# Concentration of infinitely many solutions for the finite case of a nonlinear Schrödinger equation with critical-frequency potential

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**Abstract**

We consider a nonlinear Schrödinger equation with critical frequency, $(\mathrm{P}_\varepsilon)$ : $\varepsilon^2\,\Delta v(x) - V(x)\,v(x) + |v(x)|^{p-1}\,v(x) = 0$, $x \in \mathbb{R}^N$, with $v(x) \to 0$ as $|x| \to +\infty$, for the *finite case* as described by Byeon and Wang. *Critical* means that the continuous non-negative potential $V$ verifies $\mathcal{Z} = \{V = 0\} = \{x_0\} \neq \emptyset$, and *finite* means, grossly speaking, that as one comes close to $\mathcal{Z}$, the potential decays like an positive even polynomial. As the Planck constant, $\varepsilon$, tends to zero, the finite case has a characteristic semiclassical limit problem, $(\mathrm{P}_{\mathrm{fin}})$ : $\Delta u(x) - P(x)\,u(x) + |u(x)|^{p-1}\,u(x) = 0$, $x \in \mathbb{R}^N$, with $u(x) \to 0$ as $|x| \to +\infty$, which differs from the limit problems corresponding to the *flat* and *infinite cases*. We prove the existence of an infinite number of solutions for both the original and the limit problem, via a Ljusternik-Schnirelman scheme that uses Krasnoselskii's genus. Fixed a topological level $k$ we prove that $v_{k,\varepsilon}$, a solution of $(\mathrm{P}_\varepsilon)$, subconverges, up to a scaling, to a solution of $(\mathrm{P}_{\mathrm{fin}})$, and that $v_{k,\varepsilon}$ exponentially decays out of $\mathcal{Z}$.

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## 1. Introduction

A number of phenomena involving atomic and molecular collisions can be analyzed with enough accuracy by using the asymptotic method known as semiclassical mechanics. By letting the Planck constant tend to zero, one can deal with quantum mechanics problems by transforming them into classical mechanics objects that are mathematically easier to handle, [1].

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The time-dependent nonlinear Schrödinger equation,

$$i\hbar\,\Psi_t + \frac{\hbar^2}{2}\Delta\Psi - V_0(x)\Psi + |\Psi|^{p-1}\Psi = 0, \tag{1}$$

is an appropriate tool to study the evolution of quantum systems like Bose-Einstein condensates, [2], as well as to model the propagation of light in some nonlinear optical materials, [3]. When $\hbar$ is very small, a semi-classical state of (1) is a standing-wave having the form $\Psi(x,t) = v(x)\,\exp(-iEt/\hbar)$, where $v$ verifies

$$\varepsilon^2\Delta v(x) - V(x)\,v(x) + |v(x)|^{p-1}v(x) = 0, \tag{2}$$

with

$$\varepsilon^2 = \hbar^2/2, \quad V(x) = V_0(x) - E, \quad E = \inf(V_0).$$

Let's assume that

$$\mathcal{Z} = \left\{ x \in \mathbb{R}^N \;/\; V(x) = \inf(V) \right\} \neq \emptyset.$$

The case of $\inf(V) > 0$, referred to as *non-critical case*, has a unique limit problem,

$$\Delta u - u + |u|^{p-1}u = 0, \tag{3}$$

which is well-known and was used to study (2) in a number of works (see e.g. [4], [5], [6], [7], [8], [9], [10]) by using the Lyapunov-Schmidt reduction or the variational method.

The case of $\inf(V) < 0$ is much less meaningful from the physics point of view and there is no nice limit problem, [11]. When $\mathcal{Z}$ is bounded it's not possible to find least energy solutions (mountain-pass solutions) but the problem can still be treated as in [12] and [13], at least for the one-dimensional and radial cases, respectively. In this context the case of $\inf(V) = 0$ corresponds to a critical frequency or energy and, as we shall comment below, the qualitative behaviour of the solutions of (2) changes dramatically compared with its non-critical counterpart.

In the mentioned works, [4], [5], [6], [7], [8], [9] and [10], several common characteristics were found for the non-critical case $\inf(V) > 0$:

(N1) $v_\varepsilon^*$, a solution of (2), is bounded away from zero,

$$\liminf_{\varepsilon\to 0} \max_x |v_\varepsilon(x)| > 0; \tag{4}$$

(N2) $v_\varepsilon^*$ concentrates around some critical points of $V$;

(N3) $v_\varepsilon^*$ exponentially decays to zero away from such critical points, as $\varepsilon \to 0$; and,

(N4) there is a unique limit problem, (3), and, therefore, a unique profile, as $\varepsilon \to 0$.

In this paper we continue the analytical study of the critical-frequency problem

$$\begin{cases} \varepsilon^2 \Delta v(x) - V(x)\, v(x) + |v(x)|^{p-1} v(x) = 0, & x \in \mathbb{R}^N, \\ v(x) \to 0, & \text{as } |x| \to +\infty, \end{cases} \tag{$P_\varepsilon$}$$

$p > 1$, that was initiated in [11] and continued in [14]. In [11] it's proved the existence of $v_\varepsilon$, a positive standing wave, a least energy solution, for which

(C1) property (4) stops holding, giving pass to the following behaviour:

$$\lim_{\varepsilon \to 0} \|v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 0, \tag{5}$$

$$\liminf_{\varepsilon \to 0} \frac{\|v_\varepsilon\|_{L^\infty(\mathbb{R}^N)}}{\varepsilon^{2/(p-1)}} > 0; \tag{6}$$

(C2) $v_\varepsilon$ concentrates around an isolated component of $\mathcal{Z} = \{V = 0\}$;
(C3) $v_\varepsilon$ exponentially decays out of the region $\mathcal{Z}$; and,
(C4) there is no unique limit problem and, consequently, neither is there a unique profile; actually it depends on the behavior of $V$ nearby $\mathcal{Z}$.

