_{kj})\left(\int_{D(a_k,2r)}|f(w)|^p dv_\alpha(w)\right)^{1/p}.$$

Furthermore, since

$$\begin{aligned}\sum_{j=1}^{J} v(D_{kj}) &= v(D_k)\\ &\leq v(D(a_k,r))\\ &\leq C(1-|a_k|^2)^{n+1},\end{aligned}$$

this implies that

$$\sum_{j=1}^{J} H_{kj} \leq C\eta(1-|a_k|^2)^{(pb-n-1-\alpha)/p}\left(\int_{D(a_k,2r)}|f(w)|^p dv_\alpha(w)\right)^{1/p}.$$

Finally, applying Lemma 2.5.9, $|1- < z, a_{kj} > |^b$ is comparable to $|1- < z, a_k > |^b$ and hence

$$H(z) \leq C\eta\sum_{k=1}^{\infty}\frac{(1-|a_k|^2)^{(pb-n-1-\alpha)/p}}{|1- < z, a_k > |^b}\left(\int_{D(a_k,2r)}|f(w)|^p dv_\alpha(w)\right)^{1/p}.$$

and we are done. □

We are finally ready to prove the main result of this section: the atomic decomposition of A^p_α.

Theorem 2.5.15. *Assume $p > 0$, $\alpha > -1$ and $b > 0$ such that*

$$b > n \max\left(1, \frac{1}{p}\right) + \frac{\alpha+1}{p}. \tag{2.5.13}$$

Then, for every $f \in A^p_\alpha$ there exists a sequence $\{a_k\}_k \in \mathbb{B}_n$ such that

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^{(pb-n-1-\alpha)/p}}{(1-<z, a_k>)^b}, \quad z \in \mathbb{B}_n \tag{2.5.14}$$

where $\{c_k\}_k \in l^p$ and the series converges in the norm topology of A^p_α.

Proof. Suppose that $f(z)$ is defined as (2.5.14) and $\{a_k\}_k$ is an r-lattice in the Bergman metric whose existence follows from Theorem 2.5.4. Our goal is to prove that $f \in A^p_\alpha$. To this end, we introduce the following sequence of functions

$$f_k(z) = \frac{(1-|a_k|^2)^{(pb-n-1-\alpha)/p}}{(1-<z, a_k>)^b}.$$

We notice that, if $p > 1$, we have

$$\begin{aligned} pb &> pn \max\left(1, \frac{1}{p}\right) + \alpha + 1 \\ &= pn + \alpha + 1 \\ &> n + \alpha + 1. \end{aligned}$$

If $p \leq 1$, we have

$$\begin{aligned} pb &> pn \max\left(1, \frac{1}{p}\right) + \alpha + 1 \\ &= n + \alpha + 1. \end{aligned}$$

In other words, the assumption on b implies that

$$pb > n + 1 + \alpha, \ \forall p > 0.$$

Hence, applying Theorem 1.4.4, we obtain

$$\{f_k(z)\}_k \in A^p_\alpha, \ \forall k \in \mathbb{N}.$$

We consider two cases: if $0 < p \leq 1$, then

$$||f||^p_{p,\alpha} \leq \sum_{k=1}^{\infty} |c_k|^p ||f_k||^p_{p,\alpha},$$

since $\{c_k\}_k \in l^p$ and $\{f_k(z)\}_k \in A^p_\alpha$, we get $f \in A^p_\alpha$.
If $p > 1$. We denote by D_k the family of sets from Lemma 2.5.11 and with I_{D_k} the indicator function. Then, defining

$$F(z) := \sum_{k=1}^{\infty} |c_k| v_\alpha(D_k)^{-1/p} I_{D_k}(z),$$

we obtain

$$||F||_{p,\alpha}^p = \sum_{j=1}^{\infty} \int_{D_j} \left| \sum_{k=1}^{\infty} |c_k| v_\alpha(D_k)^{-1/p} I_{D_k}(z) \right|^p dv_\alpha(z)$$

$$= \sum_{k=1}^{\infty} |c_k|^p v_\alpha(D_k) \int_{D_k} dv_\alpha(z)$$

$$= \sum_{k=1}^{\infty} |c_k|^p < \infty.$$

Since,

$$p(b-n) > \alpha + 1, \quad p > 1$$

By Theorem 1.4.6, the operator on (2.5.8) is bounded on $L^p(\mathbb{B}_n, dv_\alpha)$. Afterthat, applying the operator T to F, we get

$$TF(z) = \sum_{k=1}^{\infty} |c_k| v_\alpha(D_k)^{-1/p} \int_{D_k} \frac{(1-|w|^2)^{b-n-1}}{|1-<z,w>|^b} dv(w).$$

Then, using Lemmas 1.6.15 and 2.5.1, we find

$$v_\alpha(D_k) \sim (1-|a_k|^2)^{n+1+\alpha}$$

and

$$1-|w|^2 \sim 1-|a_k|^2, w \in D_k.$$

Furthermore, from (2.5.7) we obtain that $|1-<z,w>|$ and $|1-<z,a_k>|$ are comparable for $w \in D_k$. As a consequence, there exists a constant $\delta > 0$ such that

$$TF(z) \geq \delta \sum_{k=1}^{\infty} |c_k| \frac{(1-|a_k|^2)^{(pb-n-1-\alpha)/p}}{|1-<z,a_k>|^b}, \quad \forall z \in \mathbb{B}_n.$$

By the triangle inequality

$$|f(z)| \leq \frac{1}{\delta} TF(z), \quad z \in \mathbb{B}_n.$$

Now, since $F \in L^p(\mathbb{B}_n, dv_\alpha)$ and T is a bounded operator on $L^p(\mathbb{B}_n, dv_\alpha)$, it turns out that $f \in A_\alpha^p$ and exists a positive constant C, independent of f, such that the following estimate holds

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \leq C \sum_k |c_k|^p.$$

If $\{a_k\}_k$ is replaced by $\left\{a_{k_j}\right\}_j$, the previous proof, with some obvious adjustment, still works. We prove this fact as follows. If

$$f(z) = \sum_{k=1}^{\infty} \sum_{j=1}^{J} c_{kj} \frac{(1-|a_{k_j}|^2)^{(pb-n-1-\alpha)/p}}{(1-<z,a_{k_j}>)^b},$$

since

$$1-|a_{k_j}|^2 \sim 1-|a_k|^2 \quad \text{and} \quad |1-<z,a_{k_j}>| \sim |1-<z,a_k>|^2,$$

we find a constant $C > 0$ such that

$$|f(z)| \leq C \sum_{k=1}^{\infty} d_k \frac{(1-|a_k|^2)^{(pb-n-1-\alpha)/p}}{(1-<z,a_k>)^b},$$

where

$$d_k = \sum_{j=1}^{J} |c_{kj}|.$$

Using Holder's inequality,

$$|d_k|^p \leq J^{p/q} \sum_{j=1}^{J} |c_{kj}|^p,$$

hence

$$\sum_{k=1}^{\infty} |d_k|^p \leq J^{p/q} \sum_{k=1}^{\infty} \sum_{j=1}^{J} |c_{kj}|^p < \infty,$$

that is, $\{d_k\} \in l^p$. This fact, for what was shown previously, clearly implies that $f \in A^p_\alpha$.
The first part of the proof is completed: we proved that if f is defined by (2.5.14), then $f \in A^p_\alpha$, using a sequence $\{a_k\}_k$ whose existence is guaranteed by Theorem 2.5.4 or an associated sequence $\{a_k j\}_j$ that was constructed on Lemma 2.5.14. We remark that we have not supposed any further condition about the separation constants r and η.
In the second part of the proof, our aim is to show that if $f \in A^p_\alpha$, then f must admit a representation as in (2.5.14). In order to proceed, let an $r-$lattice $\{a_k\}_k$ in the Bergman metric and the $\eta-$lattice $\{a_{kj}\}_j$ with the corresponding finer partition $\{D_{kj}\}$ of $\mathbb{B}_n$ whose construction was explained on Lemma 2.5.14. From such lemma and the first part of this proof, there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{B}_n} |f(z) - Sf(z)|^p dv_\alpha(z) \leq C_1 \sigma^p \sum_{k=1}^{\infty} \int_{D(a_k,2r)} |f(z)|^p dv_\alpha(z)$$

where σ is as (2.5.11). Since each point of $\mathbb{B}_n$ belongs at most N of $D(a_k, 2r)$, we have

$$\int_{\mathbb{B}_n} |f(z) - Sf(z)|^p dv_\alpha(z) \leq C_1 \sigma^p N \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z).$$

Denoting with I the identity operator and choosing η small enough so that $C_1 N \sigma^p < 1$, then the operator $I - S$ has norm less than 1. Under these conditions, from standard Functional Analysis, we deduce that the operator S is invertible on A^p_α and, as a consequence, $f \in A^p_\alpha$ must be written as

$$f(z) = \sum_{k,j} c_{kj} \frac{(1 - |a_{kj}|^2)^{(pb-n-1-\alpha)/p}}{(1 - < z, a_{kj} >)^b},$$

where

$$c_{kj} = \frac{v_\beta(D_{jk}) g(a_{kj})}{(1 - |a_{kj}|^2)^{(pb-n-1-\alpha)/p}}$$

and $g = S^{-1} f$. Applying Lemma 1.6.15, we find

$$v_\beta(D_{jk}) \leq v_\beta(D_k) \sim (1 - |a_k|^2)^{n+1+\beta} = (1 - |a_k|^2)^b.$$

Then, using the equivalence between $1 - |a_k|^2$ and $1 - |a_{jk}|^2$, there exists a positive constant C_2, independent on f, such that

$$\sum_{jk} |c_{jk}|^p \leq C_2 \sum_{jk} (1 - |a_k|^2)^{n+1+\alpha} |g(a_{kj})|^p.$$

We use Lemma 2.5.7 to each $g(a_{kj})$, $1 - |a_k|^2$ is comparable to $1 - |a_{kj}|^2$ and that $D(a_{kj}, 2r) \subset D(a_k, 2r)$. Hence, we recover a further constant $C_3 > 0$ such that

$$\sum_{kj} |c_{kj}|^p \leq C_3 J \sum_{k=1}^{\infty} \int_{D(a_k,2r)} |g(z)|^p dv_\alpha(z).$$

Finally, since every point of $\mathbb{B}_n$ belongs at most N of the sets $D(a_k, 2r)$, we have

$$\sum_{kj} |c_{kj}|^p \leq C_3 JN \int_{\mathbb{B}_n} |g(z)|^p dv_\alpha(z).$$

The proof of the theorem is completed. $\square$

Remark 2.5.16. In other words, if $\{c_k\}_k \in l^p$ and f is written as in (2.5.14), there exists a positive constant C, independent of f, such that

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \leq C \sum_k |c_k|^p,$$

and we also get that $f \in A^p_\alpha$. Conversely, if $f \in A^p_\alpha$, we can pick a sequence $\{c_k\}_k$ so that f is represented as (2.5.14). Hence, we find a positive constant C, independent of f, such that the folllowing inequality holds

$$\sum_k |c_k|^p \leq C \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z).$$

We summarise the above two inequalities as follows

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \sim \inf \left\{ \sum_k |c_k|^p \;:\; f \text{ satisfies (2.5.14)} \right\}. \tag{2.5.15}$$

Essentially, we proved that the spaces A^p_α and l^p are isometrically isomorphic and, hence, as a consequence we have the following further property of the Bergman space.

Corollary 2.5.17. *Let $\alpha > -1$ and $0 < p < 1$. Then, the Bergman space A^p_α is not locally convex. Moreover, let $q > p$, then the inclusion $A^q_\alpha \subset A^p_\alpha$ is compact.*

The following two corollaries are concerned with two special cases. First, we assume that $p > 1$ and $b = n + 1 + \alpha$. Then, in the next case, we suppose $p = 1$ and $b = 2(n + 1 + \alpha)$.

Corollary 2.5.18. *For any $\alpha > -1$ and $p > 1$ we can find a sequence $\{a_k\}_k \in \mathbb{B}_n$ such that $f \in A^p_\alpha$ is represented as follows*

$$f(z) = \sum_k c_k \frac{(1 - |a_k|^2)^{(n+1+\alpha)/q}}{(1 - < z, a_k >)^{n+1+\alpha}} \tag{2.5.16}$$

where $1/p + 1/q = 1$ and $\{c_k\}_k \in l^p$.

Corollary 2.5.19. *For any $\alpha > 1$, there exists a sequence $\{a_k\}_k \in \mathbb{B}_n$ such that, for every $f \in A^1_\alpha$, the following representation holds*

$$f(z) = \sum_k c_k \frac{(1 - |a_k|^2)^{(n+1+\alpha)}}{(1 - < z, a_k >)^{2(n+1+\alpha)}}, \tag{2.5.17}$$

where $\{c_k\}_k \in l^1$.

To conclude this section, in the following corollary we provide a particular atomic decomposition in terms of elements of the Bergman space itself.

Corollary 2.5.20. *Let $\alpha > -1$, $p > 0$ and two real positive numbers r and q such that*

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}.$$

Then, under the above conditions, every $f \in A^p_\alpha$ can be decomposed as

$$f(z) = \sum_k g_k(z) h_k(z), \quad z \in \mathbb{B}_n, \tag{2.5.18}$$

where $g_k \in A^q_\alpha$ and $h_k \in A^r_\alpha$. In particular, if $0 < p \leq 1$, we can find a positive constant C, independent on f, such that the following holds

$$\sum_k ||g_k||_{q,\alpha}||h_k||_{r,\alpha} \leq C||f||_{p,\alpha}. \tag{2.5.19}$$

Proof. If $f(z) \neq 0, \forall z \in \mathbb{B}_n$, we just write

$$f(z) = f(z)^{p/q} f(z)^{p/r},$$

so, putting $h(z) := f(z)^{p/r} \in A^r_\alpha$ and $g(z) := f(z)^{p/q} \in A^q_\alpha$, the result follows easily. In general, the atomic decomposition of f is given by

$$f(z) = \sum_k f_k(z),$$

where

$$f_k(z) = c_k \frac{(1-|a_k|^2)^{b-(n+1+\alpha)/p}}{(1- < z, a_k >)^b}$$

that, when $c_k = 0$, is either identically zero or, when $c_k \neq 0$, nonvanishing on $\mathbb{B}_n$. Finally, if $c_k \neq 0$, the factorization is $f_k = f_k^{p/q} f_k^{p/r}$. □

2.6 Complex Interpolation

In this section, under the condition $1 \leq p < \infty$, we prove that the Bergman spaces A^p_α interpolates in the same manner that every $L^p(\mathbb{B}_n, dv_\alpha)$ space does. We remark that the case $p = \infty$ will be discussed in the following chapter.

Theorem 2.6.1. *Let $\alpha > -1$ and $1 \leq p_0 < p_1 < \infty$ such that, we have*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

for some $\theta \in (0,1)$. Then

$$[A^{p_0}_\alpha, A^{p_1}_\alpha]_\theta = A^p_\alpha$$

with equivalent norms.

Proof. Let $f \in A^p_\alpha$, for any complex number ζ so that $0 \leq Re\ \zeta \leq 1$, we define the function

$$h_\zeta(z) = \frac{f(z)}{|f(z)|}|f(z)|^{p\left(\frac{1-\zeta}{p_0}+\frac{\zeta}{p_1}\right)}, \quad z \in \mathbb{B}_n.$$

Pick some $\beta > \alpha$ and let $f_\zeta = P_\beta(h_\zeta)$. We notice that f_ζ is continuous in the closed strip $0 \leq Re\ \zeta \leq 1$. As a consequence of Theorem 2.2.9, for $1 \leq q < \infty$, we obtain that P_β is a bounded projection from $L^q(\mathbb{B}_n, dv_\alpha)$ onto A^q_α. Moreover, there exists a positive constant $C > 0$ such that the following holds

$$||f_\zeta||^{p_0}_{p_0,\alpha} \leq C||h_\zeta||^{p_0}_{p_0,\alpha} = C||f||^p_{p,\alpha}$$

when $Re\ \zeta = 0$. If $Re\ \zeta = 1$, we have

$$||f_\zeta||^{p_1}_{p_1,\alpha} \leq C||h_\zeta||^{p_1}_{p_1,\alpha} = C||f||^p_{p,\alpha}.$$

This fact proves that, when $||f||_\theta \leq C||f||_{p,\alpha}$, $f = f_\theta \in [A^{p_0}_\alpha, A^{p_1}_\alpha]_\theta$.
Now, we suppose that $f \in [A^{p_0}_\alpha, A^{p_1}_\alpha]_\theta$. Hence, f is holomorphic and

$$f \in [L^{p_0}(\mathbb{B}_n, dv_\alpha), L^{p_1}(\mathbb{B}_n, dv_\alpha)]_\theta = L^p(\mathbb{B}_n, dv_\alpha).$$

In other words, $f \in A^p_\alpha$. This completes the proof. □

Chapter 3

Bloch spaces

In this chapter we study the Bloch space and the little Bloch space. The Bloch space appears as the image of the bounded functions under the Bergman projection, but it also plays the role of the dual space of the Bergman spaces for small exponents ($0 < p \leq 1$).
The current chapter is organised as follows. Section 1 deals with the introduction of the Bloch space, we focus our attention on many different characterisations and some basic properties.
Next, in Section 2, we give a complete description of the little Bloch space and summarise some elementary properties that can be deduced, as a closed subspace of the Bloch space.
In Section 3 we present a classical method, perhaps the most important, to construct not-trivially function for both the Bloch space and the little Bloch space. So that, we will give some relevant applications of it.
The objective of Section 4 is to represent the dual space of A^p_α, for $0 < p \leq 1$, in terms of the Bloch space and the dual space of the little Bloch space in terms of A^1_α. Of course, we will also mention some straigthforward consequence of this representation.
The Bloch space is prominent among Mobius invariant function spaces. In fact, in Section 5, we prove that the Bloch space is the largest possible space of holomorphic functions whose seminorm is invariant under the action of the automorphism group $Aut(\mathbb{B}_n)$.
In Section 6, we give a short description of the pointwise multipliers of the Bloch space and the little Bloch space. In fact, after recalling some notion and necessary tool, we provide a useful characterisation of both spaces studied in this chapter.
In section 7, we show that the Bloch space admits an atomic decomposition that turns out to be similar to that of the Bergman spaces. This means that the results concerning the atomic decomposition of the Bergman spaces, obtained in chapter 2, will be crucial.
In Section 8 we illustrate the complex interpolation of the Bloch space and, so that, give much more evidence that such space behaves like the limit of A^p_α when $p \to \infty$.
For this chapter the main references are:
J. Garnett. Bounded Analytic Functions. Academic Press, New York, 1982.
Miroslav Pavlovic. Decompositions of L^p and Hardy Spaces of Polyharmonic Functions. 1996.
G. Ren, C. Tu. Bloch space in the Unit Ball of $\mathbb{C}^n$. 1992.
K. Zhu. Spaces of Holomorphic Functions in the Unit Ball. Springer, 2005.

3.1 The Bloch space $\mathcal{B}$

This section starts with the definition of the Bloch space, denoted by $\mathcal{B}$. We will concentrate on different aspects such as invariance under the action of the group $Aut(\mathbb{B}_n)$, separability and completeness. After that, we will prove various characterisations of the Bloch space: in terms of the holomorphic gradient, the radial derivative, as the image of the bounded functions under the Bergman projection, higher order derivatives and fractional radial derivatives. In this section we will exhibit the intrinsic connection between the Bloch space and the Bergman metric. Namely, we will prove that the Bloch space consists exactly of those holomorphic functions that are Lipschitz from $\mathbb{B}_n$ with the

Bergman metric to $\mathbb{C}$ with the Euclidean metric. We discuss the relation between the Bloch space and the Bergman space A^p_α, for $0 < p \leq \infty$.

Definition 3.1.1 (Bloch space)**.** The Bloch space, denoted by $\mathcal{B}$, is defined as

$$\mathcal{B} := \left\{ f \in H(\mathbb{B}_n) \ \mid \ ||f||_{\mathcal{B}} := \sup_{z \in \mathbb{B}_n} |\widetilde{\nabla} f(z)| < \infty \right\}. \tag{3.1.1}$$

The elements of $\mathcal{B}$ are called Bloch functions. Finally, we easily notice that $\mathcal{B}$ is a complex and convex vector linear space.

In the following proposition, as a consequence of Corollary 2.4.3, we give an important property of the Bloch space: the invariance under automorphisms of $\mathbb{B}_n$.

Proposition 3.1.2. *The Bloch space $\mathcal{B}$ is invariant under the action of $Aut(\mathbb{B}_n)$. Moreover,*

$$||f \circ \varphi||_{\mathcal{B}} = ||f||_{\mathcal{B}}, \tag{3.1.2}$$

for all $f \in \mathcal{B}$ and all $\varphi \in Aut(\mathbb{B}_n)$.

Proof. Using Corollary 2.4.3, we find

$$\begin{aligned} ||f \circ \varphi||_{\mathcal{B}} &= \sup_{z \in \mathbb{B}_n} |\widetilde{\nabla}(f \circ \varphi)(z)| \\ &= \sup_{z \in \mathbb{B}_n} |(\widetilde{\nabla} f) \circ \varphi(z)| \\ &= \sup_{w \in \mathbb{B}_n} |(\widetilde{\nabla} f)(w)| \\ &= ||f||_{\mathcal{B}}, \end{aligned}$$

and we are done. □

Remark 3.1.3. Since $|| \cdot ||_{\mathcal{B}}$ identifies functions that differ by a constant, we immediately notice that $|| \cdot ||_{\mathcal{B}}$ is not a norm but is a semi-norm. Furthermore, $\mathcal{B}$ is the largest possible space of holomorphic functions whose seminorm is invariant under the action of the automorphism group. This property will be proved in Section 5.

Our first goal is to define a invariant seminorm on the Bloch space in several equivalent ways. In order to proceed, we need the following definition.

Definition 3.1.4. Let $f \in H(\mathbb{B}_n)$, we introduce

$$Q_f(z) = \sup_{w \in \mathbb{C}^n \setminus \{0\}} \left\{ \frac{| < \nabla f(z), \overline{w} > |}{\sqrt{< B(z)w, w >}} \right\}, \ z \in \mathbb{B}_n. \tag{3.1.3}$$

Theorem 3.1.5. *Let $f \in H(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$. Then, the following quantities are equal:*
a) $Q_f(z)$.
b) $< \overline{B(z)^{-1} \nabla f(z), \nabla f(z)} >^{1/2}$.
c) $\left(\dfrac{\widetilde{\Delta}(|f|^2)(z)}{4} \right)^{1/2}$.
d) $|\widetilde{\nabla} f(z)|$.
e) $[(1 - |z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2)]^{1/2}$.

Proof. From (2.4.4), we obtain that the quantities *c*) and *d*) are equal. Then, from Lemma 2.4.1 follows that *d*) and *e*) are the same. We wish to prove that *a*) and *b*) are equals. Putting $w := B(z)^{-1/2}w$ in the definition of $Q_f(z)$ and using Cauchy-Schwarz, it turns out that

$$\begin{aligned} Q_f(z) &= \sup_{w\in\mathbb{C}^n\setminus\{0\}} \left\{ \frac{| < \nabla f(z), \overline{B(z)^{-1/2}w} > |}{|w|} \right\} \\ &= \sup_{w\in\mathbb{C}^n\setminus\{0\}} \left\{ \frac{| < \overline{B(z)^{-1/2}}\nabla f(z), \overline{w} > |}{|w|} \right\} \\ &= |\overline{B(z)^{-1/2}}\nabla f(z)| \\ &= < \overline{B(z)^{-1}}\nabla f(z), \nabla f(z) >^{1/2} . \end{aligned}$$

Finally, to prove that *b*) and *e*) are equal, using point *b*) of Proposition 1.6.3 and the previous equality, we find that

$$\begin{aligned} Q_f(z) &= < \overline{B(z)^{-1}}\nabla f(z), \nabla f(z) >^{1/2} \\ &= \left[< \overline{(1-|z|^2)(I - A(z))}\nabla f(z), \nabla f(z) > \right]^{1/2} \\ &= \left[(1-|z|^2)(|\nabla f(z)|^2 - < A(z)\nabla f(z), \nabla f(z) >) \right]^{1/2} \\ &= \left[(1-|z|^2)\left(|\nabla f(z)|^2 - < \sum_i^n z_i \frac{\partial f}{\partial z_i}, \sum_j^n z_j \frac{\partial f}{\partial z_j} > \right) \right]^{1/2} \\ &= \left[(1-|z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2) \right]^{1/2} . \end{aligned}$$

This completes the proof. □

Theorem 3.1.6. *The Bloch space $\mathcal{B}$ is complete.*

Proof. Assume that $\{f_k\}_k$ is a Cauchy sequence on $\mathcal{B}$ such that $f_k(0) = 0$. Since, from Lemma 2.4.4,

$$(1-|z|^2)|\nabla f(z)| \le |\widetilde{\nabla} f(z)|.$$

So, as a consequence, $\left\{ \frac{\partial f_k}{\partial z_i} \right\}_k$ is uniformly Cauchy on every compact set of $\mathbb{B}_n$. Hence, from Weierstrass theorem and the completeness of $\mathbb{C}$, there exists $f \in H(\mathbb{B}_n)$, with $f(0) = 0$, such that

$$\lim_{k\to\infty} f_k(z) = f(z)$$

and

$$\lim_{k\to\infty} \frac{\partial f_k}{\partial z_i}(z) = \frac{\partial f}{\partial z_i}(z), \ \ i = 1, ..., n,$$

uniformly on every compact set in $\mathbb{B}_n$. By the fact that $\{f_k\}_k$ is a Cauchy sequence, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$||f_k - f_l||_{\mathcal{B}} < \varepsilon, \quad k > N, \ l > N.$$

So that, applying part *e*) of the Theorem 3.1.5, we get

$$(1-|z|^2)(|\nabla(f_k - f_l)(z)|^2 - |R(f_k - f_l)(z)|^2) < \varepsilon^2, \quad k > N, \ l > N.$$

Finally, considering the limit $l \to \infty$ and the supremum over $z \in \mathbb{B}_n$, we obtain

$$||f - f_k||_{\mathcal{B}} < \varepsilon, \forall k > N,$$

and we are done. $\square$

Actually, more is true.

Theorem 3.1.7. *The Bloch space $\mathcal{B}$ can be made into a Banach space by introducing the norm*

$$||f|| := |f(0)| + ||f||_{\mathcal{B}} \tag{3.1.4}$$

Hence, we easily deduce that the Bloch space, endowed with $||\cdot||$, is locally convex and locally bounded.

It is inconvenient, by using the definition, to verify that a certain function belongs to the Bloch space. Hence, we will give several conditions that are equivalent to but more easily verifiable than the definition. To this end, we need a pair of tools. We start with the following lemma.

Lemma 3.1.8. *Assume $\beta \in \mathbb{R}$ and $g \in L^1(\mathbb{B}_n, dv)$. If*

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)dv(w)}{(1- < z, w >)^{\beta}}, \quad z \in \mathbb{B}_n, \tag{3.1.5}$$

then

$$|\widetilde{\nabla} f(z)| \leq \sqrt{2}|\beta|(1 - |z|^2)^{1/2} \int_{\mathbb{B}_n} \frac{|g(w)|dv(w)}{|1- < z, w > |^{\beta+1/2}}, \ \forall z \in \mathbb{B}_n.$$

Proof. If $\beta < 0$, it is clear that $f \in H(\mathbb{B}_n)$. If $\beta \geq 0$, we have

$$\begin{aligned} f(z) &= \int_{\mathbb{B}_n} \frac{g(w)dv(w)}{(1- < z, w >)^{\beta}} \\ &= \int_{\mathbb{B}_n} g(w) \sum_{k=0}^{\infty} \frac{\Gamma(|k| + \beta)}{k!\Gamma(\beta)} z^k \overline{w}^k dv(w) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(|k| + \beta)}{k!\Gamma(\beta)} z^k \int_{\mathbb{B}_n} g(w)\overline{w}^k dv(w), \end{aligned}$$

where we used the uniform convergence of $\dfrac{1}{(1- < z, w >)^{\beta}}$ on compact subsets. Furthermore

$$\begin{aligned} |f(z)| &\leq \int_{\mathbb{B}_n} \frac{|g(w)|dv(w)}{|1- < z, w > |^{\beta}} \\ &\leq \frac{1}{(1 - |z|)^{\beta}} \int_{\mathbb{B}_n} |g(w)|dv(w) < \infty, \end{aligned}$$

in other words

$$|f(z)| \leq \frac{1}{(1 - |z|)^{\beta}} ||g||_1.$$

Now, fix $a \in \mathbb{B}_n$, making the change of variables $w \to \varphi_a(w)$ and using (2.5.5), we obtain that (3.1.5) becomes as follows

$$\begin{aligned} f \circ \varphi_a(z) &= \int_{\mathbb{B}_n} \frac{(g \circ \varphi_a)(w)}{(1- < \varphi_a(z), \varphi_a(w) >)^{\beta}} \frac{(1 - |a|^2)^{n+1}}{|1- < w, a > |^{2(n+1)}} dv(w) \\ &= \frac{(1- < z, a >)^{\beta}}{(1 - |a|^2)^{\beta}} \int_{\mathbb{B}_n} (g \circ \varphi_a)(w) \frac{(1 - |a|^2)^{n+1}}{|1- < w, a > |^{2(n+1)}} \frac{(1- < a, w >)^{\beta}}{(1- < z, w >)^{\beta}} dv(w), \end{aligned}$$

so

$$f\circ\varphi_a(z)=\frac{(1-<z,a>)^\beta}{(1-|a|^2)^\beta}\int_{\mathbb{B}_n}(g\circ\varphi_a)(w)\frac{(1-|a|^2)^{n+1}}{|1-<w,a>|^{2(n+1)}}\frac{(1-<a,w>)^\beta}{(1-<z,w>)^\beta}dv(w). \tag{3.1.6}$$

Then, using the product rule and the dominated convergence theorem, we have

$$\begin{aligned}\left.\frac{\partial(f\circ\varphi_a)(z)}{\partial z_j}\right|_{z=0} &= \frac{\beta(-\overline{a}_j)}{(1-|a|^2)^\beta}\int_{\mathbb{B}_n}(g\circ\varphi_a)(w)\frac{(1-|a|^2)^{n+1}}{|1-<w,a>|^{2(n+1)}}(1-<a,w>)^\beta dv(w)+\\ &+\frac{\beta}{(1-|a|^2)^\beta}\int_{\mathbb{B}_n}(g\circ\varphi_a)(w)\frac{(1-|a|^2)^{n+1}}{|1-<w,a>|^{2(n+1)}}\overline{w}_j(1-<a,w>)^\beta dv(w)\\ &=\frac{\beta}{(1-|a|^2)^\beta}\int_{\mathbb{B}_n}(g\circ\varphi_a)(w)\frac{(1-|a|^2)^{n+1}}{|1-<w,a>|^{2(n+1)}}(\overline{w}_j-\overline{a}_j)(1-<a,w>)^\beta dv(w),\end{aligned}$$

that clearly implies

$$\widetilde{\nabla}f(a)=\frac{\beta}{(1-|a|^2)^\beta}\int_{\mathbb{B}_n}(g\circ\varphi_a)(w)\frac{(1-|a|^2)^{n+1}}{|1-<w,a>|^{2(n+1)}}(\overline{w}-\overline{a})(1-<a,w>)^\beta dv(w). \tag{3.1.7}$$

Again, making the change of variables $w\to\varphi_a(w)$, it turns out that (3.1.7) is written as follows

$$\widetilde{\nabla}f(a)=\beta\int_{\mathbb{B}_n}\frac{(\overline{\varphi_a(w)}-\overline{a})g(w)dv(w)}{(1-<a,w>)^\beta}, \tag{3.1.8}$$

so that

$$\begin{aligned}|\widetilde{\nabla}f(a)| &\le |\beta|\int_{\mathbb{B}_n}\frac{|(\overline{\varphi_a(w)}-\overline{a})||g(w)|dv(w)}{|1-<a,w>|^\beta}\\ &=|\beta|\int_{\mathbb{B}_n}\frac{\sqrt{(1-|a|^2)(|w|^2-|<w,a>|^2)}}{|1-<a,w>|}\frac{|g(w)|dv(w)}{|1-<a,w>|^\beta}\\ &\le|\beta|\int_{\mathbb{B}_n}\frac{\sqrt{(1-|a|^2)(1-|<w,a>|)(1+|<w,a>|)}}{|1-<a,w>|}\frac{|g(w)|dv(w)}{|1-<a,w>|^\beta}\\ &\le|\beta|\int_{\mathbb{B}_n}\frac{\sqrt{(1-|a|^2)2(1-|<w,a>|)}}{|1-<a,w>|}\frac{|g(w)|dv(w)}{|1-<a,w>|^\beta}\\ &\le|\beta|\sqrt{2}\int_{\mathbb{B}_n}\sqrt{(1-|a|^2)}\frac{|g(w)|dv(w)}{|1-<a,w>|^{\beta+1/2}},\end{aligned}$$

and we are done. □

The second tool is an identity that establishes the connection between the radial derivative and the integral representation formula of Lemma 2.1.3.

Proposition 3.1.9. *Let $f\in H(\mathbb{B}_n)$. Then, the following holds*

$$Rf(z)=(n+1+\alpha)\int_{\mathbb{B}_n}\frac{f(w)<z,w>}{(1-<z,w>)^{n+2+\alpha}}dv_\alpha(w). \tag{3.1.9}$$

Proof.

$$
\begin{aligned}
Rf(z) &= \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}(z) z_k \\
&= \sum_{k=1}^{n} \frac{\partial}{\partial z_k} \left\{ \int_{\mathbb{B}_n} \frac{f(w)}{(1-<z,w>)^{n+1+\alpha}} dv_\alpha(w) \right\} z_k \\
&= \sum_{k=1}^{n} \left\{ \int_{\mathbb{B}_n} \frac{\partial}{\partial z_k} \frac{f(w)}{(1-<z,w>)^{n+1+\alpha}} dv_\alpha(w) \right\} z_k \\
&= \sum_{k=1}^{n} (n+1+\alpha) \left\{ \int_{\mathbb{B}_n} \frac{f(w)\overline{w}_k}{(1-<z,w>)^{n+2+\alpha}} dv_\alpha(w) \right\} z_k \\
&= (n+1+\alpha) \int_{\mathbb{B}_n} \frac{f(w) <z,w>}{(1-<z,w>)^{n+2+\alpha}} dv_\alpha(w)
\end{aligned}
$$

□

Now, we are ready to prove the following characterisation of the Bloch space in terms of the holomorphic gradient and the radial derivative. Furthermore, in this theorem, we prove that the Bloch space can be considered as the limit case of A^p_α as $p \to +\infty$.

Theorem 3.1.10. *Assume $\alpha > -1$ and let $f \in H(\mathbb{B}_n)$. Then, the following are equivalent:*
a) $f \in \mathcal{B}$.
b) $(1-|z|^2)|\nabla f(z)|$ is bounded in $\mathbb{B}_n$.
c) $(1-|z|^2)|Rf(z)|$ is bounded in $\mathbb{B}_n$.
d) There exists $g \in L^\infty(\mathbb{B}_n)$ such that $f = P_\alpha g$.

Proof. From Lemma 2.4.4, we easily obtain that *a*) implies *b*) and *b*) implies *c*) as well.
We prove that *c*) implies *d*) as follows. Let the following function be

$$g(z) = \frac{c_{\alpha+1}}{c_\alpha}(1-|z|^2) \int_{\mathbb{B}_n} \frac{f(w)dv_\alpha(w)}{(1-<z,w>)^{n+2+\alpha}}, \ z \in \mathbb{B}_n,$$

where

$$c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}.$$

Then, using the boundedness of $(1-|z|^2)|Rf(z)|$, the integral representation formula of Lemma 2.1.3 and Proposition 3.1.9, it turns out that

$$
\begin{aligned}
g(z) &= \frac{c_{\alpha+1}}{c_\alpha} \left\{ (1-|z|^2) \int_{\mathbb{B}_n} \frac{1-<z,w>}{(1-<z,w>)^{n+2+\alpha}} f(w)dv_\alpha(w) + (1-|z|^2) \int_{\mathbb{B}_n} \frac{<z,w>}{(1-<z,w>)^{n+2+\alpha}} f(w)dv_\alpha(w) \right\} \\
&= \frac{c_{\alpha+1}}{c_\alpha} \left\{ (1-|z|^2) \int_{\mathbb{B}_n} \frac{f(w)dv_\alpha(w)}{(1-<z,w>)^{n+1+\alpha}} + (1-|z|^2) \int_{\mathbb{B}_n} \frac{<z,w>}{(1-<z,w>)^{n+2+\alpha}} f(w)dv_\alpha(w) \right\} \\
&= \frac{c_{\alpha+1}}{c_\alpha} \left\{ (1-|z|^2)f(z) + (1-|z|^2) \int_{\mathbb{B}_n} \frac{<z,w>}{(1-<z,w>)^{n+2+\alpha}} f(w)dv_\alpha(w) \right\} \\
&= \frac{c_{\alpha+1}}{c_\alpha} \left\{ (1-|z|^2)f(z) + \frac{(1-|z|^2)}{(n+1+\alpha)} Rf(z) \right\}.
\end{aligned}
$$

Now, using (1.5.3) and the fact that $(1-|z|^2)|\nabla f(z)|$ is bounded, we find

$$\begin{aligned}
|f(z)-f(0)| &= \left|\int_0^1 \frac{Rf(tz)}{t}dt\right| \\
&= \left|\int_0^1 < \nabla f(tz), \overline{z} > dt\right| \\
&\leq \int_0^1 |\nabla f(tz)| \frac{1-t^2|z|^2}{1-t^2|z|^2} dt \\
&\leq \sup_{w\in\mathbb{B}_n} \left(|\nabla f(w)|(1-|w|^2)\right) \int_0^1 \frac{1}{1-t^2|z|^2} dt \\
&= \sup_{w\in\mathbb{B}_n} \left(|\nabla f(w)|(1-|w|^2)\right) \left\{\int_0^1 \frac{1}{2}\frac{1}{1+t|z|}dt + \int_0^1 \frac{1}{2}\frac{1}{1-t|z|}dt\right\} \\
&= \sup_{w\in\mathbb{B}_n} \left(|\nabla f(w)|(1-|w|^2)\right) \left\{\frac{1}{2|z|}\int_0^1 \frac{|z|}{1+t|z|}dt - \frac{1}{2|z|}\int_0^1 \frac{-|z|}{1-t|z|}dt\right\} \\
&= \sup_{w\in\mathbb{B}_n} \left(|\nabla f(w)|(1-|w|^2)\right) \frac{1}{2|z|} \log\left(\frac{1+|z|}{1-|z|}\right).
\end{aligned}$$

This fact shows that f grows at most as fast as $\log\left(\frac{1+|z|}{1-|z|}\right)$ and, since from de l'Hopital theorem we have that

$$\lim_{|z|\to 1^-}(1-|z|^2)\log\left(\frac{1+|z|}{1-|z|}\right) = 0,$$

and by the fact that

$$\begin{aligned}
\lim_{|z|\to 0}\frac{1}{2|z|}\log\left(\frac{1+|z|}{1-|z|}\right) &= \lim_{|z|\to 0}\frac{1}{2|z|}\log\left(1+\frac{2|z|}{1-|z|}\right) \\
&= \lim_{|z|\to 0}\frac{1}{2|z|}\frac{2|z|}{1-|z|} \\
&= 1,
\end{aligned}$$

it turns out that

$$\begin{aligned}
||g||_{\infty,\alpha} &\leq \frac{c_{\alpha+1}}{c_\alpha}\sup_{z\in\mathbb{B}_n}\left\{(1-|z|^2)|f(z)| + \frac{(1-|z|^2)}{(n+1+\alpha)}|Rf(z)|\right\} \\
&\leq \frac{c_{\alpha+1}}{c_\alpha}\sup_{z\in\mathbb{B}_n}\left\{\sup_{w\in\mathbb{B}_n}\left(|\nabla f(w)|1-|w|^2\right)\frac{1}{2}(1-|z|^2)\log\left(\frac{1+|z|}{1-|z|}\right) + \frac{(1-|z|^2)}{(n+1+\alpha)}|Rf(z)|\right\} \\
&< \infty.
\end{aligned}$$

In other words $g \in L^\infty(\mathbb{B}_n)$. After that, applying Fubini's theorem, we get

$$
\begin{aligned}
P_\alpha g &= \int_{\mathbb{B}_n} \frac{1}{(1-<x,z>)^{n+\alpha+1}} dv_\alpha(z) \frac{c_{\alpha+1}}{c_\alpha}(1-|z|^2) \int_{\mathbb{B}_n} \frac{f(w)dv_\alpha(w)}{(1-<z,w>)^{n+2+\alpha}} \\
&= \int_{\mathbb{B}_n} \frac{1}{(1-<x,z>)^{n+\alpha+1}} dv_{\alpha+1}(z) \int_{\mathbb{B}_n} \frac{f(w)dv_\alpha(w)}{(1-<z,w>)^{n+2+\alpha}} \\
&= \int_{\mathbb{B}_n} f(w)dv_\alpha(w) \int_{\mathbb{B}_n} \frac{1}{(1-<x,z>)^{n+\alpha+1}(1-<z,w>)^{n+2+\alpha}} dv_{\alpha+1}(z) \\
&= \int_{\mathbb{B}_n} f(w)dv_\alpha(w) \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x^j \overline{w}^k \frac{\Gamma(|k|+n+\alpha+2)}{k!\Gamma(n+2+\alpha)} \frac{\Gamma(|j|+n+\alpha+1)}{j!\Gamma(n+1+\alpha)} \int_{\mathbb{B}_n} \overline{z}^j z^k dv_{\alpha+1}(z) \\
&= \int_{\mathbb{B}_n} f(w)dv_\alpha(w) \sum_{k=0}^{\infty} x^k \overline{w}^k \frac{\Gamma(|k|+n+\alpha+2)}{k!\Gamma(n+2+\alpha)} \frac{\Gamma(|k|+n+\alpha+1)}{k!\Gamma(n+1+\alpha)} \frac{k!\Gamma(n+\alpha+2)}{\Gamma(n+|k|+\alpha+2)} \\
&= \int_{\mathbb{B}_n} f(w)dv_\alpha(w) \sum_{k=0}^{\infty} x^k \overline{w}^k \frac{\Gamma(|k|+n+\alpha+1)}{k!\Gamma(n+1+\alpha)} \\
&= \int_{\mathbb{B}_n} \frac{f(w)dv_\alpha(w)}{(1-<x,w>)^{n+1+\alpha}} \\
&= f,
\end{aligned}
$$

We prove that $P_\alpha g = f$ and, hence, that *c*) implies *d*).
Finally, we wish to prove that *d*) implies *a*). We assume that $f = P_\alpha g$ for some $\alpha > -1$, where $g \in L^\infty(\mathbb{B}_n)$. First of all, from Proposition 2.2.4, we have that $f \in H(\mathbb{B}_n)$. Then, from Lemma 3.1.8, we deduce that there exists a positive constant C such that

$$|\widetilde{\nabla} f(z)| \leq C||g||_\infty (1-|z|^2)^{1/2} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha dv(w)}{|1-<z,w>|^{n+1+\alpha+1/2}}, \ \forall z \in \mathbb{B}_n.$$

As a consequence of Theorem 1.4.4, the integral on the right side of the inequality is bounded in $\mathbb{B}_n$. In a few words, we obtain that $|\widetilde{\nabla} f(z)|$ is bounded in $\mathbb{B}_n$. This completes the proof. □

Remark 3.1.11. In other words, the equivalence between point *a*) and *d*) says that, for $\alpha > -1$, the Bergman projection P_α is a bounded linear operator from $L^\infty(\mathbb{B}_n)$ onto $\mathcal{B}$. More is true: recalling that, from Remark 2.2.2, we had

$$\lim_{|z| \to 1^-} P_\alpha(g)(z)(1-|z|^2)^{(n+1+\alpha)/2} = 0, \ \forall \ g \in L^\infty(\mathbb{B}_n).$$

Hence, as a further consequence of the equivalence between point *d*) and *a*) in Theorem 3.1.10, we get the following boundary behaviour property of Bloch functions:

$$\lim_{|z| \to 1^-} f(z)(1-|z|^2)^{(n+1+\alpha)/2} = 0, \ \forall \ f \in \mathcal{B}.$$

The Bloch space can also be described in terms of higher order derivatives and, more generally, in terms of fractional radial derivatives. This is the content of the next theorem.

Theorem 3.1.12. *Assume that* N *is a positive integer,* $t > 0$ *and* $f \in H(\mathbb{B}_n)$. *If* α *is a real parameter such that neither* $n+\alpha$ *nor* $n+\alpha+t$ *is a negative integer. Then, the following conditions are equivalent:*
1) $f \in \mathcal{B}$.

2) *The function* $(1-|z|^2)R^{\alpha,t}f(z)$ *is bounded in* $\mathbb{B}_n$.
3) *The family of functions*

$$(1-|z|^2)^N \frac{\partial^m f}{\partial z^m}(z), \quad |m| = N,$$

are bounded in $\mathbb{B}_n$.