Three cases were considered: *flat case*, where $\text{int}(\mathcal{Z})$ is non-empty and bounded; *finite case*, where $\mathcal{Z}$ is finite and $V$ vanishes polynomially around it; and, *infinite case*, where $\mathcal{Z}$ is finite and $V$ vanishes exponentially around it. The flat and infinite cases have their limit problems defined on appropriate regions of $\mathbb{R}^N$, meanwhile the limit problem for the finite case is defined on the whole space $\mathbb{R}^N$. For the three cases it was also shown that

(C5) a scaling of the positive least-energy solution $v_\varepsilon$ converges to $u$, a positive least-energy solution of the corresponding limit problem;
(C6) the energy of $v_\varepsilon$ converges to the energy of $u$.

The work [14] focuses on the flat case, assuming that the potential verifies:

(V1) $V \in C(\mathbb{R}^N)$ is non-negative;
(V2) $V(x) \to +\infty$, as $|x| \to +\infty$;
($V_{\text{flat}}$) $\text{int}(\mathcal{Z}) \neq \emptyset$ is connected and smooth.

Here the limit problem is

$$\begin{cases} \Delta u(x) + |u(x)|^{p-1} u(x) = 0, & x \in \text{int}(\mathcal{Z}), \\ u(x) = 0, & x \in \partial \mathcal{Z}. \end{cases} \tag{$P_{\text{flat}}$}$$

The authors showed the existence of sequences of solutions, $(v_{k,\varepsilon})_{k\in\mathbb{N}}$ and $(w_k)_{k\in\mathbb{N}}$, for ($P_\varepsilon$) and ($P_{\text{flat}}$), respectively. Fixed $k$, they proved that, as $\varepsilon \to 0$, a solution $v_{k,\varepsilon}$ - not necessarily positive - behaves like the positive solution found in [11], that is, (C1), (C2) and (C3) hold. Point (C6) also holds:

$$\lim_{\varepsilon \to 0} I_\varepsilon(v_{k,\varepsilon}) = I(w_k),$$

where $I_\varepsilon$ and $I$ are standard functionals associated to ($P_\varepsilon$) and ($P_{\text{flat}}$), respectively. Point (C5) holds in the sense that a scaling of $v_{k,\varepsilon}$ converges, up to subsequences, to $u_k$ a solution of ($P_{\text{flat}}$) sharing the energy level of $w_k$:

$$I(w_k) = I(u_k).$$

**Remark 1.1.** Condition (V2) is more restrictive than that assumed in [11] were, for some $\gamma > 0$, $\liminf_{|x|\to+\infty} V(x) > 2\gamma$.

In short, in this paper we prove that the type of results of [14] also holds for the finite case. The general framework which characterize the finite case, see (V3) and ($\text{V}_{\text{fin}}$), is introduced in Section 2 in a precise way. These conditions provoke a limit problem that, as was already mentioned, differs from those for the flat and infinite cases, and creates its own interesting technical difficulties; in particular, it's required a control over the potential far away from $\mathcal{Z}$, see condition (PQ). The main results are detailed in Section 2:

- Theorem 2.8 states the existence of sequences $(v_{k,\varepsilon})_{k\in\mathbb{N}}$ and $(w_{k,\varepsilon})_{k\in\mathbb{N}}$ of different solutions, respectively, for ($P_\varepsilon$) and its limit problem

  $$\begin{cases} \Delta u(x) - P(x)u(x) + |u(x)|^{p-1}\cdot u(x) = 0, & x\in\mathbb{R}^N, \\ u(x)\to 0, & \text{as } x\to+\infty. \end{cases} \tag{$P$}$$

  This is dealt with in Section 3 by setting up a Ljusternik-Schnirelman scheme for the functionals $J_\varepsilon$ and $J$ associated to ($P_\varepsilon$) and ($P$), respectively.
- Theorem 2.9 states the convergence of $c_{k,\varepsilon}$, the energy of a scaled version of $v_{k,\varepsilon}$, to $c_k$, the energy of a scaled version of $w_k$. In the context of the Ljusternik-Schnirelman machinery, an index $k$ of a critical value represents the topological characteristic of the level set, as captured by Krasnoselskii's genus. Therefore Theorem 2.9 also says that the level sets of the functionals $J_\varepsilon$ and $J$ are equivalent. The proof of this property is the topic of Section 4.
- Theorem 2.10 states the asymptotic profiles as $\varepsilon\to 0$, i.e., up to a scaling and up to subsequences, $v_{k,\varepsilon}$ converges to $u_k$, a solution of ($P$) which shares the critical energy $c_k$. The proof of this result is presented in Section 5.
- In Theorem 2.11 is stated that, up to a scaling, $v_{k,\varepsilon}$ exponentially decays out of $\mathcal{Z}$. The proofs of this and other concentration phenomena are presented in Section 5.

## 2. General framework and main results

*2.1. Problem setting*

As was mentioned, we consider the problem

$$\begin{cases} \varepsilon^2\Delta v(x) - V(x)\,v(x) + |v(x)|^{p-1}v(x) = 0, & x\in\mathbb{R}^N, \\ v(x)\to 0, & \text{as } |x|\to+\infty, \end{cases} \tag{$P_\varepsilon$}$$

where

$$\begin{cases} 2 < 1+p < 2^* = 2N/(N-2), & \text{if } N \geq 3; \\ 2 < 1+p, & \text{if } N = 1,2, \end{cases} \tag{7}$$

and, in addition to properties (V1) and (V2), we shall assume that the potential $V(\cdot)$ verifies a couple of conditions which replace $(\mathrm{V_{flat}})$. One of them is

(V3) $\overline{\mathcal{Z}} = \{x \in \mathbb{R}^N : V(x) = 0\} = \{x_0\}$.

Actually for simplicity, we shall assume that $x_0 = 0$. The second condition, $(\mathrm{V_{fin}})$ below, differentiates our situation with that of the infinite case and corresponds, grossly speaking, to $V$ decaying at a polynomial rate as we get close to $\overline{\mathcal{Z}}$. For its statement we need the concept of $m$-homogeneous positive function, as given in [11].

$(\mathrm{V_{fin}})$ for each $x \in \mathbb{R}$, $V(x) = P(x) + Q(x)$, where, for some $m > 0$, $Q \in C(\mathbb{R}^N)$ is such that

$$\lim_{|x|\to 0} |x|^{-m} Q(x) = 0, \tag{8}$$

and $P$ is a $m$-homogeneous positive function, i.e., $P \in C(\mathbb{R})$ and

$$\begin{aligned} &\forall x \in \mathbb{R} \setminus \{0\} : && P(x) > 0; \\ &\forall x \in \mathbb{R}, \forall t \geq 0 : && P(tx) = t^m P(x). \end{aligned} \tag{9}$$

Given $\varepsilon > 0$ we shall denote

$$\begin{aligned} V_\varepsilon(x) &= \varepsilon^{-2m/(m+2)} \cdot V\left(\varepsilon^{2/(m+2)} x\right) \\ &= P(x) + \varepsilon^{-2m/(m+2)} \cdot Q\left(\varepsilon^{2/(m+2)} x\right). \end{aligned} \tag{10}$$

Therefore, since $V$ is continuous and non-negative, so is $V_\varepsilon$.

The following easy result provides a control of $P$ over $Q$ and $V_\varepsilon$ around zero that shall be useful.

**Lemma 2.1.** *Let $\alpha, \beta, \varepsilon > 0$.*

1. *There exists $R_{\varepsilon,\alpha,\beta} > 0$ such that*

$$\forall x \in B(0, R_{\varepsilon,\alpha,\beta}) : \quad \varepsilon^{-2m/(m+2)} \left| Q\left(\varepsilon^{2/(m+2)} x\right) \right| \leq \frac{\alpha}{\beta} P(x). \tag{11}$$

2. *There exists $R_{\varepsilon,\alpha} > 0$ such that*

$$\forall x \in B_{\varepsilon,\alpha} : \quad \varepsilon^{-2m/(m+2)} \left| Q\left(\varepsilon^{2/(m+2)} x\right) \right| \leq \alpha P(x) \tag{12}$$

*as well as*

$$\forall x \in B_{\varepsilon,\alpha} : \quad (1-\alpha) P(x) \leq V_\varepsilon(x) \leq (1+\alpha) P(x), \tag{13}$$

*where $B_{\varepsilon,\alpha} = B(0, R_{\varepsilon,\alpha})$. It also holds*

$$\lim_{\varepsilon \to 0} R_{\varepsilon,\alpha} = +\infty.$$

*Proof.* 1. By (8), we have that

$$\forall h > 0, \exists \delta(h) > 0 : \quad |x| < \delta(h) \Rightarrow |x|^{-m} |Q(x)| < h. \tag{14}$$

Then we put $\delta_{\alpha,\beta} = \min \{\delta(h_1), \delta(h_2)\}$, where

$$h_1 = \frac{\alpha}{\beta} \min_{|z|=1} P(z) > 0, \quad h_2 = \frac{\alpha}{\beta} \max_{|z|=1} P(z) > 0. \tag{15}$$

Let's define

$$R_{\alpha,\varepsilon,\beta} = \delta_{\alpha,\beta} \cdot \varepsilon^{-2/(m+2)}. \tag{16}$$

Let $x \in B(0, R_{\alpha,\varepsilon,\beta}) \setminus \{0\}$. Since $\left|\varepsilon^{2/(m+2)} x\right| < \delta_{\alpha,\beta}$, point (14) implies that

$$-|x|^m h_2 < \varepsilon^{-2m/(m+2)} Q\left(\varepsilon^{2/(m+2)} x\right) < |x|^m h_1,$$

so that, by (9) and (15) and taking $z = x/|x|$, we get

$$-\frac{\alpha}{\beta} P(x) \leq \varepsilon^{-2m/(m+2)} Q\left(\varepsilon^{2/(m+2)} x\right) \leq \frac{\alpha}{\beta} P(x). \tag{17}$$

By the continuity of $P$ and $Q$, it's clear that the last relation also holds for $x = 0$. Since $x$ was chosen arbitrarily, we have proved (11).

2. In (16) we put

$$\delta_\alpha = \delta_{\alpha,1}, \quad R_{\varepsilon,\alpha} = R_{\varepsilon,\alpha,1}, \tag{18}$$

so that (17) and (10) provide (13).

□

**Remark 2.2.** Observe that by (16), point (13) in Lemma 2.1 can be rewritten as

$$\forall \alpha > 0, \exists \delta_\alpha > 0, \forall x \in B(0, \delta_\alpha) : \quad (1-\alpha) P(x) \leq V(x) \leq (1+\alpha) P(x).$$

The last condition, (PQ), which we immediately introduce, is a technical one and shall be used in Section 4; it requieres the function $Q$ to be controled by $P$ far away from zero.