Proof. We start proving 1) $\Longrightarrow$ 2). Let $f \in \mathcal{B}$, by Theorem 3.1.10, there exists a function $g \in L^\infty(\mathbb{B}_n)$ such that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)dv_\beta(w)}{(1-<z,w>)^{n+1+\beta}}, \tag{3.1.10}$$

where $\beta = \alpha + K$ and K is a positive integer large enough so that $\beta > -1$. From Lemma 2.4.13, there exists a one-variable polynomial $h(<z,w>)$ and a positive constant c_α such that

$$R^{\alpha,t}f(z) = c_\alpha \int_{\mathbb{B}_n} \frac{h(<z,w>)g(w)dv_\beta(w)}{(1-<z,w>)^{n+1+\beta+t}},$$

then, applying Theorem 1.4.4, it turns out that

$$\begin{aligned}
(1-|z|^2)^t|R^{\alpha,t}f(z)| &= c_\alpha(1-|z|^2)^t \left| \int_{\mathbb{B}_n} \frac{h(<z,w>)g(w)dv_\beta(w)}{(1-<z,w>)^{n+1+\beta+t}} \right| \\
&= c_\alpha(1-|z|^2)^t \left| \int_{\mathbb{B}_n} \frac{h(<z,w>)g(w)(1-|w|^2)^\beta dv(w)}{(1-<z,w>)^{n+1+\beta+t}} \right| \\
&\le c_\alpha ||g||_{\infty,\alpha} \sup_{z,w\in\mathbb{B}_n} |h(<z,w>)|(1-|z|^2)^t \left| \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\beta dv(w)}{(1-<z,w>)^{n+1+\beta+t}} \right| \\
&\le c_\alpha ||g||_{\infty,\alpha} \sup_{z,w\in\mathbb{B}_n} |h(<z,w>)|(1-|z|^2)^t \tilde{C} \frac{1}{(1-|z|^2)^t} \\
&= c_\alpha \tilde{C} ||g||_{\infty,\alpha} \sup_{z,w\in\mathbb{B}_n} |h(<z,w>)| \\
&< \infty.
\end{aligned}$$

In other words, $(1-|z|^2)^t|R^{\alpha,t}f(z)|$ is bounded in $\mathbb{B}_n$.
A similar argument proves that 1) $\Longrightarrow$ 3). In fact, from (3.1.10), since the derivative of a polynomial is a polynomial too, applying Theorem 1.4.4 and denoting by

$$C_1 := \sup_{z\in\mathbb{B}_n} \left| \frac{\partial}{\partial z_i} h(<z,w>) \right| \quad \text{and} \quad C_2 := (n+1+\beta+t) \sup_{z\in\mathbb{B}_n} |h(<z,w>)|,$$

we have that

$$(1-|z|^2)\left|\frac{\partial f}{\partial z_i}(z)\right| = c_\alpha(1-|z|^2)\left|\int_{\mathbb{B}_n} g(w)\frac{\partial}{\partial z_i}\frac{h(<z,w>)dv_\beta(w)}{(1-<z,w>)^{n+1+\beta}}\right|$$

$$= c_\alpha(1-|z|^2)\left|\int_{\mathbb{B}_n} g(w)\left\{\frac{\frac{\partial}{\partial z_i}h(<z,w>)}{(1-<z,w>)^{n+1+\beta}} - \frac{(n+1+\beta)h(<z,w>)}{(1-<z,w>)^{n+2+\beta}}\right\}dv_\beta(w)\right|$$

$$\leq c_\alpha(1-|z|^2)||g||_{\infty,\alpha}\left|\int_{\mathbb{B}_n}\left\{\frac{\frac{\partial}{\partial z_i}h(<z,w>)}{(1-<z,w>)^{n+1+\beta}} - \frac{(n+1+\beta)h(<z,w>)}{(1-<z,w>)^{n+2+\beta}}\right\}dv_\beta(w)\right|$$

$$\leq c_\alpha(1-|z|^2)||g||_{\infty,\alpha}\left|\int_{\mathbb{B}_n}\left\{\frac{\frac{\partial}{\partial z_i}h(<z,w>)}{(1-<z,w>)^{n+1+\beta}} - \frac{(n+1+\beta)h(<z,w>)}{(1-<z,w>)^{n+2+\beta}}\right\}dv_\beta(w)\right|$$

$$\leq c_\alpha(1-|z|^2)||g||_{\infty,\alpha}\left\{C_1\left|\int_{\mathbb{B}_n}\frac{(1-|z|^2)^\beta dv(z)}{(1-<z,w>)^{n+1+\beta}}\right| + C_2\left|\int_{\mathbb{B}_n}\frac{(1-|z|^2)^\beta dv(z)}{(1-<z,w>)^{n+2+\beta}}\right|\right\}$$

$$\leq c_\alpha(1-|z|^2)||g||_{\infty,\alpha}C_1\tilde{C} + C_2\hat{C}c_\alpha||g||_{\infty,\alpha},$$

that is, $(1-|z|^2)\dfrac{\partial f}{\partial z_i}(z)$ is bounded in $\mathbb{B}_n$. Proceeding similarly, we prove that

$$(1-|z|^2)^N\frac{\partial^m f}{\partial z^m}(z), \quad |m| = N,$$

are all bounded in $\mathbb{B}_n$.
We prove that 2) implies 1). Assuming that the function $(1-|z|^2)^t R^{\alpha,t}f(z)$ is bounded in $\mathbb{B}_n$, from Remark 2.4.14, the function

$$g(z) := \frac{c_{\beta+t}}{c_\beta}(1-|z|^2)^t R^{\beta,t}f(z)$$

is also bounded in $\mathbb{B}_n$, where $\beta = \alpha + K$. Using Fubini theorem and (2.4.12), we get

$$P_\beta(g)(z) = \int_{\mathbb{B}_n}\frac{c_{\beta+t}}{c_\beta}(1-|w|^2)^t\frac{R^{\beta,t}f(w)}{(1-<z,w>)^{n+1+\beta}}c_\beta(1-|w|^2)^\beta dv(w)$$

$$= \int_{\mathbb{B}_n}\frac{R^{\beta,t}f(w)}{(1-<z,w>)^{n+1+\beta}}dv_{\beta+t}(w)$$

$$= \int_{\mathbb{B}_n}\frac{1}{(1-<z,w>)^{n+1+\beta}}dv_{\beta+t}(w)\lim_{r\to 1^-}\int_{\mathbb{B}_n}\frac{f(rx)dv_\beta(x)}{(1-<w,x>)^{n+1+\beta+t}}$$

$$= \lim_{r\to 1^-}\int_{\mathbb{B}_n}f(rx)dv_\beta(x)\int_{\mathbb{B}_n}\frac{dv_{\beta+t}(w)}{(1-<z,w>)^{n+1+\beta}(1-<w,x>)^{n+1+\beta+t}}$$

$$= \lim_{r\to 1^-}\int_{\mathbb{B}_n}\frac{f(rx)dv_\beta(x)}{(1-<z,x>)^{n+1+\beta}}$$

$$= f(z).$$

Hence, from Theorem 3.1.10, we deduce that $f \in \mathcal{B}$.
Let's prove that 1) and 3) are equivalent. Suppose that 3) holds. Then, proceeding by successive integration, we obtain that

$$(1-|z|^2)\frac{\partial f}{\partial z_k}(z),\ 1 \leq k \leq n,$$

are all bounded in $\mathbb{B}_n$. So, $f \in \mathcal{B}$. We conclude that 1) and 3) are equivalent. □

In the following proposition, we provide a description of the Bloch space in terms of derivatives and the automorphism group $Aut(\mathbb{B}_n)$.

Proposition 3.1.13. *Suppose $m = (m_1, \cdots, m_n)$ is any given multi-index of nonnegative integers with $|m| > 0$ and let $f \in H(\mathbb{B}_n)$. Then, $f \in \mathcal{B}$ if and only if*

$$\sup_{\varphi \in Aut(\mathbb{B}_n)} \left\{ \left| \frac{\partial^m (f \circ \varphi)}{\partial z^m}(0) \right| \right\} < \infty \tag{3.1.11}$$

Proof. Assume that $f \in \mathcal{B}$ and that $m = (m_1, m_2, ..., m_j + 1, ..., m_n)$. From (1.2.11), putting $a = 0$, it turns out that

$$\begin{aligned}
\left| \frac{\partial^m f}{\partial z^m}(0) \right| &= \left| \frac{m_1! \dots m_n!}{(2\pi i)^n} \int_{b_0 P(0,r)} \frac{\partial f}{\partial z_j}(\xi) \frac{d\xi_1 ... d\xi_n}{(\xi_1)^{m_1} \dots (\xi_n)^{m_n}} \right| \\
&= \left| \frac{m_1! \dots m_n!}{(2\pi i)^n} \int_{C(0,r_1)} \cdots \int_{C(0,r_n)} \frac{\partial f}{\partial z_j}(\xi) \frac{d\xi_1 ... d\xi_n}{(\xi_1)^{m_1} \dots (\xi_n)^{m_n}} \right| \\
&= \left| \frac{m_1! \dots m_n!}{(2\pi i)^n} \int_{C(0,r_1)} \cdots \int_{C(0,r_n)} \frac{\partial f}{\partial z_j}(\xi) \frac{\sqrt{1-|\xi|^2}}{\sqrt{1-|\xi|^2}} \frac{d\xi_1 ... d\xi_n}{(\xi_1)^{m_1} \dots (\xi_n)^{m_n}} \right| \\
&\leq \sup_{z \in \mathbb{B}_n} \left\{ (1-|z|^2)^{1/2} \left| \frac{\partial f}{\partial z_j}(z) \right| \right\} \left\{ \frac{m_1! \dots m_n!}{(2\pi)^n} \int_{C(0,r_1)} \cdots \int_{C(0,r_n)} \frac{1}{\sqrt{1-|\xi|^2}} \frac{d|\xi_1| \dots d|\xi_n|}{|\xi_1|^{m_1} \dots |\xi_n|^{m_n}} \right\} \\
&= \underbrace{\frac{m_1! \dots m_n!}{\sqrt{1-r^2}(r_1)^{m_1-1} \dots (r_n)^{m_n-1}}}_{=:C} \sup_{z \in \mathbb{B}_n} \left\{ (1-|z|^2)^{1/2} \left| \frac{\partial f}{\partial z_j}(z) \right| \right\} \\
&\leq C ||f||_{\mathcal{B}}.
\end{aligned}$$

Then, we replace f by $f \circ \varphi$ and the first implication is proved.
Conversely, suppose that (3.1.11) holds. Hence, choosing $m = (1, 0, ..., 0)$ and $\varphi = \varphi_z$, we obtain that there exists a positive constant C such that

$$\left| \frac{\partial (f \circ \varphi_z)}{\partial z_1}(0) \right| \leq C_1,$$

and, choosing $m = (0, ..., 0, \underbrace{1}_{j-th}, 0, ..., 0)$, it turns out that

$$\left| \frac{\partial (f \circ \varphi_z)}{\partial z_j}(0) \right| \leq C_j, j = 2, ..., n.$$

This fact implies that

$$\begin{aligned}
\sup_{z \in \mathbb{B}_n} |\nabla (f \circ \varphi_z)(0)| &= \sup_{z \in \mathbb{B}_n} \sqrt{\sum_{j=1}^n \left| \frac{\partial (f \circ \varphi_z)}{\partial z_j}(0) \right|^2} \\
&\leq \sup_{z \in \mathbb{B}_n} \sqrt{\sum_{j=1}^n C_j^2} \\
&< \infty.
\end{aligned}$$

In other words,

$$||f||_{\mathcal{B}} < \infty.$$

$\square$

The Bloch space can also be characterised in terms of the Bergman metric. In fact, the seminorm $||\cdot||_{\mathcal{B}}$ is related to the Bergman metric in a very precise way.

Theorem 3.1.14. *Let $f \in H(\mathbb{B}_n)$, then*

$$||f||_{\mathcal{B}} = \sup\left\{\frac{|f(z)-f(w)|}{\beta(z,w)} \quad : z,w \in \mathbb{B}_n, z \neq w\right\}, \tag{3.1.12}$$

where β is the Bergman metric on $\mathbb{B}_n$.

Proof. Assumed that $||f||_{\mathcal{B}} < \infty$. Fixed two points $z, w \in \mathbb{B}_n$. Let

$$\gamma = \gamma(t), \quad 0 \le t \le 1,$$

be a smooth curve from w to z in the Bergman metric. Then, using (3.1.3), it turns out that

$$\begin{aligned}
|f(z)-f(w)| &= \left|\int_0^1 \left(\sum_{k=1}^n \gamma_k'(t)\frac{\partial f}{\partial z_k}(\gamma(t))\right) dt\right| \\
&= \int_0^1 \left|\sum_{k=1}^n \gamma_k'(t)\frac{\partial f}{\partial z_k}(\gamma(t))\right| dt \\
&= \int_0^1 Q_f(\gamma(t))\sqrt{< B(\gamma(t))\gamma'(t), \gamma'(t) >}dt \\
&= ||f||_{\mathcal{B}}\beta(z,w).
\end{aligned}$$

From this estimate, we obtain

$$\sup\left\{\frac{|f(z)-f(w)|}{\beta(z,w)} \quad : z,w \in \mathbb{B}_n, z \neq w\right\} \le ||f||_{\mathcal{B}}, \ \forall f \in H(\mathbb{B}_n). \tag{3.1.13}$$

In order to prove the other inequality, we proceed as follows. Supposing that

$$C := \sup\left\{\frac{|f(z)-f(w)|}{\beta(z,w)} \quad : z,w \in \mathbb{B}_n, z \neq w\right\} < \infty.$$

Hence, putting $w = 0$ and using (1.6.7) in (3.1.12), we obtain

$$|f(z)-f(0)| \le \frac{C}{2}\log\left(\frac{1+|z|}{1-|z|}\right)$$

so that

$$\frac{|f(z)-f(0)|}{|z|} \le \frac{C}{2|z|}\log\left(\frac{1+|z|}{1-|z|}\right), \ \forall z \in \mathbb{B}_n \setminus \{0\}.$$

Let u any unit vector of $\mathbb{C}^n$ be. Then, taking the directional derivative of f at 0 in the $u-$directional yields

$$\left|\sum_{k=1}^n u_k \frac{\partial f}{\partial z_k}(0)\right| \le C \lim_{|z|\to 0^+} \frac{1}{2|z|}\log\left(\frac{1+|z|}{1-|z|}\right) = C.$$

This proves that $Q_f(0) \le C$. Finally, using the invariance of the Bergman metric under the automorphism group, we get

$$C = \sup\left\{\frac{|f\circ\varphi(z)-f\circ\varphi(w)|}{\beta(z,w)} \quad : z,w \in \mathbb{B}_n, z \neq w\right\}, \quad \forall \varphi \in \mathrm{Aut}(\mathbb{B}_n).$$

That is,

$$Q_f(z) = Q_{f\circ\varphi}(0) \leq C, \forall z \in \mathbb{B}_n.$$

We conclude that $||f||_{\mathcal{B}} \leq C$ and, hence, (3.1.12) holds. $\square$

As a consequence, we have that

Corollary 3.1.15. *Let $f \in H(\mathbb{B}_n)$. Then, $f \in \mathcal{B}$ if and only if there exists a positive constant C such that*

$$|f(z) - f(w)| \leq C\beta(z, w)$$

for all z and w in $\mathbb{B}_n$.

Remark 3.1.16. Theorem 3.1.14 has some immediate consequences. First, assume that $f \in \mathcal{B}$ and $w = 0$. Then, we obtain

$$|f(z)| \leq ||f||_{\mathcal{B}} \log\left(\frac{1+|z|}{1-|z|}\right) + |f(0)|.$$

This means that point evaluation is a bounded linear functional on the Bloch space, with a norm that is uniformly bounded on each compact subset of $\mathbb{B}_n$. This fact, together with the maximum principle, implies that if a sequence of functions converges in the Bloch norm, then it does so locally uniformly. After that, we also notice, from (3.1.12), that a Bloch function reduces lengths by a fixed factor from the hyperbolic metric on $\mathbb{B}_n$ to the Euclidean metric on $\mathbb{C}$.
Moreover, we've proved one of the important properties of Bloch functions: the growth is controlled by

$$\log\left(\frac{1+|z|}{1-|z|}\right).$$

Finally, we will show later that such growth rate is actually achieved by the following functions in $\mathcal{B}$,

$$f_w(z) = \log\left(\frac{1+ < z, w >}{1- < z, w >}\right), \ z \in \mathbb{B}_n,$$

where w is any point from $\mathbb{S}_n$.

As a further consequence of Theorem 3.1.14, we provide a characterisation of the Bloch space in terms of the involutive automorphism.

Corollary 3.1.17. *Assume that $\alpha > -1$, $p > 0$ and $f \in H(\mathbb{B}_n)$. Then $f \in \mathcal{B}$ if and only if there exists a positive constant C such that*

$$\int_{\mathbb{B}_n} |f \circ \varphi_a(z) - f(a)|^p dv_\alpha(z) \leq C, \quad a \in \mathbb{B}_n, \tag{3.1.14}$$

or equivalently,

$$\int_{\mathbb{B}_n} |f(z) - f(a)|^p \frac{(1-|a|^2)^{n+1+\alpha}}{|1- < z, a > |^{2(n+1+\alpha)}} dv_\alpha(z) \leq C, \quad a \in \mathbb{B}_n. \tag{3.1.15}$$

Proof. First of all, from Proposition 1.4.7, we deduce that (3.1.14) and (3.1.15) are equivalent. Then, assume that $f \in \mathcal{B}$. Hence, by Theorem 3.1.14 or equivalently Corollary 3.1.15, there exists a positive constant C such that

$$|f(z) - f(w)| \leq C\beta(z, w)$$

for all $z, w \in \mathbb{B}_n$. So, after some computations, using the invariance of the Bergman metric (that is $\beta(\varphi_a(z), \varphi_a(w)) = \beta(z, w)$) and the most obvious change of variables, the above inequality implies

$$\begin{aligned}
\int_{\mathbb{B}_n} |f\circ\varphi_a(z) - f(a)|^p dv_\alpha(z) &\le C^p \int_{\mathbb{B}_n} \beta(\varphi_a(z), a)^p dv_\alpha(z) \\
&= C^p \int_{\mathbb{B}_n} \beta(z, 0)^p dv_\alpha(z) \\
&\le C^p (2\pi)^n c_\alpha \int_0^1 \left[\log\left(\frac{1+r}{1-r}\right)\right]^p (1-r^2)^\alpha 2r dr \\
&= C^p (2\pi)^n c_\alpha \int_{log(2)}^{\infty} x^p exp(-x(\alpha+1))dx \\
&< \infty,
\end{aligned}$$

for all $a \in \mathbb{B}_n$. So that

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} |f\circ\varphi_a(z) - f(a)|^p dv_\alpha(z) < \infty,$$

and this shows the first implication. Regarding the reverse implication, let any $g \in H(\mathbb{B}_n)$, from Lemma 2.1.6, we can find a positive constant $C > 0$ such that

$$|\nabla g(0)|^p \le C \int_{\mathbb{B}_n} |g(z) - g(0)|^p dv_\alpha(z).$$

After that, we replace g by $f \circ \varphi_a$ to obtain

$$|\widetilde{\nabla} f(a)|^p \le C \int_{\mathbb{B}_n} |f\circ\varphi_a(z) - f(a)|^p dv_\alpha(z)$$

for all $a \in \mathbb{B}_n$, and the wished result follows easily. □

Not only can the Bloch seminorm be defined using the Bergman metric, the following result shows that the Bergman metric can also be recovered from the Bloch seminorm.

Theorem 3.1.18. *We have*

$$\beta(z, w) = \sup\{\ |f(z) - f(w)|\ :\ ||f||_{\mathcal{B}} \le 1\} \tag{3.1.16}$$

for all z and w in $\mathbb{B}_n$.

Proof. Since from Theorem 3.1.14 we have

$$|f(z) - f(w)| \le ||f||_{\mathcal{B}}\beta(z, w),$$

that clearly implies

$$\sup\{\ |f(z) - f(w)|\ :\ ||f||_{\mathcal{B}} \le 1\} \le \beta(z, w),$$

for all z and w in $\mathbb{B}_n$.
To complete this proof, we show that there exists a function such that the inverse direction of the above inequality holds. Assume that $z \ne 0$ and consider the following function in $\mathbb{B}_n$,

$$h(w) = \frac{1}{2}\log\left(\frac{|z| + < w, z >}{|z| - < w, z >}\right),\ w \in \mathbb{B}_n.$$

We start proving that $||h||_{\mathcal{B}} \le 1$. First of all, we notice that

$$h(z) = \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right), \quad h(0) = 0,$$

and

$$\frac{\partial h}{\partial w_k}(w) = \frac{\overline{z}_k|z|}{|z|^2 - < w, z >^2},\ 1 \leq k \leq n.$$

Then, defining $z' = z/|z|$, using Theorem 3.1.5 point e) and the triangle inequality, we obtain

$$\begin{aligned}
|\widetilde{\nabla} h(w)|^2 &= (1-|w|^2)(|\nabla h(w)|^2 - |Rh(w)|^2) \\
&= (1-|w|^2)\frac{|z|^2(|z|^2 - |< w, z >|^2)}{||z|^2 - < w, z >^2|^2} \\
&= (1-|w|^2)\frac{(1 - |< w, z' >|^2)}{|1 - < w, z' >^2|^2} \\
&\leq |1 - < w, z' >^2|\frac{|1 - < w, z' >^2|}{|1 - < w, z' >^2|^2} \\
&= 1,
\end{aligned}$$

for all $w \in \mathbb{B}_n$. Hence,

$$\begin{aligned}
\beta(z,0) &= \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right) \\
&= |h(z) - h(0)| \\
&= \sup\left\{|f(z) - f(0)| \ : \ ||f||_{\mathcal{B}} \leq 1\right\}.
\end{aligned}$$

Finally, using the invariance under $Aut(\mathbb{B}_n)$, the above implies

$$\beta(z,w) \leq \sup\left\{|f(z) - f(w)| \ : \ ||f||_{\mathcal{B}} \leq 1\right\}$$

for all z and w in $\mathbb{B}_n$. We are done. □

We wish to give a higher-dimensional version of the Holland-Walsh characterisation of the Bloch space. For this result, the main reference is Bloch space in the Unit Ball of $\mathbb{C}^n$, written by Guangbin Ren and Caifeng Tu.
In order to prove this characterisation, we need the following estimate concerning harmonic functions. We don't prove this result. But, the interested reader can find all the details on Decompositions of L^p and Hardy Spaces of Polyharmonic Functions, written by Miroslav Pavlovic (Journal of Mathematical Analysis and Applications 216, Article nu. AY975675, 1996).

Lemma 3.1.19. *Let $f \in J := \{f \in C^\infty(\mathbb{B}_n) \ : \ \Delta f = 0\}$ and $0 < p < \infty$. Then, there exists a positive constant $C := C(p,n)$ such that*

$$|f(x)|^p \leq \frac{C}{r^n}\int_{B(x,r)} |f(y)|^p dv_\alpha(y) \tag{3.1.17}$$

and

$$|\nabla f(x)|^p r^p \leq \frac{C}{r^n}\int_{B(x,r)} |f(y)|^p dv_\alpha(y) \tag{3.1.18}$$

Moreover, we need the following facts concerning the Bergman metric ball $D(a,\delta)$ and the involutive automorphisms. Again, the reference for their proof is Bloch space in the Unit Ball of $\mathbb{C}^n$, Lemma 2.1, by Guangbin Ren and Caifeng Tu.

Proposition 3.1.20. *For any $z, w \in \mathbb{B}_n$, with $z \neq w$, we have that*

$$\frac{1-|\varphi_z(w)|^2}{|\varphi_z(w)|^2} = \frac{(1-|z|^2)(1-|w|^2)}{|w-z|^2}. \tag{3.1.19}$$

Furthermore, the following holds

$$B\left(a, \frac{\delta(1-|a|^2)}{2}\right) \subset D(a,\delta) \tag{3.1.20}$$

and, defining the measure $d\tau(w) := (1-|w|^2)^{-n}dw$ on $\mathbb{B}_n$, we have

$$\tau(D(a,\delta)) = \tau(B(0,\delta)) = n\int_0^\delta t^{n-1}(1-t^2)^{-n}dt. \tag{3.1.21}$$

We are ready to prove the following theorem.

Theorem 3.1.21. *Let $f \in H(\mathbb{B}_n)$. Then, $f \in \mathcal{B}$ if and only if*

$$S(f) := \sup_{z,w\in\mathbb{B}_n, z\neq w} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f(z)-f(w)|}{|z-w|} < \infty. \tag{3.1.22}$$

Furthermore, the seminorms $\sup_{z\in\mathbb{B}_n}(1-|z|^2)|\nabla f(z)|$ and $S(f)$ are equivalent.