(PQ) $Q$ is non-negative and there exist $\rho, \eta > 0$ such that

$$\forall x \in \mathbb{R}^N \setminus B(0, \rho) : \quad Q(x) \leq \eta\, P(x),$$

which is equivalent to

$$\forall \varepsilon > 0, \forall |x| \geq \varepsilon^{-2/(m+2)} \rho : \quad \varepsilon^{-2m/(m+2)}\, Q(\varepsilon^{2/(m+2)} x) \leq \eta\, P(x). \tag{19}$$

**Remark 2.3.** The following problems are closely related to $(P_\varepsilon)$:

$$\begin{cases} \Delta w(x) - V_\varepsilon(x)\, w(x) + |w(x)|^{p-1} w(x) = 0, & x \in \mathbb{R}^N, \\ w(x) \to 0, & \text{as } |x| \to +\infty, \end{cases} \tag{$P'_\varepsilon$}$$

$$
\begin{cases} \Delta \hat{w}(x) - V_\varepsilon(x)\, \hat{w}(x) + 2\Lambda\, |\hat{w}(x)|^{p-1} \hat{w}(x) = 0, & x \in \mathbb{R}^N, \\ \hat{w}(x) \to 0, & \text{as } |x| \to +\infty, \end{cases} \qquad (\hat{P}_\varepsilon)
$$

where $\Lambda = \|\hat{w}\|_\varepsilon^2 /2$ and $\|\cdot\|_\varepsilon$ is given in (22) below. In fact, if $\hat{w}$ is a solution of $(\hat{P}_\varepsilon)$, then

$$
w(x) = (2\Lambda)^{1/(p-1)} \hat{w}(x), \quad x \in \mathbb{R}^N,
$$

is a solution of $(P'_\varepsilon)$, and

$$
\begin{aligned} v(x) &= \varepsilon^{2m/(m+2)(p-1)}\, w\left(\varepsilon^{-2/(m+2)}\, x\right) \\ &= \left[2\Lambda \cdot \varepsilon^{2m/(m+2)}\right]^{1/(p-1)} \hat{w}\left(\varepsilon^{-2/(m+2)}\, x\right), \qquad x \in \mathbb{R}^N, \end{aligned} \qquad (20)
$$

is a solution of $(P_\varepsilon)$.

**Remark 2.4.** Under conditions (V1), (V2), (V3) and $(\mathrm{V_{fin}})$ the limit problem of $(P_\varepsilon)$ is

$$
\begin{cases} \Delta w(x) - P(x) w(x) + |w(x)|^{p-1} \cdot w(x) = 0, & x \in \mathbb{R}^N, \\ w(x) \to 0, & \text{as } x \to +\infty. \end{cases} \qquad (P_{\text{fin}})
$$

Connected to $(P_{\text{fin}})$ is the problem

$$
\begin{cases} \Delta \hat{w}(x) - P(x) \hat{w}(x) + 2\Gamma\, |\hat{w}(x)|^{p-1}\, \hat{w}(x), & x \in \mathbb{R}^N, \\ \hat{w}(x) \to 0, & \text{as } |x| \to +\infty, \end{cases} \qquad (\hat{P}_{\text{fin}})
$$

where $\Gamma = \|\hat{u}\|_P^2 /2$ and $\|\cdot\|_P$ is given in (23) below. In fact, if $\hat{w}$ is a solution of $(\hat{P}_{\text{fin}})$, then

$$
w(x) = (2\Gamma)^{1/(p-1)} \hat{w}(x), \quad x \in \mathbb{R}^N, \qquad (21)
$$

is a solution of $(P_{\text{fin}})$.

*2.2. Main results*

We shall look for solutions of $(P_\varepsilon)$ and $(P_{\text{fin}})$ in the Hilbert spaces $\mathrm{H}_\varepsilon$ and $\mathrm{H}_P$, defined as the completions of $\mathrm{C}_0^\infty(\mathbb{R}^N)$ in the norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_P$ induced, respectively, by the inner products

$$
\begin{aligned} (u,v)_\varepsilon &= \int_{\mathbb{R}^N} \left[\nabla u(x) \cdot \nabla v(x) + V_\varepsilon(x) u(x) v(x)\right] dx, \\ (u,v)_P &= \int_{\mathbb{R}^N} \left[\nabla u(x) \cdot \nabla v(x) + P(x) u(x) v(x)\right] dx. \end{aligned}
$$

**Remark 2.5.** The non-negativity of $Q$ implies that $\|u\|_P \leq \|u\|_\varepsilon$, for all $u \in \mathrm{H}_\varepsilon$, so that the embedding $\mathrm{H}_\varepsilon \subseteq \mathrm{H}_P$ is continuous.

The following result states that a weighted Sobolev space such that the weight-function verifies (V1) and (V2) is compactly contained in a range of $\mathrm{L}^q$-spaces.

**Theorem 2.6.** *Assume that $W \in C(\mathbb{R}^N)$ is non-negative and such that $W(x) \to +\infty$ as $|x| \to +\infty$. Let $H_W$ be the Hilbert space that results of completing $C_0^\infty(\mathbb{R}^N)$ whenever is equipped with the interior product given by*

$$(v,w)_W = \int_{\mathbb{R}^N} [\nabla v(x) \nabla w(x) + W(x)\, v(x) w(x)]\; dx.$$

*Then, the embedding*

$$H_W \subseteq L^q(\mathbb{R}^N),$$

*is compact for all $q \in [2, r[$, where $r = 2^*$ if $N \geq 3$, and $r = +\infty$ if $N = 1, 2$. For $q = r$ the embedding is continuous.*

Theorem 2.6 is obtained by an application of [15, Cor.4.26 & 4.27], by compensating the non-boundedness of the domain with the property of $U$ exploding at infinity.

To state our main results we also need to define functionals associated to the problems $(P_\varepsilon)$ and $(P_{\text{fin}})$. Let's consider $J_\varepsilon : \mathcal{M}_\varepsilon \subseteq H_\varepsilon \to \mathbb{R}$ and $J : \mathcal{M} \subseteq H_P \to \mathbb{R}$ given by

$$J_\varepsilon(u) = \frac{1}{2}\, \|u\|_\varepsilon^2 = \frac{1}{2} \int_{\mathbb{R}^N} \left[ |\nabla u(x)|^2 + V_\varepsilon(x) |u(x)|^2 \right] dx, \tag{22}$$

$$J(u) = \frac{1}{2}\, \|u\|_P^2 = \frac{1}{2} \int_{\mathbb{R}^N} \left[ |\nabla u(x)|^2 + P(x) |u(x)|^2 \right] dx, \tag{23}$$

where we are considering the Nehari manifolds

$$\mathcal{M}_\varepsilon = \left\{ u \in H_\varepsilon \,/\, \|u\|_{L^{p+1}(\mathbb{R}^N)} = 1 \right\},$$
$$\mathcal{M} = \left\{ u \in H_P \,/\, \|u\|_{L^{p+1}(\mathbb{R}^N)} = 1 \right\}.$$

**Remark 2.7.** Let's observe that for $u \in H_\varepsilon$,

$$\|u\|_\varepsilon^2 = \|u\|_P^2 + \Theta_\varepsilon(u),$$

where

$$\Theta_\varepsilon(u) = \varepsilon^{-2m/(m+2)} \int_{\mathbb{R}^N} Q\left( \varepsilon^{2/(m+2)} x \right) \cdot |u(x)|^2 \, dx.$$

Lemma 2.1 implies that for all $\alpha, \varepsilon > 0$ and all $u \in H_\varepsilon$,

$$\begin{aligned} \varepsilon^{-2m/(m+2)} \int_{B_{\varepsilon,\alpha}} Q\left( \varepsilon^{2/(m+2)} x \right) \cdot |u(x)|^2 \, dx &\leq \alpha \int_{B_{\varepsilon,\alpha}} P(x) |u(x)|^2 dx \\ &\leq \alpha \int_{\mathbb{R}^N} P(x) |u(x)|^2 dx, \end{aligned}$$

so that for all $\alpha > 0$ and $u \in H_\varepsilon$,

$$\begin{aligned} \lim_{\varepsilon \to 0} \Theta_\varepsilon(u) &= \lim_{\varepsilon \to 0} \varepsilon^{-2m/(m+2)} \int_{B_{\varepsilon,\alpha}} Q\left( \varepsilon^{2/(m+2)} x \right) \cdot |u(x)|^2 \, dx \\ &\leq \alpha \int_{\mathbb{R}^N} P(x) |u(x)|^2 dx. \end{aligned}$$

Therefore, it also holds, for $\alpha > 0$ and $u \in \mathrm{H}_P$,

$$(1-\alpha)\,\|u\|_P^2 \le \lim_{\varepsilon\to 0} \|u\|_\varepsilon^2 \le (1+\alpha)\,\|u\|_P^2\,. \tag{24}$$

Let's assume that (V1), (V2), (V3), ($\mathrm{V_{fin}}$), (PQ) and (7) hold. Our first main result is a multiplicity one.

**Theorem 2.8.** *The following points are true.*

*i) Let $\varepsilon > 0$. The functional $J_\varepsilon$ has a sequence of different critical points $(\hat{w}_{k,\varepsilon})_{k\in\mathbb{N}} \subseteq \mathcal{M}_\varepsilon$. For each $k \in \mathbb{N}$ the function given by*

$$v_{k,\varepsilon}(x) = \left[2c_{k,\varepsilon}\cdot\varepsilon^{2m/(m+2)}\right]^{1/(p-1)} \hat{w}_{k,\varepsilon}\left(\varepsilon^{-2/(m+2)}\,x\right), \quad x \in \mathbb{R}^N, \tag{25}$$

*where $c_{k,\varepsilon} = J_\varepsilon(\hat{w}_{k,\varepsilon})$, is a solution of $(P_\varepsilon)$.*

*ii) The functional $J$ has a sequence of different critical points $(\hat{w}_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$. For each $k \in \mathbb{N}$ the function given by*

$$w_k(x) = (2c_k)^{1/(p-1)}\,\hat{w}_k(x),$$

*where $c_k = J(\hat{w}_k)$, is a solution of $(P_{\mathrm{fin}})$.*

To prove Theorem 2.8 we shall use a Ljusternik-Schnirelman scheme so that, in this context, the index $k$ of a critical value represents the topological characteristic of the level set, as captured by Krasnoselskii's genus. Therefore, the convergence of energies, which we are going to write, means that the critical values of $J$ and $J_\varepsilon$ are topologically equivalent.

**Theorem 2.9.** *Let $k \in \mathbb{N}$. Then*

$$\lim_{\varepsilon\to 0} c_{k,\varepsilon} = c_k.$$

To state the following result, let's recall the concept of subconvergence introduced in [11]. A family of functions $(f_\varepsilon)_{\varepsilon>0}$ is said to subconverge in a space $X$, as $\varepsilon \to 0$, iff from every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to zero, it is possible to extract a subsequence $(\varepsilon_{n_i})_{i\in\mathbb{N}}$ such that $\left(f_{\varepsilon_{n_i}}\right)_{i\in\mathbb{N}}$ converges in $X$, as $i \to \infty$.

**Theorem 2.10.** *Let $k \in \mathbb{N}$. As $\varepsilon \to 0$, $(w_{k,\varepsilon})_{\varepsilon>0}$ subconverges in $\mathrm{H}_P$ to some $u_k \in \mathcal{M}$ which is a solution of $(P_{\mathrm{fin}})$ and verifies*

$$J(\hat{u}_k) = c_k, \qquad \hat{u}_k = (2c_k)^{1/(1-p)} u_k.$$

Finally we have the result concernig the exponential decay out of $\mathcal{Z}$.