Proof. We start assuming $f \in \mathcal{B}$. Then, for any $z, w \in \mathbb{B}_n$, applying Cauchy-Schwarz, the convexity of $\mathbb{B}_n$ and using Lemma 2.4.4, we have

$$\begin{aligned}
|f(z)-f(w)| &= \left|\int_0^1 \frac{df}{dt}(tz+(1-t)w)dt\right| \\
&= \left|\sum_{k=1}^n (z_k-w_k)\int_0^1 \frac{\partial f}{\partial z_k}(tz+(1-t)w)dt\right| \\
&\leq \sqrt{\sum_{k=1}^n |z_k-w_k|^2}\sqrt{\sum_{k=1}^n\left(\int_0^1\left|\frac{\partial f}{\partial z_k}(tz+(1-t)w)\right|dt\right)^2} \\
&\leq |z-w|\sqrt{n}\int_0^1 |(\nabla f)(tz+(1-t)w)|dt \\
&\leq |z-w|\sqrt{n}||f||_{\mathcal{B}}\int_0^1 \frac{dt}{1-|tz+(1-t)w|^2} \\
&\leq |z-w|\sqrt{n}||f||_{\mathcal{B}}\int_0^1 \frac{dt}{1-|tz+(1-t)w|} \\
&\leq |z-w|\sqrt{n}||f||_{\mathcal{B}}\int_0^1 \frac{dt}{\sqrt{(1-t)(1-|w|)}\sqrt{t(1-|z|)}} \\
&= \frac{|z-w|\sqrt{n}||f||_{\mathcal{B}}\pi}{(1-|w|)^{1/2}(1-|z|)^{1/2}},
\end{aligned}$$

that implies

$$(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f(z)-f(w)|}{|z-w|} \leq \pi\sqrt{n}||f||_{\mathcal{B}}. \tag{3.1.23}$$

This proves the necessity.
Conversely, assume that $f \in H(\mathbb{B}_n)$ satisfies (3.1.22). We show that $f \in \mathcal{B}$ as follows. Fix $\delta \in (0,1)$ and, since f is harmonic, we can apply Lemma 3.1.19, so that there exists a positive constant such that

$$(1-|z|^2)|\nabla f(z)| \leq C \int_{B(z,\delta(1-|z|^2)/2)} |f(w)| d\tau(w),$$

for every $z \in \mathbb{B}_n$. Combining this result with (3.1.20), we have

$$(1-|z|^2)|\nabla f(z)| \leq C \int_{D(z,\delta)} |f(w)| d\tau(w), \ \forall z \in \mathbb{B}_n.$$

Hence, fixing $z \in \mathbb{B}_n$, replacing f by $f - f(z)$ and applying (3.1.21), it turns out that

$$\begin{aligned}(1-|z|^2)|\nabla f(z)| &\leq C \int_{D(z,\delta)} |f(w) - f(z)| d\tau(w)\\ &\leq \sup_{w \in D(z,\delta)} |f(w) - f(z)| \tau(D(z,\delta))\\ &\leq \sup_{w \in D(z,\delta), w \neq z} |f(w) - f(z)|.\end{aligned}$$

But, since for every $w \in D(z,\delta)$ we have $|\varphi_z(w)| \leq \delta$. This fact implies

$$\frac{\sqrt{1-\delta^2}}{\delta} \leq \frac{\sqrt{1-|\varphi_z(w)|^2}}{|\varphi_z(w)|}.$$

Hence, from (3.1.19), it follows that

$$\frac{\sqrt{1-\delta^2}}{\delta} \leq \frac{\sqrt{(1-|z|^2)}\sqrt{(1-|w|^2)}}{|w-z|}, \ \forall \ w \in D(z,\delta).$$

From,

$$(1-|z|^2)|\nabla f(z)| \leq C \sup_{w \in D(z,\delta), z \neq w} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2} \frac{|f(z)-f(w)|}{|z-w|}, \tag{3.1.24}$$

we easily obtain that

$$\begin{aligned}\sup_{z \in \mathbb{B}_n} (1-|z|^2)|\nabla f(z)| &\leq C \sup_{z \in \mathbb{B}_n} \sup_{w \in D(z,\delta), z \neq w} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2} \frac{|f(z)-f(w)|}{|z-w|}\\ &\leq C \sup_{z,w \in \mathbb{B}_n, z \neq w} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2} \frac{|f(z)-f(w)|}{|z-w|}.\end{aligned}$$

This fact implies that $f \in \mathcal{B}$. Finally, we notice that the two seminorms of the Bloch space $\mathcal{B}$, $\sup_{z \in \mathbb{B}_n}(1-|z|^2)|\nabla f(z)|$ and $S(f)$, are equivalent. This completes the proof of Theorem 3.1.21. □

Remark 3.1.22. In the former proof, we used that

$$1 - |tz + (1-t)w| \geq \sqrt{(1-t)(1-|w|)}\sqrt{t(1-|z|)}. \tag{3.1.25}$$

In fact, using the triangle inequality, we have

$$|tz + (1-t)w| \leq t|z| + (1-t)|w|,$$

so that,

$$\begin{aligned}1 - |tz + (1-t)w| &\geq 1 - t|z| - (1-t)|w|\\ &= \underbrace{(1-t)(1-|w|)}_{>0} + \underbrace{t(1-|z|)}_{>0}.\end{aligned}$$

Thus, for any $0 < t < 1$ and $z, w \in \mathbb{B}_n$, it turns out that

$$1 - |tz + (1-t)w| \geq (1-t)(1-|w|) \quad \text{and} \quad 1 - |tz + (1-t)w| \geq t(1-|z|),$$

that clearly implies

$$1 - |tz + (1-t)w| \geq \sqrt{(1-t)(1-|w|)}\sqrt{t(1-|z|)}.$$

So, (3.1.25) is proved.

The next results are related to the relation between the Bloch space and the Bergman spaces A^p_α, for $0 < p \leq \infty$. In the following proposition, we start showing that any bounded holomorphic function on $\mathbb{B}_n$ is in the Bloch space.

Proposition 3.1.23. *Let $\alpha > -1$, then $A^\infty_\alpha \subset \mathcal{B}$. Moreover, the following inequality holds*

$$||f||_{\mathcal{B}} \leq ||f||_{\infty,\alpha}, \ \forall \ f \in A^\infty_\alpha.$$

In order to prove this result, we need a generalisation of the Schwarz-Pick Lemma, in several variables, for holomorphic functions defined on $\mathbb{B}_n$. This is the content of the next proposition.

Proposition 3.1.24. *Let $f \in H(\mathbb{B}_n)$ such that $|f(z)| \leq 1$, for all $z \in \mathbb{B}_n$. Then*

$$\sum_{j=0}^{n} (1-|z_j|^2) \left| \frac{\partial f}{\partial z_j}(z) \right| \leq 1 - |f(z)|^2, \tag{3.1.26}$$

for any $z = (z_1, ..., z_n) \in \mathbb{B}_n$.

Proof. We write $z = (z_1, z_2, ..., z_j, ..., z_n), w_j = (z_1, z_2, ..., z_j + h_j, ..., z_n) \in \mathbb{B}_n$, for some $\{h_j\}_j \in \mathbb{D}$. So that taking the limit $h_j \to 0$, for $j = 1, ..., n$, and applying the Schwarz-Pick Lemma to

$$\sum_{j=1}^{n} \left| \frac{[f(z) - f(w_j)][1 - \overline{z_j}(z_j + h_j)]}{(1 - f(z)\overline{f(w_j)})h_j} \right| \leq 1,$$

we obtain

$$\sum_{j=1}^{n} \frac{(1-|z_j|^2)}{|1-|f(z)|^2|} \left| \frac{\partial f}{\partial z_j}(z) \right| \leq 1,$$

and the wished result follows easily. ∎

We are ready to prove Proposition 3.1.23.

Proof. We assume, without loss of generality, that $||f||_{\infty,\alpha} = 1$. So, putting $z = (z_1, 0, ..., 0) \in \mathbb{B}_n$ and considering f as function of z_1, we apply the Schwarz-Pick Lemma to

$$\left| \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)} \right| \leq |z_1|,$$

we get

$$\left| \frac{\partial f}{\partial z_1}(0) \right| \leq 1 - |f(0)|^2$$

$$\leq 1.$$

Similarly,

$$\left| \frac{\partial f}{\partial z_j}(0) \right| \leq 1, \ j = 1, ..., n.$$

Then, put $0 = (0, 0, ..., 0, 0) \in \mathbb{B}_n$ in (3.1.26), we easily obtain

$$|\nabla f(0)|^2 = \sum_{j=0}^{n} \left| \frac{\partial f}{\partial z_j}(0) \right|^2$$

$$\leq \sum_{j=0}^{n} \left| \frac{\partial f}{\partial z_j}(0) \right|$$

$$\leq 1.$$

So that, replacing f by $f \circ \varphi_z$, it turns out

$$\sup_{z \in \mathbb{B}_n} |\widetilde{\nabla} f(z)| \leq ||f||_{\infty,\alpha}.$$

The proof is completed. □

Remark 3.1.25. The containment $A^{\infty}_{\alpha} \subset \mathcal{B}$ is proper. In fact, the function $f(z) = Log(1 - z)$, where Log denotes the principal branch of the logarithm, is an example of a function of the Bloch space $\mathcal{B}$ that is not bounded. We prove this fact, in the one-dimensional case, as follows. A simple computation shows that

$$||f||_{\alpha,\infty} = \left| \sum_{n=1}^{\infty} \frac{1}{n} \right|$$

$$= \infty.$$

Then, using point $b)$ of Theorem 3.1.12, we find

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)|f'(z)| \leq \sup_{z \in \mathbb{B}_n} (1 - |z|^2) \frac{1}{(1 - |z|)}$$

$$= \sup_{z \in \mathbb{B}_n} (1 + |z|)$$

$$= 2.$$

A further consequence of the characterisation of the seminorm $|| \cdot ||_{\mathcal{B}}$ in terms of the Bergman metric is that the Bloch space belongs to every Bergman space A^p_{α}, for $0 < p < \infty$. This is proved in the next corollary.

Corollary 3.1.26. *The Bloch space $\mathcal{B}$ satisfies*

$$\mathcal{B} \subset A^p_{\alpha}, \tag{3.1.27}$$

for $0 < p < \infty$ and $\alpha > -1$.

Proof. Using (3.1.12), with $w = 0$, and after some computations, we find

$$\begin{aligned}
\int_{\mathbb{B}_n} |f(z)|^p c_\alpha (1-|z|^2)^\alpha dv(z) &\leq \int_{\mathbb{B}_n} |f(z) - f(0)|^p (1-|z|^2)^\alpha c_\alpha dv(z) + |f(0)|^p \\
&\leq c_\alpha \int_{\mathbb{S}_n} d\sigma \int_0^1 |f(z) - f(0)|^p (1-r^2)^\alpha 2nr^{2n-1} dr + |f(0)|^p \\
&\leq c_\alpha Cn ||f||_{\mathcal{B}}^p \int_0^1 \left[\log\left(\frac{1+r}{1-r}\right)\right]^p (1-r^2)^\alpha 2r dr + |f(0)|^p \\
&\leq c_\alpha 2 \max\{1, 2^\alpha\} Cn ||f||_{\mathcal{B}}^p \int_0^1 \left[\log\left(\frac{1+r}{1-r}\right)\right]^p (1-r)^\alpha dr + |f(0)|^p \\
&\leq 2c_\alpha \max\{1, 2^\alpha\} Cn ||f||_{\mathcal{B}}^p \int_0^1 \left[\log\left(\frac{2}{1-r}\right)\right]^p (1-r)^\alpha dr + |f(0)|^p \\
&= 2^{\alpha+2} c_\alpha \max\{1, 2^\alpha\} Cn ||f||_{\mathcal{B}}^p \int_{log(2)}^\infty x^p exp(-x(\alpha+1)) dx + |f(0)|^p \\
&< \infty,
\end{aligned}$$

where, in the last step, the change of variables is given by

$$x = \log\left(\frac{2}{1-r}\right).$$

□

Remark 3.1.27. We remark that the containment of (3.1.27) is proper. For example the function

$$f(z) = (Log(1-z))^2$$

is not a member of the Bloch space, while it is in A^p_α, for $0 < p < \infty$. This fact is proved as follows:

$$\begin{aligned}
\sup_{z\in\mathbb{B}_n} (1-|z|^2)|\nabla f(z)| &= 2 \sup_{z\in\mathbb{B}_n} (1-|z|^2) \frac{|Log(1-z)||z|}{|1-z|} \\
&= 2 \sup_{z\in\mathbb{B}_n} (1+|z|)|Log(1-z)||z| \\
&= 4 \sup_{z\in\mathbb{B}_n} |Log(1-z)| \\
&= +\infty.
\end{aligned}$$

This shows that $f \notin \mathcal{B}$. After that,

$$\begin{aligned}
\int_{\mathbb{B}_n} |Log(1-z)|^{2p} (1-|z|^2)^\alpha c_\alpha dv(z) &\leq \int_{\mathbb{B}_n} (\log(|1+|z|))^{2p} (1-|z|^2)^\alpha c_\alpha dv(z) \\
&\leq \log(2).
\end{aligned}$$

We conclude that $f \in A^p_\alpha$, for $0 < p < +\infty$.

In the next result we prove a fundamental and curious property of the Bloch space $\mathcal{B}$.

Proposition 3.1.28. *The Bloch space $\mathcal{B}$ is not separable.*

Proof. Fix $w = rexp(it) = r_1exp(it_1)...r_nexp(it_n) \in \mathbb{C}$, such that $\sum_{j=1}^n r_j^2 = 1$, we start considering the following set of holomorphic functions,

$$E := \left\{ f_t(z) = \frac{\overline{r(exp(it)}}{2} \log\left(\frac{1 + \sum_{j=1}^n z_j r_j exp(-it_j)}{1 - \sum_{j=1}^n z_j r_j exp(-it_j)} \right) \; : \; t \in [0, 2\pi)^n \right\}.$$

Clearly, E is uncountable. Moreover,

$$\frac{\partial f_t}{\partial z_j}(z) = \frac{rexp(-it)\ r_j exp(-it_j)}{\left(1 - \left(\sum_{j=1}^n z_j r_j exp(-it_j)\right)^2\right)}$$

so that

$$|\nabla f_t(z)|^2 = \frac{1}{\left|1 - \left(\sum_{j=1}^n z_j r_j exp(-it_j)\right)^2\right|^2} \quad \text{and} \quad |Rf_t(z)|^2 = \frac{|\sum_{j=1}^n z_j r_j exp(-it_j)|^2}{\left|1 - \left(\sum_{j=1}^n z_j r_j exp(-it_j)\right)^2\right|^2}.$$

Then, following the same lines as in the proof of Theorem 3.1.21, it turns out that

$$||f_t||_{\mathcal{B}} \leq 1.$$

This fact shows that $f_t \in \mathcal{B}, \forall\ t \in [0, 2\pi)^n$ and, in particular,

$$E \subset \mathcal{B}.$$

Now we demonstrate that the elements of E are always at a distance greater than ε , for some $\varepsilon > 0$. For this goal, let $t, s \in [0, 2\pi)^n$ such that $t \neq s$. Denoting by $z = |z|exp(it) \in \mathbb{B}_n$, we have

$$|\widetilde{\nabla}(f_t - f_s)(z)|^2 = (1 - |z|^2)(|\nabla(f_t - f_s)(z)|^2 - |R(f_t - f_s)(z)|^2)$$

$$= (1 - |z|^2) \left(\frac{\sum_{j=1}^n r_j^2 |exp(-it_j) - exp(-is_j)|^2}{\left|1 - \left(\sum_{j=1}^n z_j r_j exp(-it_j)\right)^2\right|^2} - \frac{|\sum_{j=1}^n z_j r_j (exp(-it_j) - exp(-is_j))|^2}{\left|1 - \left(\sum_{j=1}^n z_j r_j exp(-it_j)\right)^2\right|^2} \right)$$

$$= (1 - |z|^2) \left(\frac{2 - 2\sum_{j=1}^n r_j^2 Re(exp(i(t_j - s_j)))}{\left|1 - |z|^2 \left(\sum_{j=1}^n r_j\right)^2\right|^2} - \frac{|z|^2 |\sum_{j=1}^n r_j (1 - exp(i(t_j - s_j)))|^2}{\left|1 - |z|^2 \left(\sum_{j=1}^n r_j\right)^2\right|^2} \right)$$

to obtain

$$\sup_{z \in \mathbb{B}_n} |\widetilde{\nabla}(f_t - f_s)(z)|^2 = \sup_{z \in \mathbb{B}_n} (1 - |z|^2) \frac{2 - 2\sum_{j=1}^n r_j^2 Re(exp(i(t_j - s_j)))}{\left|1 - |z|^2 \left(\sum_{j=1}^n r_j\right)^2\right|^2}$$

$$\geq \sup_{z \in \mathbb{B}_n} \left(2 - 2\sum_{j=1}^n r_j^2 Re(exp(i(t_j - s_j))) \right)$$

$$= 2.$$

that clearly implies

$$\sup_{z\in\mathbb{B}_n} |\widetilde{\nabla}(f_t - f_s)(z)|^2 \geq 2.$$

In particular, the open balls $B(f_t, \frac{1}{2})$ are disjoint and uncountable in number. Now assume there is any dense subset of $\mathcal{B}$, say S, then for all $f_t \in E$, there exists $x \in S$ such that $||x - f_t||_{\mathcal{B}} < \frac{1}{2}$ which implies that $x \in B(f_t, \frac{1}{2})$. Hence, if we allow elements of S to these open balls, it follows that as the set S is uncountable. So, any dense subset in $\mathcal{B}$ cannot be countable. In other words, $\mathcal{B}$ is not separable. □

Concerning topological properties of the Bloch space, in the following proposition we prove the lack of strictly convexity.

Proposition 3.1.29. *The Bloch space $\mathcal{B}$, equipped with the norm $||f|| = |f(0)| + ||f||_{\mathcal{B}}$, is not strictly convex.*

Proof. We prove that

$$||f + g|| = ||f|| + ||g||, \ f \neq 0, \ g \neq 0,$$

doesn't implie $f = cg, c > 0$. Assume, without loss of generality, that $f(0) = 0$ and choose
$g(z) = \lambda, \ \lambda \in \mathbb{C}$, we obtain

$$||f + g|| = |\lambda| + \sup_{z\in\mathbb{B}_n} (1 - |z|^2)(|\nabla f(z)|^2 - |R(f)(z)|^2).$$

By the other hand,

$$||f|| + ||g|| = |\lambda| + \sup_{z\in\mathbb{B}_n} (1 - |z|^2)(|\nabla f(z)|^2 - |R(f)(z)|^2),$$

and this completes the proof. □

We conclude this section with the following remark. We prove that, if $f, g \in \mathcal{B}$, then, in general, it is not true that $f \circ g \in \mathcal{B}$. This fact, for sake of simplicity, is proved in the one-dimensional case.

Remark 3.1.30. Let $g(z) = Log(1 - z)$ and $f(z) = z^2$. Then, as proved in Remark 3.1.25, we have that

$$||g||_{\mathcal{B}} \leq 2 \quad \text{and} \quad ||f||_{\mathcal{B}} = 2.$$

That is,

$$f, g \in \mathcal{B}.$$

But, as proved in Remark 3.1.27, we have that

$$\begin{aligned} ||f \circ g(z)||_{\mathcal{B}} &= ||(Log(1 - z))^2||_{\mathcal{B}} \\ &= 4 \sup_{z\in\mathbb{B}_n} |Log(1 - z)| \\ &= \infty. \end{aligned}$$

That is, $f \circ g \notin \mathcal{B}$.

3.2 The Little Bloch Space $\mathcal{B}_0$

We proved in the previous section that the Bloch space is not separable. In this section we discuss a separable subspace of the Bloch space: the little Bloch space. This section is organised as follows. We start giving its formal definition. Then, we collect some fundamental properties that follow from the fact that the little Bloch space is a closed subspace of the Bloch space: completeness, boundedness of point evaluation and boundary behaviour. We show a peculiar property of the little Bloch space:

density of polynomials. We study the connection with other functional spaces such as $A(\overline{\mathbb{B}_n})$, the family of Bergman spaces A^p_α, for $0 < p \leq \infty$, and, of course, the Bloch space itself. We give the little Bloch version of many characterisations that have been proved for the Bloch space: in terms of the holomorphic gradient, radial derivative, image of the Bergman projection of $C_0(\mathbb{B}_n)$, higher order derivatives, fractional derivatives, on both $C_0(\mathbb{B}_n)$ and $C(\overline{\mathbb{B}_n})$, in terms of the bergman metric and, conversely, we show that the Bergman metric can be recovered from the Bloch seminorm. Finally, as a further characterisation, we provide the little Bloch version of Theorem 3.1.21 that was proved by Guangbin Ren and Caifeng Tu in 1996.

Definition 3.2.1. The little Bloch space, denoted by $\mathcal{B}_0$, is defined as

$$\mathcal{B}_0 := \left\{ f \in \mathcal{B} \ \mid \ \lim_{|z|\to 1^-} |\widetilde{\nabla} f(z)| = 0 \right\} \tag{3.2.1}$$

Moreover, $\mathcal{B}_0$ is equipped with the Bloch seminorm $||\cdot||_{\mathcal{B}}$.

Remark 3.2.2. Since $|\widetilde{\nabla} f(z)|$ is continuous in $\mathbb{B}_n$, (3.2.1) implies that

$$|\widetilde{\nabla} f(z)| \in C_0(\mathbb{B}_n).$$

More is true, as a consequence of point $e)$ in Theorem 3.1.5, we have

Corollary 3.2.3. *Suppose f is holomorphic in a neighborhood of $\overline{\mathbb{B}}_n$. Then*

$$f \in \mathcal{B}_0. \tag{3.2.2}$$

Proof. Basically, the fact that f is holomorphic in a neighborhood of $\overline{\mathbb{B}}_n$ guarantees that

$$\lim_{|z|\to 1^-} \left(|\nabla f(z)|^2 - |Rf(z)|^2 \right) < \infty,$$

so that

$$\begin{aligned} \lim_{|z|\to 1^-} |\widetilde{\nabla} f(z)| &= \lim_{|z|\to 1^-} \left((1-|z|^2) \left| |\nabla f(z)|^2 - |Rf(z)|^2 \right| \right)^{1/2} \\ &= 0. \end{aligned}$$

□

Proposition 3.2.4. *$\mathcal{B}_0$ is a closed subspace of $\mathcal{B}$. Furthermore, the set of polynomials is dense in $\mathcal{B}_0$ with respect to the Bloch seminorm.*

Proof. Let $f_n \in \mathcal{B}_0$ such that

$$\lim_{|z|\to 1^-} |\widetilde{\nabla}(f_n - f)(z)| = 0,$$

then, from the completeness of $\mathcal{B}$, we easily deduce that $f \in \mathcal{B}$. Then,

$$\begin{aligned} \lim_{|z|\to 1^-} |\widetilde{\nabla} f(z)| &\leq \lim_{|z|\to 1^-} |\widetilde{\nabla}(f - f_n)(z)| + \lim_{|z|\to 1^-} |\widetilde{\nabla} f_n(z)| \\ &= 0. \end{aligned}$$

In other words, $f \in \mathcal{B}_0$. After that, let $f \in \mathcal{B}_0$ and $f_r(z) := f(rz)$, where $r \in [0,1)$, the dilation function of f. Hence, we easily obtain $f, f_r \in \mathcal{B}$ and, clearly, $(f - f_r)(z) \in \mathcal{B}$. So, using point $e)$ of Theorem 3.1.5 (or equivalently Lemma 2.4.1), it turns out that

$$\begin{aligned} \lim_{r\to 1^-} ||f - f_r||_{\mathcal{B}} &= \lim_{r\to 1^-} \sup_{z\in\mathbb{B}_n} \left((1-|z|^2) \left(|\nabla(f-f_r)(z)|^2 - |R(f-f_r)|^2 \right) \right)^{1/2} \\ &= \sup_{z\in\mathbb{B}_n} \lim_{r\to 1^-} \left((1-|z|^2) \left(|\nabla(f-f_r)(z)|^2 - |R(f-f_r)|^2 \right) \right)^{1/2} \\ &= 0. \end{aligned}$$

Now, in Proposition 2.1.8 we proved that each f_r can be uniformly approximated by polynomials and using the fact that the sup-norm in $\mathbb{B}_n$ dominates the Bloch seminorm, we get a sequence of polynomials f_r^N, choosing N large enough, such that

$$\begin{aligned}\lim_{r\to 1^-} \|f - f_r^N\|_{\mathcal{B}} &\le \lim_{r\to 1^-} \|f - f_r\|_{\mathcal{B}} + \lim_{r\to 1^-} \|f_r - f_r^N\|_{\mathcal{B}} \\ &\le \left(\lim_{r\to 1^-} \|f - f_r\|_{\alpha,\infty} + \lim_{r\to 1^-} \|f_r - f_r^N\|_{\alpha,\infty} \right) \\ &< \varepsilon.\end{aligned}$$

By the arbytrariness of ε, the desired result follows. □

Remark 3.2.5. In other words, the little Bloch space is the closure of the polynomials with respect to the Bloch seminorm. Hence, since $\mathcal{B}_0$ is a closed subspace of $\mathcal{B}$, $\mathcal{B}_0$, endowed with $\|\cdot\|_{\mathcal{B}}$, is a complete space. So, the little Bloch space is invariant under the action of the group of automorphisms of $\mathbb{B}_n$. Moreover, the little Bloch space, endowed with the norm $\|\cdot\|$, inherits, from the Bloch space, the following properties: local convexity and local boundedness.
However, we remark that, in spite of the polynomials density, there exist functions f in $\mathcal{B}_0$ such that f cannot be approximated by their Taylor polynomials in the seminorm topology of $\mathcal{B}$. This fact will be proved when we talk about duality. That is, Section 3.4.

The following result is the Little Bloch version of Theorem 3.1.10: we provide a characterisation of the Little Bloch space in terms of the holomorphic gradient, the radial derivative and as the image of the space $C_0(\mathbb{B}_n)$ under the Bergman projection.

Theorem 3.2.6. *Assume that $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. Then, the following conditions hold*
a) $f \in \mathcal{B}_0$.
b) $(1-|z|^2)|\nabla f(z)|$ belongs to $C_0(\mathbb{B}_n)$.
c) $(1-|z|^2)|Rf(z)|$ is bounded in $C_0(\mathbb{B}_n)$.
d) There exists $g \in C_0(\mathbb{B}_n)$ such that $f = P_\alpha g$.