**Theorem 2.11.** *Let $\mu, \delta, c > 0$. Then there exist $\mathring{\varepsilon}, C > 0$ such that for all $\varepsilon \in ]0, \mathring{\varepsilon}[$ and $|x| > \mu + \delta\,\varepsilon^{-2/(m+2)}$ it holds*

$$|w_{k,\varepsilon}(x)| \le C\cdot\exp\left(-c\,\varepsilon^{-m/(m+2)}\left[|x| - \mu - \delta\,\varepsilon^{-2/(m+2)}\right]\right).$$

To finish this section let's mention that in the path of proving Theorem 2.11, for each $k \in \mathbb{N}$, we shall get the properties

$$\lim_{\varepsilon \to 0} \left\| v_{k,\varepsilon} \right\|_{L^\infty(\mathbb{R}^N)} = 0,$$
$$\liminf_{\varepsilon \to 0} \frac{\left\| v_{k,\varepsilon} \right\|_{L^\infty(\mathbb{R}^N)}}{\varepsilon^{2m/[(p-1)(m+2)]}} \geq \eta_k > 0,$$

which are analogous to (5) and (6).

## 3. Multiplicity

In this section we show how a Ljusternik-Schnirelman scheme provides Theorem 2.8 in a very direct way. Given $E$, a Banach space, we write

$$\Sigma_E = \left\{ A \subseteq E : \, A = \overline{A}, \, A = -A, \, 0 \notin A \right\}.$$

The genus of $A \in \Sigma_E$, denoted by $\gamma(A)$ is the least natural number $k$ for which there exists a continuous odd function $f : A \to \mathbb{R}^k \backslash \{0\}$, see e.g. [16] and [9]). If there is not such $k$, then $\gamma(A) = +\infty$, and, by definition, $\gamma(\emptyset) = 0$.

**Remark 3.1.** It's important to keep in mind that if $A \in \Sigma_E$, then $A$ is closed in the $\| \cdot \|_E$-norm.

Krasnoselskii's genus generalizes the notion of dimension: $\gamma\left(\mathbb{S}^{m-1}\right) = m$ and $\gamma\left(\mathbb{S}^\infty_Y\right) = +\infty$, where $\mathbb{S}^{m-1}$ is the unit-sphere in $\mathbb{R}^m$ and $\mathbb{S}^\infty_Y$ is the unit-sphere in an infinite-dimensional Banach space $Y$. In the following proposition (see e.g. [16]) the basic properties of $\gamma$ are stated.

**Proposition 3.2.** *Let $A, B \in \Sigma_E$. Then*

$$\begin{aligned} x \neq 0 \; &\Rightarrow \; \gamma(\{x\} \cup \{-x\}) = 1, \\ f \in C(A, B) \text{ odd} \; &\Rightarrow \; \gamma(A) \leq \gamma(B), \\ A \subseteq B \; &\Rightarrow \; \gamma(A) \leq \gamma(B), \\ \gamma(A \cup B) &\leq \gamma(A) + \gamma(B), \\ A \text{ compact} \; &\Rightarrow \; \gamma(A) < +\infty. \end{aligned} \tag{26}$$

**Remark 3.3.** Let $M$ be a $C^1$ manifold in $X$, a Banach space, and $\phi \in C^1(M)$. Let's recall that $(y_n)_{n \in \mathbb{N}} \subseteq M$ is a Palais-Smale (PS) sequence iff $(\phi(y_n))_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is bounded, and $\left\| \phi'(y_n) \right\|_{X^*} \to 0$, as $n \to +\infty$. We say that $(M, \phi)$ verifies (PS) condition if any (PS) sequence has a converging subsequence.

The following theorem, [16], is our main tool.

**Theorem 3.4.** *Let $M \in \Sigma_E$ be a $C^1$ manifold of $E$ and let $f \in C^1(E)$ be even. Suppose that $(M, f)$ satisfy the Palais-Smale (PS) condition and let*

$$C_k(f) = \inf_{A \in \mathcal{A}_k(M)} \max_{u \in A} f(u),$$
$$\mathcal{A}_k(M) = \left\{ A \in \Sigma_E \cap M : \, \gamma(A) \geq k \right\}.$$

*Let's denote by $K_c$ the set of critical points of $f$ corresponding to the value $c$. Then* *a)* $\gamma(M) \leq \sum_{c\in\mathbb{R}} \gamma(K_c)$ *so that* $f$ *has at least* $\gamma(M)$ *pairs of critical points on* $M$.

*b) If* $C_k(f) \in \mathbb{R}$, *then* $C_k(f)$ *is a critical value for* $f$. *Moreover, if*

$$c = C_k(f) = \cdots = C_{k+m}(f),$$

*then* $\gamma(K_c) \geq m+1$. *In particular, if* $m > 1$, *then* $K_c$ *contains infinitely many elements.*

The potentials $P$, $V$ and $V_\varepsilon$ verify the conditions of Theorem 2.6 so that, in particular, the result holds for $\mathrm{H}_P$ and $\mathrm{H}_\varepsilon = \mathrm{H}_{V_\varepsilon}$. With this ingredient, it is proved that the functionals $J$ and $J_\varepsilon$ are of class $\mathrm{C}^1$ and satisfy the Palais-Smale condition on $\mathcal{M}$ and $\mathcal{M}_\varepsilon$, respectively. Then, in the context of Theorem 3.4 and having in mind Remark 3.1, we write, for $k \in \mathbb{N}$ and $\varepsilon > 0$,

$$\Sigma_\varepsilon = \Sigma_{\mathrm{H}_\varepsilon} = \left\{A \subseteq \mathrm{H}_\varepsilon \;/\; A = \overline{A},\, A = -A,\, 0 \notin A\right\},$$
$$\Sigma = \Sigma_{\mathrm{H}_P} = \left\{A \subseteq \mathrm{H}_P \;/\; A = \overline{A},\, A = -A,\, 0 \notin A\right\},$$
$$\mathcal{A}_{k,\varepsilon} = \mathcal{A}_k(\mathcal{M}_\varepsilon) = \left\{A \in \Sigma_\varepsilon \;/\; \gamma(A) \geq k \;\;\wedge\;\; \forall u \in A:\, \|u\|_{\mathrm{L}^{p+1}(\mathbb{R}^N)} = 1\right\},$$
$$\mathcal{A}_k = \mathcal{A}_k(\mathcal{M}) = \left\{A \in \Sigma \;/\; \gamma(A) \geq k \;\;\wedge\;\; \forall u \in A:\, \|u\|_{\mathrm{L}^{p+1}(\mathbb{R}^N)} = 1\right\}.$$

The $k$-th critical values are achieved:

$$c_{k,\varepsilon} = C_k(J_\varepsilon) = \inf_{A\in\mathcal{A}_{k,\varepsilon}} \max_{u\in A} J_\varepsilon(u) = J_\varepsilon(\hat{w}_{k,\varepsilon}), \tag{27}$$

$$c_k = C_k(J) = \inf_{A\in\mathcal{A}_k} \max_{u\in A} J(u) = J(\hat{w}_k). \tag{28}$$

**Remark 3.5.** In the context just presented we have used the fact that $\gamma(\mathcal{M}) = \gamma(\mathcal{M}_\varepsilon) = +\infty$. The assertion that $v_{k,\varepsilon}$ and $w_k$ are solutions of $(P_\varepsilon)$ and $(P_{\mathrm{fin}})$, in Theorem 2.8, comes by the changes of variables (20) and (21), respectively. Also observe that in the proof of Theorem 2.8 we didn't use condition (PQ).

## 4. Convergence of energies

The proof of Theorem 2.9,

$$\forall k \in \mathbb{N}: \quad \lim_{\varepsilon\to 0} c_{k,\varepsilon} = c_k, \tag{29}$$

is given by Propositions 4.2 and 4.3, below.

**Lemma 4.1.** *Let* $k \in \mathbb{N}$ *and* $\alpha, \varepsilon > 0$. *Then,* $\mathrm{H}_P = \mathrm{H}_\varepsilon$ *and the norms* $\|\cdot\|_\varepsilon$ *and* $\|\cdot\|_P$ *are equivalent.*

*Proof.* To ease the proof let's introduce the following notation for annular regions of $\mathbb{R}^N$. For $\varepsilon > 0$ and $\mu_2 > \mu_1 > 0$,

$$G_{\mu_1,\mu_2} = \overline{B}(0,\mu_2) \setminus B(0,\mu_1), \qquad G^\varepsilon_{\mu_1,\mu_2} = \overline{B}(0,\mu_2 \cdot \varepsilon^{-2/m+2}) \setminus B(0,\mu_1 \cdot \varepsilon^{-2/m+2}).$$

Let's assume that $\delta_\alpha < \rho$, where $\delta_\alpha$ is given in (18). The case of $\rho \leq \delta_\alpha$ is easier so that it's omitted. Let $u \in \mathrm{H}_P$. Then, by (12) and (19), it follows that

$$\begin{aligned}
\Theta_\varepsilon(u) &= \varepsilon^{-2m/(m+2)} \int_{\mathbb{R}^N} Q\left(\varepsilon^{2/(m+2)} x\right) \cdot |u(x)|^2 \, dx \\
&= \varepsilon^{-2m/(m+2)} \left[ \int_{B_{\varepsilon,\alpha}} \cdots + \int_{G^\varepsilon_{\delta_\alpha,\rho}} \cdots + \int_{\mathbb{R}^N \setminus B(0,\rho\,\varepsilon^{-2/(m+2)})} \cdots \right] \\
&\leq \alpha \int_{B_{\varepsilon,\alpha}} P(x)|u(x)|^2 dx + \eta \int_{\mathbb{R}^N \setminus B(0,\rho\,\varepsilon^{-2/(m+2)})} P(x)|u(x)|^2 dx + \\
&\quad + \varepsilon^{-2m/(m+2)} \, \|Q\|_{\mathrm{L}^\infty\left(G_{\delta_\alpha,\rho}\right)} \cdot \int_{G^\varepsilon_{\delta_\alpha,\rho}} |u(x)|^2 dx \\
&\leq \tau \int_{\mathbb{R}^N} P(x) \, |u(x)|^2 \, dx, \qquad (30)
\end{aligned}$$

where

$$\tau = \max\left\{ \alpha, \eta, \frac{\|Q\|_{\mathrm{L}^\infty\left(G_{\delta_\alpha,\rho}\right)}}{\inf\limits_{y \in G_{\delta_\alpha,\rho}} P(y)} \right\},$$

and we have used the relation

$$0 < \inf_{y \in G^\varepsilon_{\delta_\alpha,\rho}} P(y) = \varepsilon^{-2m/(m+2)} \inf_{y \in G_{\delta_\alpha,\rho}} P(y),$$

which directly comes from the homegeneity of $P$. Then, by (30), we get

$$\begin{aligned}
\|u\|^2_\varepsilon &= \|u\|^2_P + \Theta_\varepsilon(u) \\
&\leq \|u\|^2_P + \tau \int_{\mathbb{R}^N} P(x) \, |u(x)|^2 \, dx \\
&\leq (1+\tau) \, \|u\|^2_P \, ,
\end{aligned}$$

which shows that the immersion $\mathrm{H}_P \subseteq \mathrm{H}_\varepsilon$ is continuous as $u$ was chosen arbitrarily. The last together with Remark 2.5 let us conclude the proof. □