Proof. Again, from Lemma 2.4.4, we easily obtain that *a)* implies *b)* and *b)* implies *c)* as well. To prove that *c)* implies *d)*, following the same lines as in the proof of Theorem 3.1.10, we consider the function

$$g(z) = \frac{c_{\alpha+1}}{c_\alpha}\left[(1-|z|^2)f(z) + \frac{(1-|z|^2)Rf(z)}{n+1+\alpha}\right]$$

or equivalently

$$g(z) = \frac{c_{\alpha+1}}{c_\alpha}(1-|z|^2)\int_{\mathbb{B}_n} \frac{f(w)dv_\alpha(w)}{(1-<z,w>)^{n+2+\alpha}}, \ z \in \mathbb{B}_n,$$

where $f \in \mathcal{B}$ and, as proved, $f = P_\alpha g$. Then, since every function of $\mathcal{B}$ grows at most logarithmically near $\mathbb{S}_n$, we have that

$$\begin{aligned}\lim_{|z|\to 1^-} |g(z)| &\le \frac{c_{\alpha+1}}{c_\alpha}\left[\lim_{|z|\to 1^-} (1-|z|^2)|f(z)| + \lim_{|z|\to 1^-} \frac{(1-|z|^2)|Rf(z)|}{n+1+\alpha}\right] \\ &\le \frac{c_{\alpha+1}}{c_\alpha}\left[\lim_{|z|\to 1^-}(1-|z|^2)\|f\|_{\mathcal{B}}\log\left(\frac{1+|z|}{1-|z|}\right) + \lim_{|z|\to 1^-}|f(0)|(1-|z|^2)\right] \\ &= 0.\end{aligned}$$

That is, $g \in C_0(\mathbb{B}_n)$.
Finally, we wish to prove that *d)* implies *a)*. If *d)* holds, there exists $g \in C_0(\mathbb{B}_n) \subset C(\overline{\mathbb{B}}_n)$. Hence, by the Stone-Weierstrass approximation theorem, we can approximate g uniformly on $\mathbb{B}_n$ by a finite

linear combination of functions of the form $h(z) = z^m \overline{z}^{m'}$. We compute the Bergman projection of $h(z)$ as follows. If $m > m'$, we have

$$\begin{aligned}
P_\alpha h(z) &= \int_{\mathbb{B}_n} \frac{w^m \overline{w}^{m'} dv_\alpha(w)}{(1- < z, w >)^{n+1+\alpha}} \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(|k| + n + \alpha + 1)}{k!\Gamma(n + 1 + \alpha)} z^k \int_{\mathbb{B}_n} w^m \overline{w}^{m'+k} dv_\alpha(w) \\
&= z^{m-m'} \frac{\Gamma(|m - m'| + n + \alpha + 1)}{(m - m')!\Gamma(n + 1 + \alpha)} \frac{m!\Gamma(n + \alpha + 1)}{\Gamma(n + |m| + \alpha + 1)} \\
&= \frac{m - m' + 1}{m + 1} z^{m-m'}.
\end{aligned}$$

Otherwise, if $m > m'$, it turns out

$$P_\alpha h(z) = 0.$$

Hence, $P_\alpha h$ is a holomorphic polynomial. So, we get

$$P_\alpha h \in \mathcal{B}_0,$$

and, by the fact that P_α maps $L^\infty(\mathbb{B}_n)$ boundedly into the Bloch space and the little Bloch space is closed in $\mathcal{B}$, we obtain

$$f = P_\alpha g \in \mathcal{B}_0.$$

□

Remark 3.2.7. A straightforward consequence of the Proposition 3.2.4 and Remark 3.1.11 is that

$$\lim_{|z| \to 1^-} f(z)(1 - |z|^2)^{(n+1+\alpha)/2} = 0, \ \forall f \in \mathcal{B}_0. \tag{3.2.3}$$

In the study of the little Bloch space, the space $C_0(\mathbb{B}_n)$ can be replaced by the space $C(\overline{\mathbb{B}}_n)$. This is proved in the following theorem.

Theorem 3.2.8. *Assume that $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. Then, the following conditions hold*
a) $f \in \mathcal{B}_0$.
b) $|\widetilde{\nabla} f(z)|$ belongs to $C(\overline{\mathbb{B}}_n)$.
c) $(1 - |z|^2)|\nabla f(z)|$ belongs to $C(\overline{\mathbb{B}}_n)$.
d) $(1 - |z|^2)|Rf(z)|$ is bounded in $C(\overline{\mathbb{B}}_n)$.
e) There exists $g \in C(\overline{\mathbb{B}}_n))$ such that $f = P_\alpha g$.

Proof. It's clear that *a*) implies *b*). From Lemma 2.4.4, we easily obtain that *b*) implies *c*) and *c*) implies *d*). Following the same construction used in the proof of Theorem 3.2.6 and Theorem 3.1.10, we obtain that *d*) implies *e*). Finally, to prove that *e*) implies *a*), we follow the same lines as in the proof of Theorem 3.2.6. This completes our proof. □

Remark 3.2.9. We recall that

$$A(\mathbb{B}_n) := C\left(\overline{\mathbb{B}}_n\right) \cap H(\mathbb{B}_n),$$

and, hence, the previous theorem implies that the following inclusion holds

$$A(\mathbb{B}_n) \subset \mathcal{B}_0.$$

Remark 3.2.10. The containment $\mathcal{B}_0 \subset \mathcal{B}$ is proper. To prove this fact, we show that, for every point $w \in \mathbb{S}_n$, the function $f(z) = \log(1- < z, w >)$ belongs to the Bloch space, but not to the little Bloch space. Since

$$\frac{\partial f}{\partial z_j}(z) = \frac{-\overline{w}_j}{(1- < z, w >)},$$

it turns out that

$$\begin{aligned}
[(1-|z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2)]^{1/2} &= \left[(1-|z|^2)\left(\frac{1}{|1- < z, w > |^2} - \frac{| < z, w > |^2}{|1- < z, w > |^2}\right)\right]^{1/2} \\
&= \left[(1-|z|^2)\left(\frac{1 - | < z, w > |^2}{|1- < z, w > |^2}\right)\right]^{1/2} \\
&\leq \left[(1-|z|^2)\left(\frac{1 + | < z, w > |}{1 - | < z, w > |}\right)\right]^{1/2}.
\end{aligned}$$

That implies,

$$\sup_{z \in \mathbb{B}_n} [(1-|z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2)]^{1/2} \leq 2.$$

That is, $f \in \mathcal{B}$.
By the other hand, choosing z and w such that they are linear dipendent, namely $z = rw$ for some $r \in (0,1)$. We obtain

$$\begin{aligned}
\lim_{|z| \to 1^-} [(1-|z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2)]^{1/2} &= \lim_{|z| \to 1^-} \left[(1-|z|^2)\left(\frac{1}{|1- < z, w > |^2} - \frac{| < z, w > |^2}{|1- < z, w > |^2}\right)\right]^{1/2} \\
&= \lim_{|z| \to 1^-} \left[(1-|z|^2)\left(\frac{1 - | < z, w > |^2}{|1- < z, w > |^2}\right)\right]^{1/2} \\
&= \lim_{r \to 1^-} \left[(1-r^2)\left(\frac{1-r^2}{(1-r)^2}\right)\right]^{1/2} \\
&= \lim_{r \to 1^-} (1+r) \\
&= 2.
\end{aligned}$$

This means that $f \notin \mathcal{B}_0$.
We conclude this remark observing that the above process has another interesting consequence. In fact, we prove that if $f, g \in \mathcal{B}_0$, then it doesn't implie that $f \circ g \in \mathcal{B}_0$. For sake of simplicity, we prove this fact in the bidemensional case. Fix $w \in \mathbb{S}_n$ and define the following two functions

$$f(z) = \log(z+2) \quad \text{and} \quad g(z) = -1- < z, w > .$$

Hence,

$$\begin{aligned}
\lim_{|z| \to 1^-} (1-|z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2) &= \lim_{|z| \to 1^-} (1-|z|^2)\left(\frac{|z|^2}{|z+2|^2} - \frac{2|z_1 z_2|}{|z+2|^2}\right) \\
&\leq \lim_{|z| \to 1^-} (1-|z|^2)\left(\frac{1}{(2-|z|)^2}\right) \\
&= 0.
\end{aligned}$$

Similarly,

$$\lim_{|z|\to 1^-}(1-|z|^2)(|\nabla g(z)|^2-|Rg(z)|^2) = \lim_{|z|\to 1^-}(1-|z|^2)\left(|w|^2-|<z,w>|^2\right)$$

$$\leq \lim_{|z|\to 1^-}(1-|z|^2)^2|w|^2$$

$$= 0.$$

That is $f, g \in \mathcal{B}_0$. But, the composition is

$$(f \circ g)(z) = \log(1-<z,w>) \notin \mathcal{B}_0.$$

In the following theorem, we provide a characterisation of the Bloch space in terms of higher order derivatives and fractional derivatives. In some sense, this is the analog version of Theorem 3.1.12 for the space $\mathcal{B}_0$.

Theorem 3.2.11. *Assume that N is a positive integer, $t > 0$ and $f \in H(\mathbb{B}_n)$. If α is a real parameter such that neither $n+\alpha$ nor $n+\alpha+t$ is a negative integer. Then, the following conditions are equivalent:*
1) $f \in \mathcal{B}_0$.
2) *The function $(1-|z|^2)R^{\alpha,t}f(z)$ is in $C_0(\mathbb{B}_n)$.*
3) *The function $(1-|z|^2)R^{\alpha,t}f(z)$ is in $C(\overline{\mathbb{B}}_n)$.*
4) *The family of functions*

$$(1-|z|^2)^N\frac{\partial^m f}{\partial z^m}(z) \in C_0(\mathbb{B}_n),$$

for every multi-index m such that $|m| = N$.
5) *The family of functions*

$$(1-|z|^2)^N\frac{\partial^m f}{\partial z^m}(z) \in C(\overline{\mathbb{B}}_n),$$

for every multi-index m such that $|m| = N$.

Proof. Essentially, this proof follows the same lines as in the proof of Theorem 3.1.12. □

The next result is the little Bloch version of Theorem 3.1.21. The main reference is Bloch space in the Unit Ball of $\mathbb{C}^n$, written by Guangbin Ren and Caifeng Tu.

Theorem 3.2.12. *Let $f \in H(\mathbb{B}_n)$. Then, $f \in \mathcal{B}_0$ if and only if*

$$\lim_{|z|\to 1^-}\sup_{w\in\mathbb{B}_n, z\neq w}(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f(z)-f(w)|}{|z-w|} = 0. \tag{3.2.4}$$

Proof. We start assuming that $f \in \mathcal{B}_0$. Let $f_r(z) := f(rz)$, $r \in (0,1)$. We apply (3.1.23) to obtain the following first estimate,

$$(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|(f-f_r)(z)-(f-f_r)(w)|}{|z-w|} \leq C||f-f_r||_{\mathcal{B}}.$$

The second estimate is

$$(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f_r(z)-f_r(w)|}{|z-w|} = \frac{r(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}{(1-|rz|^2)^{1/2}(1-|rw|^2)^{1/2}}\underbrace{(1-|rz|^2)^{1/2}(1-|rw|^2)^{1/2}\frac{|f(rz)-f(rw)|}{|rz-rw|}}_{\leq C||f||_{\mathcal{B}}}$$

$$\leq C\frac{r(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}{(1-r^2)^{1/2}(1-r^2)^{1/2}}||f||_{\mathcal{B}}$$

$$= C\frac{r(1-|z|^2)^{1/2}}{(1-r^2)}||f||_{\mathcal{B}}$$

Hence, by the triangle inequality and using the above estimates, we thus find

$$\lim_{|z|\to 1^-}(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f(z)-f(w)|}{|z-w|} \le \lim_{|z|\to 1^-} C\frac{r(1-|z|^2)^{1/2}}{(1-r^2)}||f||_{\mathcal{B}} + C||f-f_r||_{\mathcal{B}}$$

$$\le C\varepsilon.$$

By the arbitrariness of ε, the result follows easily.
Conversely, suppose $f \in H(\mathbb{B}_n)$ such that (3.2.4) holds. To prove that $f \in \mathcal{B}_0$, we proceed as follows. (3.2.4) implies that for any $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\sup_{w\in\mathbb{B}_n, z\neq w}(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f(z)-f(w)|}{|z-w|} < \varepsilon,$$

whenever $|z| > \delta$. In particular,

$$\sup_{w\in D(z,\delta), z\neq w}(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f(z)-f(w)|}{|z-w|} < \varepsilon,$$

whenever $|z| > \delta$. Combining the above with (3.1.24), we find

$$(1-|z|^2)|\nabla f(z)| < C\varepsilon$$

for any $|z| > \delta$. In other words,

$$\lim_{|z|\to 1^-}(1-|z|^2)|\nabla f(z)| = 0.$$

This completes the proof. □

Remark 3.2.13. It is clear that the pointwise estimate of Remark 3.1.16 holds for function of the little Bloch space too. That is

$$|f(z)| \le ||f||_{\mathcal{B}} \log\left(\frac{1+|z|}{1-|z|}\right) + |f(0)|,$$

where $f \in \mathcal{B}_0$.

The next corollary is a significant consequence of the density of polynomials on $\mathcal{B}_0$ and the previous remark.

Corollary 3.2.14. *Let $f \in \mathcal{B}_0$. Then, the following limit holds*

$$\lim_{|z|\to 1^-}\frac{f(z)}{\log\dfrac{1}{1-|z|^2}} = 0. \tag{3.2.5}$$

Proof. Let $f_N(z) = \displaystyle\sum_{k=0}^{N} a_k z^k$ a holomorphic polynomial be, we obtain that

$$\lim_{|z|\to 1^-}\frac{|f_N(z)|}{\log\dfrac{1}{1-|z|^2}} \le \lim_{|z|\to 1^-}\frac{\sum_{k=0}^{N}|a_k|}{\log\dfrac{1}{1-|z|^2}}$$

$$= 0.$$

That is, (3.2.4) holds for holomorphic polynomials. Then, from Proposition 3.2.4, we consider a sequence of polynomials $\{f_N(z)\}_{N\in\mathbb{N}}$ that converges, with respect to the Bloch seminorm, to f so that

$$\begin{aligned}
\lim_{|z|\to 1^-} \frac{|f(z)|}{\log \dfrac{1}{1-|z|^2}} &\leq \lim_{|z|\to 1^-} \frac{|f(z)-f_N(z)|}{\log \dfrac{1}{1-|z|^2}} + \lim_{|z|\to 1^-} \frac{|f_N(z)|}{\log \dfrac{1}{1-|z|^2}} \\
&\leq \lim_{|z|\to 1^-} \frac{||f-f_N||_{\mathcal{B}} \log\left(\dfrac{1+|z|}{1-|z|}\right)}{\log \dfrac{1}{1-|z|^2}} + \lim_{|z|\to 1^-} \frac{\sum_{k=0}^{N} |a_k|}{\log \dfrac{1}{1-|z|^2}} \\
&< \varepsilon,
\end{aligned}$$

choosing N large enough. By the arbitrariness of ε, the desired result follows easily. □

The following theorem is the little Bloch version of Theorem 3.1.18. That is, in the little Bloch space, we show that the Bergman metric can also be recovered from the Bloch seminorm.

Theorem 3.2.15. *We have*

$$\beta(z,w) = \sup\{\, |f(z)-f(w)| \;:\; ||f||_{\mathcal{B}} \leq 1, f \in \mathcal{B}_0\} \tag{3.2.6}$$

for all z and w in $\mathbb{B}_n$.

Proof. Essentially, this proof is the same as that of Theorem 3.1.18. The only difference is that we use the family of functions

$$h_r(z) = \frac{1}{2}\log\left(\frac{|w| + r < z, w >}{|w| - r < z, w >}\right), z \in \mathbb{B}_n,$$

where $w \in \mathbb{B}_n \setminus \{0\}$ is fixed and $r \in (0,1)$, instead of

$$h(z) = \frac{1}{2}\log\left(\frac{|w| + < z, w >}{|w| - < z, w >}\right).$$

Since we notice that

$$\begin{aligned}
\lim_{|z|\to 1^-} |\widetilde{\nabla} h_r(z)|^2 &= \lim_{|z|\to 1^-} (1-|z|^2) \underbrace{\left[r^2|\nabla h(rz)|^2 - |Rh(rz)|^2\right]}_{<+\infty,\ r\in(0,1)} \\
&= 0,
\end{aligned}$$

so that we obtain

$$h_r \in \mathcal{B}_0.$$

Moreover,

$$\begin{aligned}
|\widetilde{\nabla} h_r(z)|^2 &= (1-|z|^2) \underbrace{\left[r^2|\nabla h(rz)|^2 - |Rh(rz)|^2\right]}_{<+\infty,\ r\in(0,1)} \\
&\leq (1-|z|^2)\left[|\nabla h(rz)|^2 - |Rh(rz)|^2\right] \\
&= |\widetilde{\nabla} h(rz)|^2.
\end{aligned}$$

Hence, from Theorem 3.1.18, we get

$$||h_r||_{\mathcal{B}} \leq ||h||_{\mathcal{B}}$$
$$\leq 1$$

and

$$|h_r(z) - h_r(0)| \leq \sup\{|f(z) - f(0)| \ : \ f \in \mathcal{B}_0, \ ||f||_{\mathcal{B}} \leq 1\}.$$

Letting $r \to 1^-$, it turns out

$$\beta(z, 0) \leq \sup\{|f(z) - f(0)| \ : \ f \in \mathcal{B}_0, \ ||f||_{\mathcal{B}} \leq 1\}.$$

Finally, the reversed inequality follows from Theorem 3.1.18 and we obtain the desired result. □

In terms of characterisations, we give the analog version of Corollary 3.1.17 for the little Bloch space.

Corollary 3.2.16. *Assume that $\alpha > -1$, $p > 0$ and $f \in H(\mathbb{B}_n)$. Then $f \in \mathcal{B}_0$ if and only if*

$$\lim_{|a| \to 1^-} \int_{\mathbb{B}_n} |f \circ \varphi_a(z) - f(a)|^p dv_\alpha(z) = 0 \tag{3.2.7}$$

or equivalently,

$$\lim_{|a| \to 1^-} \int_{\mathbb{B}_n} |f(z) - f(a)|^p \frac{(1 - |a|^2)^{n+1+\alpha}}{|1 - < z, a > |^{2(n+1+\alpha)}} dv_\alpha(z) = 0. \tag{3.2.8}$$

Proof. From Proposition 1.4.7, we know that (3.2.7) and (3.2.8) are equivalent. To prove that if $f \in \mathcal{B}_0$, then (3.2.7) holds, we follow the same approach used in the proof of Corollary 3.1.17 to obtain

$$\int_{\mathbb{B}_n} |f(z) - f(a)|^p \frac{(1 - |a|^2)^{n+1+\alpha}}{|1 - < z, a > |^{2(n+1+\alpha)}} dv_\alpha(z) < \infty, \forall a \in \mathbb{B}_n,$$

that means

$$|f(z) - f(a)|^p \frac{(1 - |a|^2)^{n+1+\alpha}}{|1 - < z, a > |^{2(n+1+\alpha)}} \in L^1(\mathbb{B}_n, dv_\alpha) \ \forall a \in \mathbb{B}_n.$$

So that, applying the dominated convergence theorem, we easily find the wished result.
Conversely, assume that (3.2.7) holds, following the same lines as in proof of Corollary 3.1.17, it turns out that

$$|\widetilde{\nabla} f(a)| \leq \int_{\mathbb{B}_n} |f(z) - f(a)|^p \frac{(1 - |a|^2)^{n+1+\alpha}}{|1 - < z, a > |^{2(n+1+\alpha)}} dv_\alpha(z), \forall a \in \mathbb{B}_n.$$

and taking the limit $|a| \to 1^-$, to both sides, we complete the proof. □

As a consequence of Proposition 3.1.28, since $\mathcal{B}_0 \subset \mathcal{B}$ is a closed subspace, we have the following

Corollary 3.2.17. *The little Bloch space $\mathcal{B}_0$ is not strictly convex.*

Proof. We just consider the same counter example of Proposition 3.1.28. □

In the previous section we proved that $\mathcal{B} \subset A^p_\alpha$, for $0 < p < \infty$. Hence, since $\mathcal{B}_0 \subset \mathcal{B}$, we have the following trivial corollary.

Corollary 3.2.18. *Let, $\alpha > -1$. Then, the following containment holds*

$$\mathcal{B}_0 \subset A^p_\alpha, \tag{3.2.9}$$

for $0 < p < \infty$.

A fundamental question about $\mathcal{B}_0$ is whether there is a connection between $\mathcal{B}_0$ and A^∞_α. It turns out that neither $\mathcal{B}_0$ is contained in A^∞_α nor is A^∞_α contained in $\mathcal{B}_0$. We start with the following example.

Example 3.2.19. The function
$$f(z) := \exp\left(\frac{z+1}{1-z}\right)$$
is in A^∞_α but not in $\mathcal{B}_0$.
For sake of simplicity, we consider the one-dimensional case. To prove that $f \in A^\infty_\alpha$, we just observe that

$$\begin{aligned}|f(z)| &\leq |f(z)|_{\mathbb{S}_n}| \\ &= \left|\exp\left(\frac{z+1}{1-z}\frac{1-\overline{z}}{1-\overline{z}}\right)\right| \\ &= \left|\exp\left(\frac{2iIm(z)}{2-2Re(z)}\right)\right| \\ &= 1.\end{aligned}$$

That is, $||f||_{\alpha,\infty} \leq 1$ and, hence, $f \in A^\infty_\alpha$.
Then, in order to prove that
$$\lim_{|z|\to 1^-} f'(z)(1-|z|^2) \neq 0,$$
we just consider tha above limit along the real line $\mathbb{R}$. It turns out that

$$\begin{aligned}\lim_{r\in\mathbb{R},\ r\to 1^-}(1-r^2)|f'(r)| &= \lim_{r\in\mathbb{R},\ r\to 1^-}(1-r^2)\frac{2}{(1-r)^2}\exp\left(\frac{r+1}{1-r}\right) \\ &= 2\lim_{r\in\mathbb{R},\ r\to 1^-}\frac{1+r}{1-r}\exp\left(\frac{r+1}{1-r}\right) \\ &= +\infty.\end{aligned}$$

To show an unbounded function of $\mathcal{B}_0$, first it is necessary to introduce some notions. This will be discussed in the next section.

3.3 Construction of non-trivial functions in $\mathcal{B}$ and $\mathcal{B}_0$.

Goal of this part is to describe a method that is often used to construct non-trivial functions in the Bloch space, or the little Bloch space, of $\mathbb{B}_n$. A crucial consequence will be the construction of unbounded functions that belong to the little Bloch space.

First we observe that if m is any integer such that $1 \leq m \leq n$, and if f is a function in the Bloch space, or the little Bloch space, on $\mathbb{B}_n$. Then, the function
$$\varphi(z_1, ..., z_m) := f(z_1, ..., z_m, ..., z_n)$$
belongs to the Bloch, or the little Bloch space, of $\mathbb{B}_m$. This property follows from condition *b*), or *c*), in Theorems 3.1.10 and 3.2.6 respectively. In particular, functions in the Bloch space, or little Bloch space, of the unit disk $\mathbb{D}$ can be lifted to functions in the Bloch, or little Bloch space, of $\mathbb{B}_n$. Hence, we explain this contruction in the one dimensional case and, so that, will easily deduce the extension to several variables.
In order to proceed, we need to recall the notion of lacunary series.

Definition 3.3.1 (Lacunary series)**.** Consider an increasing sequence of positive integers $\{\lambda_n\}_{n\in\mathbb{N}}$. Denote by $\{\lambda'_n\}_{n\in\mathbb{N}}$ the increasing sequence consisting of the positive integers not contained in $\{\lambda_n\}_{n\in\mathbb{N}}$. Both sequences are assumed to be infinite. The zero coefficients in the series

$$\sum_n a_{\lambda_n} x^{\lambda_n} = \sum_m c_m x^m, \tag{3.3.1}$$

where

$$c_m = \begin{cases} a_{\lambda_n}, & \text{when } m = \lambda_n, \\ 0, & \text{when } m = \lambda'_n, \end{cases} \tag{3.3.2}$$

are called lacunae. A series of the form (3.3.1) is called a lacunary series and the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ is called a gap sequence.

Remark 3.3.2. First of all, we notice that a lacunary series is an holomorphic function that cannot be analytically continued anywhere outside the radius of convergence within which it is defined. More is true, the monotonically increasing sequence of positive natural numbers $\{\lambda_n\}_n$ specifies the powers of z which are in the power series for $f(z)$.

Let's take a look to a simple example.

Example 3.3.3. Consider the following lacunary function,

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}.$$

Comparing f with the geometric series, we prove that f is absolutely convergent on $\mathbb{D}$ and uniformly convergent on every compact subset of $\mathbb{D}$ and, hence, $f \in H(\mathbb{D})$. However, f has a singularity at every point on $\mathbb{S}$, and cannot be analytically continued outside of $\mathbb{D}$. In fact, it is clear that f has a singularity at $z = 1$. But since,

$$f(z^2) = f(z) - z, \quad f(z^4) = f(z^2) - z^2, \quad , f(z^8) = f(z^4) - z^4 \ \dots\ ,$$

we deduce that f has a singularity at a point z when $z^2 = 1$, and also when $z^4 = 1$. So, proceeding by induction, f must have a singularity at each of the 2^n-th roots of unity for all natural numbers n. Such set is dense on $\mathbb{S}$, and, by continuous extension, every point on $\mathbb{S}$ must be a singularity of f.

A further tool that we need is given by the following identity.

Proposition 3.3.4. *Let $f \in H(\mathbb{D})$, assume that the Taylor series of f is*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \ \forall z \in \mathbb{D}.$$

Then, we have that

$$a_{k+1} = (k+2) \int_{\mathbb{D}} (1 - |z|^2) f'(z) \overline{z}^k dA(z) \tag{3.3.3}$$

Proof. To prove this result, we can use two different approachs. We show both of them. The first one is a direct computation and proceeds as follows. First of all, from the Taylor expansion, we have

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$$

$$= \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k.$$

So that, using the uniform convergence on compact subsets of f and Corollary 1.4.8, it turns out that

$$
\begin{aligned}
(k+2)\int_{\mathbb{D}}(1-|z|^2)f'(z)\overline{z}^k dA(z) &= \frac{(k+2)}{2}\sum_{j=0}^{\infty}(j+1)a_{j+1}\int_{\mathbb{D}}2(1-|z|^2)z^j\overline{z}^k dA(z) \\
&= a_{k+1}\frac{(k+2)(k+1)}{2}\frac{k!\Gamma(3)}{\Gamma(k+3)} \\
&= a_{k+1}\frac{(k+2)(k+1)}{2}\frac{k!2!}{(k+2)!} \\
&= a_{k+1}.
\end{aligned}
$$

The second approach follows from the Cauchy integral formula. In fact, using Corollary 1.2.21 (actually equation (1.2.11)) where we replace f by its derivative f', we have

$$
\begin{aligned}
a_{k+1}(k+1) &= \frac{1}{k!}\frac{d^k}{d^k z}f'(z)\bigg|_{z=0} \\
&= \frac{1}{2\pi i}\int_{C(0,r)}\frac{f'(\xi)}{\xi^{k+1}}d\xi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{f'(rexp(i\theta))}{r^k exp(ik\theta)}d\theta.
\end{aligned}
$$

After that, multiplying to both sides by $2r^{2k+1}(1-r^2)dr$ and integrating in polar coordinates, we obtain

$$
\begin{aligned}
(k+1)\frac{k!\Gamma(2)}{\Gamma(k+1+2)}a_{k+1} &= \frac{1}{2\pi}\int_0^{2\pi}\frac{f'(rexp(i\theta))}{r^k exp(ik\theta)}d\theta\int_0^1 2r(1-r^2)dr \iff \\
(k+1)\frac{k!}{(k+2)!}a_{k+1} &= \frac{1}{2\pi}\int_0^{2\pi}f'(rexp(i\theta))r^k exp(-ik\theta)d\theta\int_0^1 2r(1-r^2)dr \iff \\
\frac{(k+1)k!}{(k+2)(k+1)k!}a_{k+1} &= \int_{\mathbb{D}}(1-|z|^2)f'(z)\overline{z}^k dA(z) \iff \\
a_{k+1} &= (k+2)\int_{\mathbb{D}}(1-|z|^2)f'(z)\overline{z}^k dA(z).
\end{aligned}
$$

This completes the proof. □

We are ready to prove the following result concerning a classical way of constructing non-trivial Bloch functions in the unit disk $\mathbb{D}$ using lacunary series.

Theorem 3.3.5. *Assume $\{n_k\}_k$ is a series a positive integers such that*

$$n_{k+1} \geq \lambda n_k, \forall k \geq 1, \tag{3.3.4}$$

where λ is a constant greater than 1. Let $f \in H(\mathbb{D})$ whose Taylor series is

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k},\ z \in \mathbb{D}. \tag{3.3.5}$$

Then,

$$f \in \mathcal{B} \iff \{a_k\}_k \text{ is bounded.}$$

Furthermore,

$$f \in \mathcal{B}_0 \iff \lim_{k\to\infty} a_k = 0.$$

Proof. Suppose $f \in \mathcal{B}$. Using (3.3.3), we find

$$\begin{aligned}
|a_{k+1}| &= (k+2)\left|\int_{\mathbb{D}} (1-|z|^2) f'(z)\overline{z}^k dA(z)\right| \\
&\leq (k+2)||f||_{\mathcal{B}} \left|\int_{\mathbb{D}} \overline{z}^k dA(z)\right| \\
&= (k+2)||f||_{\mathcal{B}} \int_0^{2\pi} d\theta \int_0^1 r^{k+1} dr \\
&= 2\pi ||f||_{\mathcal{B}}.
\end{aligned}$$

In other words,

$$|a_{k+1}| \leq 2\pi ||f||_{\mathcal{B}}, \forall k \geq 0,$$

that clearly means that the sequence $\{a_k\}_k$ is bounded.
Conversely, assume that $\{a_k\}_k$ is bounded. So, there exists a positive (finite) constant M that satisfies

$$|a_k| \leq M, \ \forall \ k \geq 1,$$

and n_k as in (3.3.4). Then, we choose $C = \dfrac{\lambda}{\lambda - 1}$ so that $1 < C < \infty$ and

$$\begin{aligned}
C(n_{k+1} - n_k) &= \frac{\lambda}{\lambda - 1}(n_{k+1} - n_k) \\
&= \frac{\lambda n_{k+1}}{\lambda - 1} - \frac{\lambda n_k}{\lambda - 1} \\
&\geq \frac{\lambda n_{k+1}}{\lambda - 1} - \frac{n_{k+1}}{\lambda - 1} \\
&= n_{k+1}.
\end{aligned}$$

That is,

$$n_{k+1} \leq C(n_{k+1} - n_k).$$

So that,

$$\begin{aligned}
n_{k+1}|z|^{n_{k+1}-1} &\leq C(n_{k+1} - n_k)|z|^{n_{k+1}-1} \\
&\leq C(|z|^{n_k} + \cdots + |z|^{n_{k+1}-1}),
\end{aligned}$$

for all $k \geq 1$. In particular,

$$n_1|z|^{n_1-1} \leq C(1 + |z| + \cdots |z|^{n_1-1}).$$

Thus,

$$\begin{aligned}
|f'(z)| &\leq M \sum_{k=1}^{\infty} n_k |z|^{n_k - 1} \\
&\leq MC \sum_{l=0}^{\infty} |z|^l \\
&= \frac{MC}{1 - |z|},
\end{aligned}$$

for all $z \in \mathbb{D}$. Hence, $f \in \mathcal{B}$.
If $f \in \mathcal{B}_0$, for every $\varepsilon > 0$ there exists $0 < \delta < 1$ such that

$$|f'(z)|(1-|z|^2) < \varepsilon, \; 1-\delta < |z| < 1.$$

Using this fact and (3.3.3), a_{k+1} can be estimated as follows:

$$|a_{k+1}| = \leq (k+2)\left|\int_{0<|z|<1-\delta}(1-|z|^2)f'(z)\overline{z}^k dA(z)\right| + (k+2)\left|\int_{1-\delta<|z|<1}(1-|z|^2)f'(z)\overline{z}^k dA(z)\right|$$

$$\leq (k+2)||f||_{\mathcal{B}}\int_0^{2\pi}d\theta\int_0^{1-\delta}r^{k+1}dr + \varepsilon(k+2)\int_0^{2\pi}d\theta\int_{1-\delta}^1 r^{k+1}dr$$

$$= ||f||_{\mathcal{B}}2\pi(1-\delta)^{k+2} + \varepsilon 2\pi(1-(1-\delta)^{k+2}).$$

That is,

$$|a_{k+1}| \leq ||f||_{\mathcal{B}}2\pi(1-\delta)^{k+2} + \varepsilon 2\pi(1-(1-\delta)^{k+2}).$$

Hence, taking the limit $k \to \infty$, the above implies

$$\lim_{k\to+\infty}|a_{k+1}| \leq \varepsilon 2\pi,$$

and, by the arbitrariness of ε, we deduce

$$\lim_{k\to+\infty}|a_{k+1}| = 0.$$

To prove that $f \in \mathcal{B}_0$, assuming that f is defined by a lacunary series whose coefficients tend to 0 and proceeding in the same manner as above we find the wished result. $\square$

As a consequence, we can easily extend this method to several variables. In particular, as a further consequence of the former construction, we can prove that the little Bloch space $\mathcal{B}_0$ is not contained in A^∞_α. This is the content of the next example that, again for sake of simplicity, we treat in the one-dimensional case.

Example 3.3.6. Let the following function be,

$$f(z) = \sum_{k=1}^{\infty}\frac{z^{2^k}}{\sqrt{k}},$$

where, using the notation of Theorem 3.3.5, $a_k = \dfrac{1}{\sqrt{k}}$ and $n_k = 2^k$. We have that

$$\lim_{k\to+\infty} a_k = 0 \quad \text{and} \quad n_{k+1} \geq \lambda n_k, \; \lambda \geq 1.$$

Hence, we obtain that $f \in \mathcal{B}_0$. Then, arguing as well as in Example 3.3.3, f has a singularity at every point on $\mathbb{S}$, and cannot be analytically continued outside of $\mathbb{D}$. In other words, $f \notin A^\infty_\alpha$.

3.4 Duality

In this section we proceed to identify the dual space of A^p_α, when $0 < p \leq 1$. Furthermore, we shall also find that A^1_α is the dual of the little bloch space. After the discussion of these representations, we will obtain some fundamental consequences for the spaces $\mathcal{B}$, $\mathcal{B}_0$ and A^p_α, for $0 < p \leq 1$: lack of reflexivity, uniform convexity and norm convergence of Taylor series.
We will think of the Bloch space as a Banach space and will use norms, but not semi-norms, on it. That is, we will consider the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$ endowed with

$$||f|| := ||f||_{\mathcal{B}} + |f(0)|.$$

In spite of the fact that A^p_α is not a Banach space when $0 < p < 1$, we can consider its dual space. In fact, we define the dual space of A^p_α for $0 < p < 1$ in exactly the same way as we do for $p \geq 1$. Thus the dual space of A^p_α consists of all linear functionals $F : A^p_\alpha \to \mathbb{C}$ such that

$$|F(f)| \leq C||f||_{p,\alpha}, \ f \in A^p_\alpha,$$

where C is a positive constant depending on F. Moreover, when we write $||F||_{(A^p_\alpha)^*}$, we mean

$$||F||_{(A^p_\alpha)^*} = \sup\left\{F(f) \mid f \in A^p_\alpha, ||f||_{p,\alpha} = 1\right\}.$$

We start identifying the dual space of A^p_α, when $0 < p \leq 1$.

Theorem 3.4.1. *Assume $\alpha > -1$, $0 < p \leq 1$ and*

$$\beta = \frac{n+1+\alpha}{p} - (n+1).$$

Then, we can identify the dual space of A^p_α with $\mathcal{B}$, with equivalent norms, under the integral pairing

$$< f, g >_\beta = \lim_{r \to 1^-} \int_{\mathbb{B}_n} f(rz)\overline{g(z)} dv_\beta(z), \ f \in A^p_\alpha, g \in \mathcal{B}. \tag{3.4.1}$$

In particular, the limit in (3.4.1) *always exist.*

Proof. Let $g \in \mathcal{B}$, from Theorem 3.1.10, point d), there exists a function $h \in L^\infty(\mathbb{B}_n)$ such that

$$g(z) = \int_{\mathbb{B}_n} \frac{h(w) dv_\beta(w)}{(1- < z, w >)^{n+1+\beta}}, \ z \in \mathbb{B}_n, \tag{3.4.2}$$

so that, there exists a positive constant C, independent of g, such that

$$||h||_\infty \leq C||g||.$$

Hence, thanks to this estimate, we can exchange the limit with the integral in (3.4.1) and, then, using Fubini's theorem, the reproducing formula of Lemma 2.1.3 and Lemma 2.4.5, it turns out that the integral pairing of (3.4.1) can be written as follows

$$\begin{aligned}
| < f, g >_\beta | &= \left| \lim_{r \to 1^-} \int_{\mathbb{B}_n} f(rz)\overline{g(z)} dv_\beta(z) \right| \\
&= \left| \int_{\mathbb{B}_n} f(z) dv_\beta(z) \overline{\int_{\mathbb{B}_n} \frac{h(w) dv_\beta(w)}{(1- < z, w >)^{n+1+\beta}}} \right| \\
&= \left| \int_{\mathbb{B}_n} f(z) dv_\beta(z) \int_{\mathbb{B}_n} \frac{\overline{h(w)} dv_\beta(w)}{(1- < w, z >)^{n+1+\beta}} \right| \\
&= \left| \int_{\mathbb{B}_n} \overline{h(w)} dv_\beta(w) \int_{\mathbb{B}_n} \frac{f(z) dv_\beta(z)}{(1- < w, z >)^{n+1+\beta}} \right| \\
&= \left| c_\beta \int_{\mathbb{B}_n} f(w)\overline{h(w)}(1 - |w|^2)^{\frac{n+1+\alpha}{p} - (n+1)} dv(w) \right| \\
&\leq ||h||_\infty \left| c_\beta \int_{\mathbb{B}_n} f(w)(1 - |w|^2)^{\frac{n+1+\alpha}{p} - (n+1)} dv(w) \right| \\
&\leq C||g||||f||_{p,\alpha}
\end{aligned}$$

That is, g induces a bounded linear functional on A^p_α under the integral pairing $<,>_\beta$. Conversely, let $F \in (A^p_\alpha)^*$ and $f \in A^p_\alpha$, then

$$f_r(z) = \int_{\mathbb{B}_n} \frac{f_r(w)dv_\beta(w)}{(1-<z,w>)^{n+1+\beta}},\ r \in (0,1),$$

so that, using the homogeneous expansion of the kernel function, we find

$$\begin{aligned} F(f_r(z)) &= F\left(\int_{\mathbb{B}_n} \frac{f_r(w)dv_\beta(w)}{(1-<z,w>)^{n+1+\beta}}\right) \\ &= \int_{\mathbb{B}_n} f_r(w) \underbrace{F\left(\frac{1}{(1-<z,w>)^{n+1+\beta}}\right)}_{:=\overline{g(w)}} dv_\beta(w) \\ &= \int_{\mathbb{B}_n} f_r(w)\overline{g(w)}dv_\beta(w) \\ &= < f_r, g >_\beta \end{aligned}$$

So, our aim to show that $g \in \mathcal{B}$. We proceed as follows. It is clear that $g \in H(\mathbb{B}_n)$. Then, interchanging the differentiation and the application of F, justified by using the homogeneous expansion of the kernel function, and the fact that F is bounded on A^p_α, we find

$$\begin{aligned} \left|\frac{\partial \overline{g(w)}}{\partial w_i}\right| &= \left|F\left(\frac{\partial}{\partial z_i}\frac{1}{(1-<z,w>)^{n+1+\beta}}\right)\right| \\ &= (n+1+\beta)\left|F\left(\frac{\overline{w_i}}{(1-<z,w>)^{n+2+\beta}}\right)\right| \\ &\leq (n+1+\beta)||F||_{(A^p_\alpha)^*}\left[\int_{\mathbb{B}_n} \frac{dv_\alpha(z)}{|1-<z,w>|^{p(n+2+\beta)}}\right]^{1/p} \\ &\leq (n+1+\beta)||F||_{(A^p_\alpha)^*}\left[\int_{\mathbb{B}_n} \frac{(1-|z|^2)^\alpha dv(z)}{|1-<z,w>|^{n+1+p+\alpha}}\right]^{1/p} \\ &\leq C(n+1+\beta)||F||_{(A^p_\alpha)^*}\frac{1}{(1-|w|^2)}, \end{aligned}$$

so that, we have

$$\begin{aligned} \sup_{w\in\mathbb{B}_n}(1-|w|^2)|\nabla g(w)|^2 &\leq \sup_{w\in\mathbb{B}_n}(1-|w|^2)(n+1+\beta)\sqrt{n}C||F||_{(A^p_\alpha)^*}\frac{1}{(1-|w|^2)} \\ &= (n+1+\beta)\sqrt{n}C||F||_{(A^p_\alpha)^*} \\ &< \infty. \end{aligned}$$

Hence, using point e) in Theorem 3.1.10, we get $g \in \mathcal{B}$. This completes our proof. □

Putting $p = 1$ in the above result, we obtain the following.

Corollary 3.4.2. *Let $\alpha > -1$. Then, we can identify the dual space of A^1_α with $\mathcal{B}$, with equivalent norms, under the integral pairing*

$$< f, g >_\alpha = \lim_{r\to 1^-}\int_{\mathbb{B}_n} f(rz)\overline{g(z)}dv_\alpha(z),\ f \in A^1_\alpha, g \in \mathcal{B}. \tag{3.4.3}$$

In particular, the limit in (3.4.3) *always exist.*

We proceed identifying the dual of the little Bloch space $\mathcal{B}_0$. After that, we will see some remarkable consequence of this result and the previous theorem.

Theorem 3.4.3. *Assume $\alpha > -1$. Then, the dual space of $\mathcal{B}_0$ can be identified with A^1_α, with equivalent norms, under the integral pairing*

$$< f, g >_\alpha = \lim_{r \to 1^-} \int_{\mathbb{B}_n} f(z)\overline{g(rz)} dv_\alpha(z), \ f \in \mathcal{B}_0, \ g \in A^1_\alpha. \tag{3.4.4}$$

In particular, the limit above always exists.

Proof. Let $f \in \mathcal{B}_0$. From point $d)$ in Theorem 3.2.6, or equivalently point $e)$ in Theorem 3.2.8, there exists $h \in C_0(\mathbb{B}_n)$ such that

$$P_\alpha h(z) = f(z), \ z \in \mathbb{B}_n.$$

Moreover, h can be chosen so that there exists a positive constant C, independent of f, such that

$$||h||_\infty \leq C||f||_{\mathcal{B}}. \tag{3.4.5}$$

Let $g \in A^1_\alpha$ and $g_r(z) = g(rz)$, where $r \in (0, 1)$ and $z \in \mathbb{B}_n$, the dilation function be. Then, defining

$$< f, g_r >_\alpha = \int_{\mathbb{B}_n} h(w)\overline{g_r(w)} dv_\alpha(w),$$

so that, using Holder's inequality and (3.4.5), it turns out

$$\begin{aligned} | < f, g_r >_\alpha | &\leq ||h||_\infty ||g||_{\alpha,1} \\ &\leq C||f||_{\mathcal{B}} ||g||_{\alpha,1}. \end{aligned}$$

In other words, we have proved that every function $g \in A^1_\alpha$ induces a bounded linear functional on $\mathcal{B}_0$ via the integral pairing $<,>_\alpha$.
Next, our aim is to show that every bounded linear functional on $\mathcal{B}_0$ arises from a function in A^1_α via the integral pairing $<,>_\alpha$. In order to proceed, we fix a sufficiently large positive parameter b and consider the operator T defined by

$$Tf(z) = \frac{c_{b+\alpha}}{c_\alpha}(1 - |z|^2)^b \int_{\mathbb{B}_n} \frac{f(w) dv_\alpha(w)}{(1- < z, w >)^{n+1+\alpha+b}}.$$

Now, let $f \in \mathcal{B}_0$, using (3.4.5) and Theorem 1.4.4, we have that there exists a positive constant C such that

$$\begin{aligned} |Tf(z)| &\leq \frac{c_{b+\alpha}}{c_\alpha}(1 - |z|^2)^b C||f|| \int_{\mathbb{B}_n} \frac{dv_\alpha(w)}{|1- < z, w > |^{n+1+\alpha+b}} \\ &\leq \frac{c_{b+\alpha}}{c_\alpha}(1 - |z|^2)^b ||f|| C \frac{1}{(1 - |z|^2)^b} \\ &= \frac{c_{b+\alpha} C||f||}{c_\alpha}. \end{aligned}$$

That is, $Tf \in L^\infty(\mathbb{B}_n)$ and, hence, $T : \mathcal{B}_0 \longrightarrow L^\infty(\mathbb{B}_n)$ is a bounded operator. On the other hand, follows the same lines as in the proof of Theorem 3.1.10, it turns out that

$$P_\alpha(Tf)(z) = f(z), \forall f \in \mathcal{B}_0, z \in \mathbb{B}_n.$$

Hence, there exists a positive constant C, independent of f, such that

$$||f|| \leq C||Tf||_\infty.$$

We conclude that T is an embedding of $\mathcal{B}$ into $L^\infty(\mathbb{B}_n)$.
If f is a polynomial, then, follows the same lines as in Theorem 3.2.6, we easly check that Tf is $(1-|z|^2)^b$ times a polynomial, which is a function in $C_0(\mathbb{B}_n)$. By the fact that $C_0(\mathbb{B}_n)$ is closed in $L^\infty(\mathbb{B}_n)$, we get that T is an embedding of $\mathcal{B}_0$ into $C_0(\mathbb{B}_n)$. Denoting by

$$T(\mathcal{B}_0) = X,$$

then, X is a closed subspace of $C_0(\mathbb{B}_n)$.
Now, let $F \in (\mathcal{B}_0)^*$, then $F \circ T^{-1} \in (X)^*$. Using the Hahn-Banach theorem, we extend continuously $F \circ T^{-1}$ to the whole space $C_0(\mathbb{B}_n)$. After that, we apply the classical Riesz representation theorem for $C_0(\mathbb{B}_n)$. So, we obtain a finite complex Borel measure μ on $\mathbb{B}_n$ such that

$$F \circ T^{-1}(f) = \int_{\mathbb{B}_n} f(z) d\mu(z), \ f \in X,$$

or equivalently,

$$F(f) = \int_{\mathbb{B}_n} Tf(z) d\mu(z), \ f \in \mathcal{B}_0. \tag{3.4.6}$$

If f is a polynomial, using Fubini's theorem, (3.4.6) can be written as

$$\begin{aligned} F(f) &= \int_{\mathbb{B}_n} \frac{c_{b+\alpha}}{c_\alpha}(1-|z|^2)^b d\mu(z) \int_{\mathbb{B}_n} \frac{f(w) dv_\alpha(w)}{(1-<z,w>)^{n+1+\alpha+b}} \\ &= \int_{\mathbb{B}_n} f(w) \left\{ \frac{c_{b+\alpha}}{c_\alpha} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^b d\mu(z)}{(1-<z,w>)^{n+1+\alpha+b}} \right\} dv_\alpha(w) \\ &= \int_{\mathbb{B}_n} f(w) \overline{\left\{ \frac{c_{b+\alpha}}{c_\alpha} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^b d\overline{\mu}(z)}{(1-<w,z>)^{n+1+\alpha+b}} \right\}} dv_\alpha(w) \\ &= \int_{\mathbb{B}_n} f(w)\overline{g(w)} dv_\alpha(w). \end{aligned}$$

By Theorem 1.4.6, we deduce that $g \in A^1_\alpha$. Finally, since polynomials are dense in the little Bloch space $\mathcal{B}_0$, we complete the proof. □

Remark 3.4.4. As a consequence of Theorem 3.4.1 and Theorem 3.4.3, we have

$$\begin{aligned} (\mathcal{B}_0)^{**} &= (A^1_\alpha)^* \\ &= \mathcal{B} \\ &\supset \mathcal{B}_0. \end{aligned}$$

In other words, the little Bloch space $\mathcal{B}_0$ is not reflexive. After that, if we assume that the Bloch space is reflexive, since $\mathcal{B}_0 \subset \mathcal{B}$ is a closed subspace, we would have that the little Bloch space $\mathcal{B}_0$ is reflexive. That is a contradiction. So, the Bloch space $\mathcal{B}$ is not reflexive. Finally, it is a well-known fact that, for a Banach space X, X is reflexive if and only if its dual X^* is reflexive. Hence, we apply this result putting $X = \mathcal{B}_0$, so that $X^* = A^p_\alpha$ is not reflexive. We summarise all these results in the following corollary.

Corollary 3.4.5. *The Bloch space $\mathcal{B}$, the little Bloch space $\mathcal{B}_0$ and the Bergman spaces A^p_α, for $\alpha > -1$ and $0 < p \leq 1$, are not reflexive.*

A further consequence, from Milman-Pettis theorem, is

Corollary 3.4.6. *The Bloch space $\mathcal{B}$, the little Bloch space $\mathcal{B}_0$ and the Bergman spaces A^p_α, for $\alpha > -1$ and $0 < p \leq 1$, are not uniformly convex.*

In Theorem 2.1.11, we showed that there exist functions in A^1_α whose Taylor series do not converge in norm. We recall that an example, not proved, of such functions is

$$f_a(z) = \frac{1-|a|^2}{(1-\overline{a}z)^3}.$$

Hence, as a consequence of Theorem 3.4.3, we have the following corollary.

Corollary 3.4.7. *There exist functions in the little Bloch space $\mathcal{B}_0$ whose Taylor series do not converge in norm.*

We just give a hint of the proof. All the details can be found on, for example, Duality of Bloch Spaces and Norm Convergence of Taylor Series, written by Kehe Zhu.

Proof. For sake of simplicity, we consider the one-dimensional case and $\alpha = 0$. It suffices to show that the operators

$$S_N : \sum_{k=0}^{\infty} b_k z^k \longrightarrow \sum_{k=0}^{N} b_k z^k$$

are not uniformly bounded on A^1_α. That is, there is no constant $C > 0$ such that

$$||S_N||_1 \leq C, \forall n \geq 1.$$

From Theorem 1.4.4, there exists a positive constant C such that

$$||f||_1 \leq C,\ \forall a \in \mathbb{D}.$$

After that, since the Taylor expansion of f_a is given by

$$f_a(z)(1-|a|^2)\sum_{k=0}^{\infty}(k+1)(k+2)\overline{a}^k z^k,$$

so that, $S_N f_a$ can be written as follows

$$S_N f_a(z) = (1-|a|^2)\left[\underbrace{-\frac{(n+2)(n+3)(\overline{a}z)^{n+1}}{1-\overline{a}z}}_{:=A_1(z)} + \underbrace{\frac{2(n+3)(\overline{a}z)^{n+2}}{(1-\overline{a}z)^2}}_{:=A_2(z)} + \underbrace{\frac{(1-\overline{a}z)^{n+3}}{(1-\overline{a}z)^3}}_{:=A_3(z)}\right].$$

After some computations, it turns out that there exist two positive finite constants C_1, C_2 such that

$$||A_3(z)||_1 \leq C_1 \quad \text{and} \quad ||A_2(z)||_1.$$

But, concerning the function $A_1(z)$, using polar coordinates, Theorem 1.4.4 and integrting by parts, we find that, respect to with $||\cdot||_1$, is unbounded as a function of n and a. In fact,

$$\begin{aligned}
||A_1||_1 &= (n+2)(n+3)(1-|a|^2)|a|^{n+1}\int_{\mathbb{D}}\frac{|z|^{n+1}dA(z)}{|1-\overline{a}z|} \\
&= (n+2)(n+3)(1-|a|^2)|a|^{n+1}\frac{1}{\pi}\int_0^1 r^{n+2}dr\int_0^{2\pi}\frac{dt}{|1-r|a|e^{it}|} \\
&\geq c(n+2)(n+3)(1-|a|^2)|a|^{n+1}\frac{1}{\pi}\int_0^1 r^{n+2}\log\left(\frac{1}{1-r|a|}\right)dr \\
&= c(n+2)(1-|a|^2)|a|^{n+1}\log\left(\frac{1}{1-r|a|}\right) - \frac{c(n+2)(1+|a|)|a|^{n+2}}{n+4}.
\end{aligned}$$

Finally, the second term above is bounded in a and n, but the first term tends to ∞ if $a = n/n+1$ and $n \to +\infty$. This finishes the proof. $\square$

3.5 Maximality

In this section we show that the Bloch space $\mathcal{B}$ is the largest Mobius-invariant linear space of holomorphic functions that can be equipped with a Mobius-invariant seminorm in such a way that there is at least one nonzero bounded linear functional on the space. One simple consequence of the main result is that there are no nontrivial Mobius-invariant closed subspaces of $H(\mathbb{B}_n)$ equipped with the topology of uniform convergence on compact subsets. We start with the following definition.

Definition 3.5.1 (Mobius invariant Banach space)**.** Let $X := (X, ||\cdot||_X)$ a seminormed linear space of holomorphic functions in $\mathbb{B}_n$. We say that X is a Mobius invariant Banach space if the following property holds

$$||f \circ \varphi||_X = ||f||_X, \; f \in X, \; \varphi \in Aut(\mathbb{B}_n). \tag{3.5.1}$$

Remark 3.5.2. We will suppose that X is already complete in the semi-norm. In fact, if necessary, we can consider its completion. Moreover, the map

$$\phi(\theta_1, ..., \theta_n) := f(|z_1|exp(i\theta_1), ..., |z_n|exp(i\theta_n)), \tag{3.5.2}$$

is assumed to be a continuous function from $[0, 2\pi]^n$ to X.

In order to prove the main result of this section, we will need the following Lemma.

Lemma 3.5.3. *Assume that X is a Mobius invariant Banach space such that contains nonconstant functions. Then, all the polynomials are contained in X.*

Proof. Let $f \in X$ a nonconstant function. Suppose the Taylor expansion of f is

$$f(z) = \sum_{m=0}^{\infty} a_m z^m, \; \forall z \in \mathbb{B}_n. \tag{3.5.3}$$

Hence, there exists some nonzero multi-index m such that

$$a_m \neq 0.$$

We fix such index $m = (m_1, ..., m_n)$. Let F be the following function,

$$F(z) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(|z_1|exp(i\theta_1), ..., |z_n|exp(i\theta_n))exp(-i(m_1\theta_1 + \cdots + m_n\theta_n))d\theta_1 \ldots d\theta_n.$$

So that, by a direct computation, we obtain

$$\begin{aligned} F(z) &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(|z_1|exp(i\theta_1), ..., |z_n|exp(i\theta_n))exp(-i(m_1\theta_1 + \cdots + m_n\theta_n))d\theta_1 \ldots d\theta_n \\ &= a_m \frac{1}{(2\pi)^n} \int_0^{2\pi} z_1^{m_1} d\theta_1 \cdots \int_0^{2\pi} z_n^{m_n} d\theta_n \\ &= a_m z^m. \end{aligned}$$

In other words, $F(z) = a_m z^m$. Furthermore, using the fact that X is Mobius invariant, we get

$$||F||_X \leq ||f||_X.$$

Hence, X contains the monomial z^m.
Then, considering the composition between z^m with all the possible unitary transformations and so, using the Mobius invariance of X, X contains all homogeneous polynomials of degree $|m|$. As a consequence, in particular, for every $\varphi \in Aut(\mathbb{B}_n)$, it turns out that

$$z_1^{|m|} \circ \varphi \in X.$$

Choose $\varphi = \varphi_a$, where $a = (\lambda, 0, ..., 0)$ with $|\lambda| < 1$. We obtain,

$$z_1^{|m|} \circ \varphi = \left(\frac{\lambda - z_1}{1 - \bar{\lambda} z_1} \right)^{|m|}. \tag{3.5.4}$$

For every nonnegative integer l, we can find some λ such that the Taylor coefficient of z_1^l of the function in (3.5.4) is nonzero. Following the same approach used in the first paragraph of this proof, we find

$$z_1^l \in X, \forall\ l \geq 0.$$

Combining this with the remarks in the previous paragraph, we conclude that X contains all polynomials. □

We now show that the Bloch space is maximal among Mobius invariant Banach spaces. Then, we discuss some related consequence.

Theorem 3.5.4. *Assume X is a Mobius invariant Banach space in $\mathbb{B}_n$. If there exists a nonzero bounded linear functional L on X. Then,*

$$X \subset \mathcal{B}.$$

Moreover, there exists a positive constant C such that

$$||f||_{\mathcal{B}} \leq C||f||_X,\ \forall f \in X. \tag{3.5.5}$$

Finally, if L is such that $L(1) \neq 0$. Then,

$$X \subset A_\alpha^\infty(\mathbb{B}_n), \tag{3.5.6}$$

and there exists a positive constant C such that

$$||f||_\infty \leq C||f||_X,\ \forall f \in X. \tag{3.5.7}$$

Proof. Let L be a nonzero bounded linear functional on X. That is,

$$|L(f)| \leq C||f||_X,\ f \in X.$$

We start supposing $L(1) \neq 0$. Since the mean value property, in the origin, is given by

$$f(0) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(z) dt_1 \ldots dt_n,$$

so that, applying L to both sides, we obtain

$$f(0)L(1) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} L(f(z)) dt_1 \ldots dt_n.$$

Hence, from the boundedness of L on X, it follows that

$$|f(0)||L(1)| \leq C||f||_X.$$

Then, we replace f by $f \circ \varphi_z$ so that, using the Mobius invariance of X, we get

$$|f(z)||L(1)| \leq C||f||_X,\ \forall z \in \mathbb{B}_n.$$

We proved that $f \in A_\alpha^\infty(\mathbb{B}_n)$ and, clearly, that

$$||f||_\infty \leq \frac{C}{|L(1)|}||f||_X.$$

After that, we assume $L(1) = 0$ and X contains a non constant function. Hence, from Lemma 3.5.3, X must contain all the polynomials. Our aim is to show that there exists some $\varphi \in Aut(\mathbb{B}_n)$ such that the linear functional $L_\varphi(f) := L(f \circ \varphi)$ satisfies $L_\varphi(z_1) \neq 0$. To prove this fact, we proceed as follows. Let $r \in (0,1)$ and $a = (r, 0, ..., 0)$. We consider the following involutive automorphism

$$\varphi_a(z) = \left(\frac{r - z_1}{1 - rz_1}, \frac{\sqrt{1-r^2}z_2}{1rz_1}, \dots, -\frac{\sqrt{1-r^2}z_n}{1 - rz_1} \right), \tag{3.5.8}$$

so that

$$\begin{aligned} z_1 \circ \varphi_a(z) &= \frac{r - z_1}{1 - rz_1} \\ &= r + (r^2 - 1) \sum_{k=1}^{\infty} r^{k-1} z_1^k. \end{aligned}$$

That is,

$$z_1 \circ \varphi_a(z) = r + (r^2 - 1) \sum_{k=1}^{\infty} r^{k-1} z_1^k. \tag{3.5.9}$$

Arguing by contradiction, assume that $L(z_1) = 0, \forall \varphi \in Aut(\mathbb{B}_n)$, and applying L to (3.5.9) we find

$$\begin{aligned} 0 &= L_{\varphi_a}(z_1) \\ &= L(z_1 \circ \varphi_a) \\ &= L(1)r + (r^2 - 1) \sum_{k=1}^{\infty} r^{k-1} L(z_1^k). \end{aligned}$$

So that, we deduce

$$\sum_{k=1}^{\infty} r^{k-1} L(z_1^k) = 0, \forall\ r \in (0,1),$$

and it follows that

$$L(z_1^k) = 0,\ \forall k \geq 1.$$

Then, replacing L by L_φ, it turns out

$$L_\varphi(z_1^k) = 0,\ \forall k \geq 1.$$

This fact implies that

$$L(z^m) = 0, \forall m = (m_1, \dots, m_n) \text{ s.t. } |m| > 0.$$

Combining this with $L(1) = 0$, we find

$$L = 0,$$

on $H(\mathbb{B}_n)$. That is, a contradiction. This means that we can assume $L(z_1) \neq 0$. For $f \in X$, let the following function be

$$F(f) = \frac{1}{2\pi} \int_0^{2\pi} L(f(|z|exp(it)))exp(-it)dt.$$

Using the fact that L is continuous on X, we get

$$|F(f)| \leq C||f||_X,\ f \in X.$$

On the other hand, expanding f in homogenous series and using its uniform convergence on compact subsets, it turns out

$$F(f) = \frac{1}{(2\pi)^n}\int_0^{2\pi} L\left(\sum_{j=0}^{\infty}\frac{1}{j!}\frac{\partial^j f}{\partial z^j}(0)|z_1|^{j_1}\dots|z_n|^{j_n}exp(ijt)\right)exp(-it)dt$$

$$= \frac{1}{(2\pi)^n}\sum_{j=0}^{\infty}\frac{1}{j!}\frac{\partial^j f}{\partial z^j}(0)L\left(|z_1|^{j_1}\dots|z_n|^{j_n}\right)\int_0^{2\pi} exp(it(j-1))dt$$

$$= \sum_{k=0}^{n} L\left(|z_k|\right)\frac{\partial f}{\partial z_k}(0).$$

That is, $F(f)$ can be written as follows

$$F(f) = \sum_{k=0}^{n} L\left(|z_k|\right)\frac{\partial f}{\partial z_k}(0). \tag{3.5.10}$$

After that, denoting by $w = (w_1, ..., w_n)$ a unit vector of $\mathbb{C}^n$, there exists a positive constant δ such that (3.5.10) can be written as

$$F(f) = \delta < \nabla f(0), w > . \tag{3.5.11}$$

Furthermore, for every $1 \leq k \leq n$, we can find a unitary matrix U_k such that

$$U_k(w) = e_k,$$

where $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of $\mathbb{C}^n$. Hence, after some computations

$$F(f \circ U_k) = \delta < \nabla(f \circ U_k)(0), w >$$

$$= \delta < U_k \nabla(f)(0), w >$$

$$= \delta < \nabla f(0), U_k w >$$

$$= \delta \frac{\partial f}{\partial z_k}(0).$$

Namely,

$$F(f \circ U_k) = \delta \frac{\partial f}{\partial z_k}(0), \forall f \in X. \tag{3.5.12}$$

We replace f by $f \circ \varphi$, where $\varphi \in Aut(\mathbb{B}_n)$, so that, for $1 \leq k \leq n$, (3.5.12) can be estimated as follows

$$\left|\frac{\partial f}{\partial z_k}(0)\right| \leq \frac{|F(f \circ \varphi \circ U_k)|}{\delta}(0)$$

$$\leq \frac{C}{\delta}||f \circ \varphi \circ U_k||_X$$

$$= \frac{C}{\delta}||f||_X.$$

Therefore, we obtain

$$|\widetilde{\nabla} f(z)| \leq n\frac{C}{\delta}||f||_X, \forall f \in X,\ z \in \mathbb{B}_n.$$

We have proved that

$$f \in \mathcal{B} \quad \text{and} \quad ||f||_{\mathcal{B}} \leq C'||f||_X,$$

for some positive constant C'. ☐

In the remainder of this section, we denote the seminorms of the Mobius invariant Banach space X and the Bloch space, respectively, by

$$p(f) := ||f||, \quad \text{and} \quad p_{\mathcal{B}}(f) := ||f||_{\mathcal{B}}, \ \forall f \in X.$$

Corollary 3.5.5. *Assume that X is a Mobius invariant Banach space in $\mathbb{B}_n$ that satisfies the same conditions of Theorem* 3.5.4*. Then, the kernel $p^{-1}(0)$ is contained in the set of constant functions.*

Proof. We just observe that the kernel of $p_{\mathcal{B}}$ is given by the constant functions. □

Corollary 3.5.6. *Let $H(\mathbb{B}_n)$ equipped with the topology of uniform convergence on compact subsets of $\mathbb{B}_n$. Then, the only closed Mobius-invariant subspaces of $H(\mathbb{B}_n)$ are $\{0\}$, $H(\mathbb{B}_n)$, and the constant functions.*

Proof. Let E be a closed Mobius-invariant subspace of $H(\mathbb{B}_n)$ with $E \neq H(\mathbb{B}_n)$. Then, there exists a nonzero continuous linear functional L, with $L(f) = 0$ for all $f \in E$. Let $f \in H(\mathbb{B}_n)$, we set

$$p(f) = \sup \{L(f \circ \varphi) \mid \varphi \in Aut(\mathbb{B}_n)\}.$$

After that, we define

$$X := \{f \in H(\mathbb{B}_n) \mid p(f) < \infty\}.$$

Then, since (X, p) satisfies the hyphoteses of Theorem 3.5.4, by Corollary 3.5.5 $p^{-1}(0)$ is contained in the constant functions. Besides, let $f \in E$, then

$$\begin{aligned} p(f) &= \sup \{L(f \circ \varphi) \mid \varphi \in Aut(\mathbb{B}_n)\} \\ &= L(f) \\ &= 0. \end{aligned}$$

In other words, we have that

$$E \subseteq p^{-1}(0) \subseteq X.$$

Thus, E is contained in the set of constant functions. That is, E is either $\{0\}$ or the constant functions, as required. □

3.6 Pointwise Multipliers

The aim of this section is to characterise the pointwise multipliers of the Bloch space and the little Bloch space. The pointwise multipliers of the Bloch space and the little Bloch space were first characterised by Arazy in the case of the open unit disc of $\mathbb{C}$ and later rediscovered by Kehe Zhu in the case of $\mathbb{B}_n$. This section is organised as follows. We start recalling the definition of Pointwise Multipliers. After that, we provide a crucial tool: we prove that the pointwise multipliers of every Banach space of holomorphic functions in $\mathbb{B}_n$, such that every point evaluation is a bounded linear functional, can be embedded into A_α^∞. As a consequence, we apply this result to the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$. Then, we illustrate the goal of this section: let $f \in H(\mathbb{B}_n)$, we show that f is a pointwise multiplier of the Bloch space, and the little Bloch space, if and only if both f and $(1 - |z|^2)|\nabla f(z)| \log \frac{1}{1-|z|^2}$ are bounded in $\mathbb{B}_n$.

Formally, the definition of Pointwise Multipliers is

Definition 3.6.1 (Pointwise Multipliers)**.** Let X a space of functions. A function f is called a pointwise multiplier of a space X if for every $g \in X$ the pointwise product fg also belongs to X.
In this section, we denote a pointwise multiplier f of a space X by

$$fX \subset X.$$

Throughout this section, we endow $\mathcal{B}$ with the following norm

$$||f|| = |f(0)| + \sup_{z\in\mathbb{B}_n} |\nabla f(z)|,\ f \in \mathcal{B}. \tag{3.6.1}$$

In order to proceed, we need the following Lemma.

Lemma 3.6.2. *Let X be a Banach space of holomorphic functions in $\mathbb{B}_n$. Assume that X contains the constant functions and that every point evaluation is a bounded linear functional on X. Then, every pointwise multiplier of X is in A^∞_α.*

Proof. Let f be a pointwise multiplier of X. Since X contains the constant function, we obtain that $f \in X$. After that, we define the linear operator of multiplication by f as

$$M_f(g)(z) = f(z)g(z), g \in X.$$

Since every point evaluation is a bounded linear functional on X, we have that

$$\begin{aligned}|M_f(g)(z)| &= |f(z)||g(z)| \\ &\leq ||f||\ ||g||.\end{aligned}$$

We deduce that $M_f(g)$ is bounded.
Consequently, we denote by e_z the point evaluation at z, where $z \in \mathbb{B}_n$ and, hence, by assumption, $e_z \in X^*$. So that, considering the action of the bounded linear operator on X^* given by the adjoint operator of $M_f(g)(z)$, denoted by $M^*_f(g)$, it turns out that

$$\begin{aligned}M^*_f(e_z)(g) &= e_z(M_f(g)) \\ &= f(z)g(z) \\ &= f(z)e_z(g).\end{aligned}$$

In other words,

$$M^*_f(e_z)(g) = f(z)e_z(g),$$

that implies

$$\begin{aligned}|f(z)| &= \frac{|M^*_f(e_z)(g)|}{|e_z(g)|} \\ &\leq ||M^*_f||.\end{aligned}$$

This proves that

$$|f(z)| \leq ||M^*_f||,\ \forall z \in \mathbb{B}_n,$$

and we are done. □

As a consequence of this result, putting $X = \mathcal{B}$ or $X = \mathcal{B}_0$, we have the following corollary.

Corollary 3.6.3. *Assume that f is a pointwise multiplier of $\mathcal{B}$, or $\mathcal{B}_0$. Then, f is bounded in $\mathbb{B}_n$.*

We are ready to prove the main result of this section: the characterisation of the pointwise multipliers of the Bloch space and the little Bloch space.

Theorem 3.6.4. *Let $f \in H(\mathbb{B}_n)$. Then, the following conditions are equivalent:*
a) $f\mathcal{B} \subset \mathcal{B}$.
b) $f\mathcal{B}_0 \subset \mathcal{B}_0$.
c) $f \in A^\infty_\alpha$ and the function

$$(1 - |z|^2)|\nabla f(z)| \log \frac{1}{1 - |z|^2}$$

is bounded in $\mathbb{B}_n$.

Proof. We start proving $a) \implies c)$. If $f\mathcal{B} \subset \mathcal{B}$, from the previous corollary, we have that $f \in A^\infty_\alpha$ and, hence, there exists a positive constant $C > 0$ such that

$$||fg|| \leq C||g||, \forall g \in \mathcal{B}. \tag{3.6.2}$$

After that, since the Leibniz rule is

$$g\nabla f = \nabla(fg) - f\nabla g.$$

So that, assuming without loss of generality that $g(0) = 0$ and usin Theorem 3.1.18 (actually (3.1.16)), we have that

$$\begin{aligned}
(1-|z|^2)|\nabla f(z)|\frac{1}{2}\log\frac{1}{1-|z|^2} &\leq (1-|z|^2)|\nabla f(z)|\frac{1}{2}\log\frac{1+|z|}{1-|z|} \\
&= (1-|z|^2)|\nabla f(z)|\frac{1}{2}\beta(z,0) \\
&= \frac{1}{2}(1-|z|^2)|\nabla f(z)| \sup\{\ |g(z)| \mid ||g|| \leq 1\} \\
&\leq |f(z)|\frac{(1-|z|^2)}{2}\sup\{\ |\nabla g(z)| \mid ||g|| \leq 1\} + \frac{(1-|z|^2)}{2}\sup\{\ |\nabla(fg)(z)| \mid ||g|| \leq 1\} \\
&\leq \frac{1}{2}\left(||g||\ ||f||_{\infty,\alpha} + ||fg||\right) \\
&\leq \frac{1}{2}\left(||f||_{\infty,\alpha} + C\right).
\end{aligned}$$

In other words,

$$(1-|z|^2)|\nabla f(z)|\log\frac{1}{1-|z|^2} \leq ||f||_{\infty,\alpha} + C, \forall z \in \mathbb{B}_n,$$

and the implication $a) \implies c)$ is proved.
To prove that $b) \implies c)$, we follow the same lines as in the previous implication but, instead of Theorem 3.1.18, we use Theorem 3.2.15.
Next, we wish to prove that $c)$ implies $a)$. So, assume that $c)$ holds. Since $f \in A^\infty_\alpha$ and $g \in \mathcal{B}$, we proceed with the following estimate,

$$\begin{aligned}
\frac{(1-|z|^2)}{2}|\nabla(fg)(z)| &\leq \frac{1}{2}(1-|z|^2)|f(z)||\nabla g(z)| + (1-|z|^2)|g(z)||\nabla f(z)| \\
&\leq \frac{1}{2}||f||_{\infty,\alpha}||g|| + ||g||\frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right)(1-|z|^2)|\nabla f(z)| \\
&\leq \frac{1}{2}||f||_{\infty,\alpha}||g|| + ||g||\frac{1}{2}\log\left(\frac{1}{1-|z|^2}\right)(1-|z|^2)|\nabla f(z)| \\
&< \infty, \forall z \in \mathbb{B}_n.
\end{aligned}$$

That is $fg \in \mathcal{B}$.
Finally, to prove that $c)$ implies that $a)$, we proceed as follows. Recalling (3.2.5) of Corollary 3.2.14,

we have that

$$
\begin{aligned}
\lim_{|z|\to 1^-} \frac{(1-|z|^2)}{2}|\nabla(fg)(z)| &\leq \lim_{|z|\to 1^-} \frac{1}{2}(1-|z|^2)|f(z)||\nabla g(z)| + \frac{1}{2}\lim_{|z|\to 1^-}(1-|z|^2)|g(z)||\nabla f(z)| \\
&\leq \frac{1}{2}||f||_{\infty,\alpha} \lim_{|z|\to 1^-}(1-|z|^2)|\nabla g(z)| + \frac{1}{2}\lim_{|z|\to 1^-}(1-|z|^2)|g(z)||\nabla f(z)| \\
&= \frac{1}{2}\lim_{|z|\to 1^-}(1-|z|^2)|g(z)||\nabla f(z)| \\
&\leq \varepsilon(1-|z|^2)\log\left(\frac{1}{1-|z|^2}\right)|\nabla f(z)| \\
&\leq \varepsilon C.
\end{aligned}
$$

By the arbitrariness of ε, we deduce

$$\lim_{|z|\to 1^-}(1-|z|^2)|\nabla(fg)(z)| = 0,$$

that is,

$$fg \in \mathcal{B}_0.$$

$\square$

We conclude this section with the following remark. We prove that, if $f, g \in \mathcal{B}$, then, in general, it is not true that $fg \in \mathcal{B}$. For sake of simplicity, this fact is proved in the one-dimensional case.

Remark 3.6.5. Let $f = g = Log(1-z)$. Then,

$$
\begin{aligned}
\sup_{z\in\mathbb{B}_n}(1-|z|^2)|\nabla(fg)(z)| &= 2\sup_{z\in\mathbb{B}_n}(1-|z|^2)|f(z)||f'(z)| \\
&\geq 2(1-|z|^2)|f'(z)|\sup_{z\in\mathbb{B}_n}|f(z)| \\
&= \frac{2(1-|z|^2)|z|}{|1-z|}\left|\sum_{n=1}^{\infty}\frac{1}{n}\right| \\
&\geq 2|1-z||z|\left|\sum_{n=1}^{\infty}\frac{1}{n}\right| \\
&= \infty.
\end{aligned}
$$

That is, $fg \notin \mathcal{B}$. Moreover,

$$
\begin{aligned}
||f||_{\infty,\alpha} &= \left|\sum_{n=1}^{\infty}\frac{1}{n}\right| \\
&= \infty.
\end{aligned}
$$

Namely, we proved, with a counterexample, that if condition c) doesn't hold then condition a) fails.

3.7 Atomic Decomposition of $\mathcal{B}$ and $\mathcal{B}_0$

The aim of this section is to prove that the Bloch space admits an atomic decomposition. We showed that the Bloch space $\mathcal{B}$ is identified as the dual space of the Bergman space A^1_α, for $\alpha > -1$. Hence, it turns out that this decomposition is similar to that of the Bergman spaces. This section is organised as follows. We start recalling some tools, notions and a pair of operators, actually introduced in Section 2.5, that are necessary. Then, we provide a lemma that describes the action of the operators, previously introduced, in the Bloch space $\mathcal{B}$. Such lemma will be crucial for the most impotant result of this section. That is, the atomic decomposition of the Bloch space and, consequently, the little Bloch space.

We begin by recalling some facts studied in Section 2.5. Fix a parameter $b > n$ and consider a sequence $\{a_k\}_k$ that satisfies the condition of Theorem 2.5.4. According to Lemma 2.5.11, we proved that such sequence induces a partition, denoted by $\{D_k\}_k$, of $\mathbb{B}_n$. Moreover, after Lemma 2.5.11, we described a further partition of each D_k, denoted by $D_{k1}, ..., D_{kJ}$, into a finite number of disjoint pieces.
Denoting by $\alpha = b-(n+1)$, we are going to use the following two operators that act, respectively, on $L^1(\mathbb{B}_n, dv_\alpha)$ and $H(\mathbb{B}_n)$.

Definition 3.7.1. Let $f \in L^1(\mathbb{B}_n, dv_\alpha)$, we introduce

$$Tf(z) = \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{b-n-1}}{|1-<z,w>|^b} f(w)dv(w).$$

Then, let $f \in H(\mathbb{B}_n)$, we define

$$Sf(z) = \sum_{k=1}^{\infty}\sum_{j=1}^{J} \frac{v_\alpha(D_{kj})f(a_{kj})}{(1-<z,a_{kj}>)^b},$$

where $\{a_{kj}\}$ is the refinement lattice, in the Bergman metric, of $\{a_k\}$.

The following Lemma is a particular case of Lemma 2.5.14 and, of course, plays a fundamental role in the atomic decomposition of the Bloch space and the little Bloch space as well.

Lemma 3.7.2. *There exists a constant $C > 0$, independent of the separation constant r, for $\{a_k\}$, and the separation constant η, for $\{a_{kj}\}$, such that*

$$|f(z) - Sf(z)| \leq C\sigma T(|f|)(z), \tag{3.7.1}$$

for all $r \leq 1$, $z \in \mathbb{B}_n$, $f \in H(\mathbb{B}_n)$ and where

$$\sigma = \eta + \frac{tanh(\eta)}{tanh(r)}. \tag{3.7.2}$$

Proof. Putting $p = 0$ and $\alpha = 0$ in Lemma 2.5.14, so that we get

$$|f(z) - Sf(z)| \leq C\sigma \sum_{k=1}^{\infty} \frac{(1-|a_k|^2)^{b-(n+1)}}{|1-<z,a_k>|^b} \int_{D(a_k,2r)} |f(w)|dv(w).$$

After that, using (2.5.7), we can find a positive constant C_1 such that

$$|f(z) - Sf(z)| \leq C_1\sigma \sum_{k=1}^{\infty} \int_{D(a_k,2r)} \frac{(1-|w|^2)^{b-(n+1)}}{|1-<z,w>|^b} |f(w)|dv(w). \tag{3.7.3}$$

By the fact that each point z of $\mathbb{B}_n$ belongs at most N of $D(a_k, 2r)$, (3.7.3) must implie that

$$|f(z) - Sf(z)| \leq C_1\sigma N \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{b-(n+1)}}{|1-<z,w>|^b} |f(w)|dv(w),$$

and we are done. $\square$

We can prove the main result of this section: the atomic decomposition of the Bloch space $\mathcal{B}$.

Theorem 3.7.3. *For every $b > n$ there exists a sequence $\{a_k\}_k \subset \mathbb{B}_n$ such that every function f of the Bloch space $\mathcal{B}$ can be decomposed as follows*

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^b}{(1-<z, a_k>)^b}, \tag{3.7.4}$$

where $\{c_k\}_k \in l^\infty$. Furthermore, since $\mathcal{B}$ is identified as the dual space of A^1_α, for $\alpha = b-n-1$, the series on (3.7.4) converges in the weak-star topology of $\mathcal{B}$.

Proof. Let f defined by (3.7.4) and let $\{a_k\}_k$ be a sequence that satisfies the conditions of Theorem 2.5.15. In the first part of the proof we show that $f \in \mathcal{B}$. To this end, we start proving that the series (3.7.4) converges uniformly on compact subsets of $\mathbb{B}_n$, whenever $\{c_k\}_k$ is bounded, as follows. From Lemma 1.6.15, Lemma 2.5.1 and Corollary 2.5.2, there exists a positive constant C_1 such that

$$\begin{aligned}\sum_{k=1}^{\infty} |c_k|(1-|a_k|^2)^b &\leq C_1 \sum_{k=1}^{\infty} \int_{D(a_k, r/4)} (1-|z|^2)^\alpha dv(z) \\ &< C \int_{\mathbb{B}_n} (1-|z|^2)^\alpha dv(z) \\ &< \infty.\end{aligned}$$

Moreover, assuming that $\{c_k\}_k$ is bounded. Applying Theorem 1.4.4, we have that there exists a positive constant C_2 such that

$$\begin{aligned}||f||_{1,\alpha} &= \sum_{k=1}^{\infty} |c_k|(1-|a_k|^2)^b \int_{\mathbb{B}_n} \frac{dv_\alpha(z)}{|1-<z, a_k>|^b} \\ &\leq C_2 \sum_{k=1}^{\infty} |c_k|(1-|a_k|^2)^b \log\left(\frac{2}{1-|a_k|^2}\right).\end{aligned}$$

Then, from the estimate in the previous paragraph, it follows that for any $b' \in (n, b)$ there exists a positive constant C_3 such that

$$\begin{aligned}\sum_{k=1}^{\infty} |c_k|(1-|a_k|^2)^b \int_{\mathbb{B}_n} \frac{dv_\alpha(z)}{|1-<z, a_k>|^b} &\leq C_3 \sum_{k=1}^{\infty} |c_k|(1-|a_k|^2)^{b'} \\ &< \infty.\end{aligned}$$

In other words, we proved that (3.7.4) converges in the norm topology of A^1_α. To conclude this first part, we wish to prove that f induces a bounded linear functional on A^1_α under the integral pairing given by

$$<g, f>_\alpha = \int_{\mathbb{B}_n} g(z)\overline{f(z)} dv_\alpha(z), \tag{3.7.5}$$

where $g \in A^\infty_\alpha$. Replacing (3.7.4) in (3.7.5), using the fact that (3.7.4) converges uniformly on compact subsets and the reproducing formula of Lemma 2.1.3, we obtain

$$\begin{aligned}<g, f>_\alpha &= \int_{\mathbb{B}_n} g(z)\overline{f(z)} dv_\alpha(z) \\ &= \sum_{k=1}^{\infty} \overline{c_k}(1-|a_k|^2)^b \int_{\mathbb{B}_n} g(z) \frac{1}{(1-<a_k, z>)^{\alpha+n+1}} dv_\alpha(z) \\ &= \sum_{k=1}^{\infty} \overline{c_k}(1-|a_k|^2)^b g(a_k).\end{aligned}$$

So, the integral pairing is well defined. Hence, using Lemma 1.6.15 and Lemma 2.5.7, there exists a positive constant C_4 such that the above inner product can be estimated as follows,

$$
\begin{aligned}
| < g, f >_\alpha | &\leq \sum_{k=1}^{\infty} \int_{D(a_k, r/4)} |g(z)||f(z)| dv_\alpha(z) \\
&\leq \sum_{k=1}^{\infty} \int_{D(a_k, r/4)} |g(z)| \underbrace{\left\{ \int_{D(a_k, r/4)} |f(w)| dv_\alpha(w) \right\}}_{\leq \sup_{D(a_k,r/4)} |f(w)| v_\alpha(D(a_k, r/4))} \frac{1}{(1-|z^2|)^{\alpha+n+1}} dv_\alpha(z) \\
&\leq \sum_{k=1}^{\infty} \left[\sup_{D(a_k, r/4)} |f(w)| v_\alpha(D(a_k, r/4)) \right] \int_{D(a_k, r/4)} |g(z)| \frac{1}{(1-|z^2|)^{\alpha+n+1}} dv_\alpha(z) \\
&\leq \sup_k \left\{ \sup_{D(a_k, r/4)} |f(w)| \right\} \sum_{k=1}^{\infty} \int_{D(a_k, r/4)} |g(z)| \underbrace{\left\{ \frac{v_\alpha(D(a_k, r/4))}{(1-|z^2|)^{\alpha+n+1}} \right\}}_{Lemma\ 1.6.15} dv_\alpha(z) \\
&\leq \underbrace{C_4 \sup_k \left\{ \sup_{D(a_k, r/4)} |f(w)| \right\}}_{=:\hat{C}} \sum_{k=1}^{\infty} \int_{D(a_k, r/4)} |g(z)| dv_\alpha(z) \\
&= \hat{C} ||g||_{1,\alpha}.
\end{aligned}
$$

That is, using the fact that A_α^∞ is dense in A_α^1, we've proved that f induces a bounded linear functional on A_α^1. Hence, as a consequence of Theorem 3.4.1, f must be a Bloch function.
Furthermore, with some obvious minor adjustments, if the sequence $\{a_k\}_k$ is replaced by the more dense sequence $\{a_{kj}\}_{kj}$, then, the previous argument still works.
We wish to prove the other half of the theorem. Namely, if $f \in \mathcal{B}$, then f must consist of the form of (3.7.4). To this end, we need to introduce the following space of holomorphic fuctions

$$X := \left\{ f \in H(\mathbb{B}_n) \mid ||f||_X := \sup_{z \in \mathbb{B}_n} (1-|z|^2)|f(z)| < \infty \right\}.$$

X, equipped with $|| \cdot ||_X$ defined above, is a Banach space. After that, let S and T be as in Definition 3.7.1. If $f \in X$, replacing b by $b+1$ in Lemma 3.7.2, then, we must have that there exists a positive constant C_5, independent of the separation constant r for $\{a_k\}_k$ and the separation constant η for $\{a_{kj}\}_{kj}$, such that

$$|f(z) - Sf(z)| \leq C_5 \sigma \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{b-n} |f(w)| dv(w)}{|1- < z, w > |^{b+1}}, \forall z \in \mathbb{B}_n.$$

Moreover, from Theorem 1.4.4 there exists a positive constant C_6 such that

$$
\begin{aligned}
||f - Sf||_X &= \sup_{z\in\mathbb{B}_n}(1-|z|^2)|f(z)-Sf(z)| \\
&\le C_5\sigma \sup_{z\in\mathbb{B}_n}(1-|z|^2)\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{b-n}|f(w)|dv(w)}{|1-<z,w>|^{b+1}} \\
&\le C_5\sigma \sup_{z\in\mathbb{B}_n}(1-|z|^2)\int_{\mathbb{B}_n}\left\{\sup_{w\in\mathbb{B}_n}|f(w)|(1-|w|^2)\right\}\frac{(1-|w|^2)^{b-n-1}dv(w)}{|1-<z,w>|^{b+1}} \\
&= C_5\sigma||f||_X \sup_{z\in\mathbb{B}_n}(1-|z|^2)\underbrace{\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{b-n-1}dv(w)}{|1-<z,w>|^{n+1+(b-n-1)+1}}}_{\le(1-|z|^2)^{-1}} \\
&\le C_5C_6\sigma||f||_X.
\end{aligned}
$$

We've proved that

$$||f-Sf||_X \le C_5C_6\sigma||f||_X,\ \ f\in X.$$

Assumed that the separation constant η and r are so that $C_6\sigma<1$, then, denoting by I the identity operator, the operator $I-S$ has norm less than 1 on X. So, the operator S is invertible on X.
Let $f\in\mathcal{B}$, putting $\alpha=b-(n+1)$, we define the holomorphic function

$$g(z)=R^{\alpha,1}f(z), z\in\mathbb{B}_n.$$

By the fact that $R^{\alpha,1}f$ is a differential operator of order 1 having polynomial coefficients, we obtain

$$g\in X.$$

Since S is invertible, defining $h=S^{-1}g\in X$, g admits the following representation

$$g(z)=\sum_{k=1}^{\infty}\sum_{j=1}^{J}\frac{v_\beta(D_{kj})h(a_{kj})}{(1-<z,a_{kj}>)^{b+1}}, \tag{3.7.6}$$

where

$$\beta=(b+1)-(n+1)=b-n.$$

Applying the inverse of $R^{\alpha,1}$, denoted by $R_{\alpha,1}$, to (3.7.6) so that, from Proposition 1.5.10, we get that the following representation holds

$$
\begin{aligned}
f(z) &= R_{\alpha,1}g(z) \\
&= R_{\alpha,1}\sum_{k=1}^{\infty}\sum_{j=1}^{J}\frac{v_\beta(D_{kj})h(a_{kj})}{(1-<z,a_{kj}>)^{b+1}} \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{J}v_\beta(D_{kj})h(a_{kj})R_{\alpha,1}\left(\frac{1}{(1-<z,a_{kj}>)^{b+1}}\right) \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{J}\frac{v_\beta(D_{kj})h(a_{kj})}{(1-<z,a_{kj}>)^{b}} \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{J}c_{kj}\frac{(1-|a_{kj}|^2)^{b}}{(1-<z,a_{kj}>)^{b}},
\end{aligned}
$$

where

$$c_{kj} = \frac{v_\beta(D_{kj})h(a_{kj})}{(1-|a_{kj}|^2)^b}.$$

Finally, since $h \in X$ and

$$\begin{aligned} v_\beta(D_{kj}) &\le v_\beta(D_k) \\ &\sim (1-|a_k|^2)^{n+1+\beta} \\ &= (1-|a_k|^2)^{b+1} \\ &\sim (1-|a_{kj}|^2)^{b+1}. \end{aligned}$$

That is

$$\{c_{kj}\} \in l^\infty.$$

This completes the proof. □

From the proof of the preceding theorem we obtain the following corollary.

Corollary 3.7.4. *The Bloch norm*

$$||f|| := |f(0)| + \sup_{z\in\mathbb{B}_n} |\nabla(f \circ \varphi_z)(0)|,$$

is comparable to

$$\inf\left\{ ||\{c_k\}||_\infty \;:\; f(z) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^b}{(1-<z,a_k>)^b}, \right\}$$

Our nex goal is to obtain the atomic decomposition of the little Bloch space. To this end, with some adjustments, we adopt the proof of Theorem 3.7.3.

Theorem 3.7.5. *For any $b > n$ there exists a sequence $\{a_k\}$ in $\mathbb{B}_n$ such that the little Bloch space $\mathcal{B}_0$ consists exactly of functions of the form*

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^b}{(1-<z,a_k>)^b}, \tag{3.7.7}$$

where

$$\lim_{k\to\infty} c_k = 0.$$

Proof. We proceed following the same lines as the proof of Theorem 3.7.3. Assume that f admits the decomposition of (3.7.7) and, then, we want to prove that $f \in \mathcal{B}_0$. From Theorem 3.7.3, there exists a positive constant $C > 0$ suh that

$$\left\| \sum_{k=1}^{\infty} c_k \left(\frac{1-|a_k|^2}{1-<z,a_k>} \right)^b \right\| \le C \sup_{k\ge 1} |c_k|,$$

where we recall that $\|\cdot\|$ is the Bloch norm. If $c_k \to 0$, then

$$\lim_{N\to\infty} ||f - f_N|| = 0,$$

where $\{f_N\}$ denotes the partial sum of f. By the fact that each f_N is an element of $\mathcal{B}_0$, we deduce that $f \in \mathcal{B}_0$ whenever $c_k \to 0$.

After that, we prove that every function of $\mathcal{B}_0$ must admit the decomposition of (3.7.7). So, we introduce the following holomorphic functional space:

$$X_0 := \left\{ f \in H(\mathbb{B}_n) \mid (1-|z|^2)f(z) \in C_0(\mathbb{B}_n) \right\},$$

and consider the action of the operator S on X_0, with paremeter $b+1$ instead of b. We notice that the differential operator $R^{\alpha,1}$ is an invertible operator from $\mathcal{B}_0$ onto X_0. Furthermore, when the separation constant r for $\{a_k\}$ is small enough, S is invertible on X_0. Then, from the proof of Theorem 3.7.3, since $f \in \mathcal{B}_0 \subset \mathcal{B}$, f admits the following representation

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^b}{(1-<z,a_k>)^b},$$

where

$$c_{kj} = \frac{v_\beta(D_{kj})h(a_{kj})}{(1-|a_{kj}|^2)^b}, \ \forall h \in X_0.$$

Since,

$$v_\beta(D_{kj}) \leq v_\beta(D_k)$$

$$\sim (1-|a_k|^2)^{b+1},$$

and $1-|a_{kj}|^2$ is comparable to $1-|a_k|^2$, the condition

$$\lim_{k\to\infty}(1-|a_{kj}|^2)h(a_{kj}) = 0$$

implies that there exists a finite positive constant C such that

$$\lim_{k\to\infty} c_{kj} \leq \lim_{k\to\infty} \frac{v_\beta(D_{kj})h(a_{kj})}{(1-|a_{kj}|^2)^b}$$

$$\leq \lim_{k\to\infty} \frac{(1-|a_k|^2)^{b+1}h(a_{kj})}{(1-|a_{kj}|^2)^b}$$

$$\leq C \lim_{k\to\infty} \frac{(1-|a_k|^2)^{b+1}h(a_{kj})}{(1-|a_k|^2)^b}$$

$$= C \lim_{k\to\infty}(1-|a_k|^2)h(a_{kj})$$

$$= 0.$$

That is,

$$\lim_{k\to\infty} c_{kj} = 0.$$

□

3.8 Complex Interpolation

The Bloch space behaves like the limit of the Bergman space A^p_α, when $p \to \infty$. In fact, in this section, we further remark this type of behaviour illustrating the complex interpolation between $\mathcal{B}$ and the weighted Bergman spaces A^p_α, for $\alpha > -1$. So that, the fact that $\mathcal{B}$ belongs to every Bergman space is fundamental. We will see that the Bergman projection and the complex interpolation of L^p spaces will be crucial tools for our aim.

In the following theorem we prove the result concerning the complex interpolation of $\mathcal{B}$ with A^p_α, for $\alpha > -1$.

Theorem 3.8.1. *Assume* $\alpha > -1$ *and*

$$\frac{1}{p} = \frac{1-\theta}{p'},$$

where $\theta \in (0,1)$ *and* $1 \leq p' < \infty$. *Then, the Bloch space interpolates with the Bergman spaces as follows*

$$\left[A^{p'}_\alpha, \mathcal{B}\right] = A^p_\alpha \tag{3.8.1}$$

with equivalent norms.

Proof. Fix a real number β so that $\beta > \alpha$. According to Theorem 2.2.9, the Bergman projection P_β is bounded from $L^q(\mathbb{B}_n, dv_\alpha)$ onto A^q_α, for $1 \leq q < \infty$. Furthermore, from Theorem 3.1., P_β maps $L^\infty(\mathbb{B}_n)$ boundedly onto $\mathcal{B}$.
If $f \in A^p_\alpha \subset L^p(\mathbb{B}_n, dv_\alpha)$, we know, from the complex interpolation of L^p spaces, that there exists a family of functions h_ζ in

$$L^p(\mathbb{B}_n, dv_\alpha) + L^\infty(\mathbb{B}_n) = L^p(\mathbb{B}_n, dv_\alpha) \tag{3.8.2}$$

such that the following conditions hold
a) h_ζ depends on the parameter ζ continuously in $0 \leq Re\ \zeta \leq 1$ and analytically in $0 < Re\ \zeta < 1$.
b) $h_\zeta \in L^{p'}(\mathbb{B}_n, dv_\alpha)$ for $Re\ \zeta = 0$ and $h_\zeta \in L^\infty(\mathbb{B}_n)$ for $Re\ \zeta = 1$, with

$$\sup\left\{||h_\zeta||^{p'}_{p',\alpha} : Re\ \zeta = 0\right\} \leq ||h||^p_{p,\alpha},$$

and

$$\sup\left\{||h||_\infty : Re\ \zeta = 1\right\} \leq ||h||_{p,\alpha},$$

c) $f = h_\theta$. Let $f_\zeta = P_\beta h_\zeta$. Then $f_\zeta \in A^{p'}_\alpha$ when $Re\ \zeta = 0$, $f_\zeta \in \mathcal{B}$ for $Re\ \zeta = 1$, and $f_\theta = f$. Appropriate norm estimates also holds for $Re\ \zeta = 0$ and $Re \zeta = 1$. This shows that

$$f \in [A^{p'}_\alpha, \mathcal{B}]_\theta.$$

Conversely, if $f \in [A^{p'}_\alpha, \mathcal{B}]_\theta$, then there exists a family of functions f_ζ in

$$A^{p'}_\alpha + \mathcal{B} = A^{p'}_\alpha,$$

where the parameter ζ satisfies $0 \leq Re\ \zeta \leq 1$, such that

1) f_ζ depends on the parameter ζ continuously in $0 \leq Re\ \zeta \leq 1$ and analytically in $0 < Re\ \zeta < 1$.

2) $||f_\zeta||_{p',\alpha}$ for all $Re\ \zeta = 0$ and $||f_\zeta||_{\mathcal{B}} \leq ||f||_\theta$ for all $Re\ \zeta = 1$.

3) $f = f_\theta$.

Define

$$h_\zeta(z) = \frac{c_{\beta+1}}{c_\beta}(1-|z|^2)\left(f_\zeta(z) + \frac{Rf_\zeta(z)}{n+1+\beta}\right),$$

where $0 \leq Re\ \zeta \leq 1$. Using Theorem 2.4.8

$$||h_\zeta||_{p',\alpha} \leq C||f_\zeta||_{p',\alpha},\ Re\ \zeta = 0,$$

and by Theorem 2.2.9

$$||h_\zeta||_\infty \leq C||f_\zeta||_{\mathcal{B}},\ Re\ \zeta = 1.$$

Using the complex interpolation for L^p spaces , we have

$$h_\theta \in L^p(\mathbb{B}_n, dv_\alpha).$$

Since $f_\theta = P_\beta h_\theta$, we conclude that

$$f \in A^p_\alpha.$$

This completes the proof. □

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