**Proposition 4.2.** *Let $k \in \mathbb{N}$ and $\varepsilon > 0$. Then, it holds*

$$c_k \leq c_{k,\varepsilon}.$$

*Proof.* By Lemma 4.1, a set $W \subseteq \mathrm{H}_P$ open (closed) in the $\|\cdot\|_\varepsilon$-norm is also open (closed) in the $\|\cdot\|_P$-sense. Then, having in mind Remarks 2.5 and 3.1 as well as point (26), it follows that $\mathcal{A}_{k,\varepsilon} \subseteq \mathcal{A}_k$ and

$$\begin{aligned}
c_k &= \inf_{A \in \mathcal{A}_k} \max_{u \in A} J(u) \\
&\leq \inf_{A \in \mathcal{A}_{k,\varepsilon}} \max_{u \in A} J(u) \\
&\leq \inf_{A \in \mathcal{A}_{k,\varepsilon}} \max_{u \in A} J_\varepsilon(u) \\
&= c_{k,\varepsilon}.
\end{aligned}$$

□

**Proposition 4.3.** *Let $k \in \mathbb{N}$ and $\sigma > 0$. Then*

$$\limsup_{\varepsilon \to 0} c_{k,\varepsilon} \leq c_k + \sigma.$$

*Proof.* Let $\varepsilon > 0$. By Lemma 4.1, a set $W \subseteq \mathrm{H}_P = \mathrm{H}_\varepsilon$ open (closed) in the $\|\cdot\|_P$-norm is also open (closed) in the $\|\cdot\|_\varepsilon$-sense. Then it follows that $\mathcal{A}_k \subseteq \mathcal{A}_{k,\varepsilon}$ and, for all $\tilde{A} \in \mathcal{A}_k$,

$$\begin{aligned} c_{k,\varepsilon} &= \inf_{A \in \mathcal{A}_{k,\varepsilon}} \max_{u \in A} J_\varepsilon(u) \\ &\leq \inf_{A \in \mathcal{A}_k} \max_{u \in A} J_\varepsilon(u) \\ &\leq \max_{u \in \tilde{A}} J_\varepsilon(u). \end{aligned} \tag{31}$$

Now we choose $A_\sigma \in \mathcal{A}_k$ such that

$$\max_{u \in A_\sigma} J(u) \leq \inf_{A \in \mathcal{A}_k} \max_{u \in A} J(u) + \frac{\sigma}{2} = c_k + \frac{\sigma}{2}. \tag{32}$$

Let's pick

$$\alpha = \frac{\sigma/2}{c_k + \sigma/2} > 0. \tag{33}$$

Then, by (24), (31), (32) and (33), we get

$$\begin{aligned} \limsup_{\varepsilon \to 0} c_{k,\varepsilon} &\leq \limsup_{\varepsilon \to 0} \max_{u \in A_\sigma} J_\varepsilon(u) \\ &\leq \max_{u \in A_\sigma} \limsup_{\varepsilon \to 0} J_\varepsilon(u) \\ &\leq (1+\alpha) \max_{u \in A_\sigma} J(u) \\ &\leq \left(1 + \frac{\sigma/2}{c_k + \sigma/2}\right) \cdot (c_k + \sigma/2) \\ &= c_k + \sigma, \end{aligned} \tag{34}$$

where we have used the relation

$$\limsup_{\varepsilon \to 0} \max_{u \in A_\sigma} J_\varepsilon(u) \leq \max_{u \in A_\sigma} \limsup_{\varepsilon \to 0} J_\varepsilon(u). \tag{35}$$

To show (35) let's pick $(M_r)_{r \in \mathbb{N}} \subseteq \mathbb{R}$ such that

$$\forall r \in \mathbb{N}: \quad \max_{u \in A_\sigma} \limsup_{\varepsilon \to 0} J_\varepsilon(u) < M_r,$$

and

$$\lim_{r \to +\infty} M_r = \max_{u \in A_\sigma} \limsup_{\varepsilon \to 0} J_\varepsilon(u).$$

Let's fix $r \in \mathbb{N}$. Then, for all $u \in A_\sigma$,

$$\limsup_{\varepsilon \to 0} J_\varepsilon(u) < M_r.$$

Therefore, for all $u \in A_\sigma$ there exists $\varepsilon_u > 0$ such that

$$\forall \varepsilon \in ]0, \varepsilon_u[: \qquad J_\varepsilon(u) < M_r.$$

By a contradiction argument we prove that $\varepsilon_\sigma = \inf_{u \in A_\sigma} \varepsilon_u > 0$. Then

$$\forall \varepsilon \in ]0, \varepsilon_\sigma[, \forall u \in A_\sigma : \quad J_\varepsilon(u) < M_r,$$

and

$$\forall \varepsilon \in ]0, \varepsilon_\sigma[: \quad \limsup_{\varepsilon \to 0} \max_{u \in A_\sigma} J_\varepsilon(u) \leq M_r,$$

whence we obtain (35) by letting $r \to +\infty$. □

## 5. Asymptotic profiles and concentration phenomena

Let's prove the asymptotic profiles stated in Theorem 2.10, that is, for a fixed $k \in \mathbb{N}$, as $\varepsilon \to 0$, $(w_{k,\varepsilon})_{\varepsilon > 0}$ subconverges in $\mathrm{H}_P$ to some $u_k \in \mathcal{M}$ which is a solution of $(P_{\text{fin}})$ and verifies

$$\begin{aligned} J(\hat{u}_k) &= c_k, \\ u_k &= (2c_k)^{1/(1-p)} \hat{u}_k. \end{aligned} \tag{36}$$

***Proof of Theorem 2.10.***

1. Let us prove that $w_{k,\varepsilon}$ weakly subconverges to some $u_k \in H_P$. Let $\delta > 0$. By (29) there is $\bar{\varepsilon}_\delta > 0$ such that

   $$\forall \varepsilon \in ]0, \bar{\varepsilon}_\delta[: \quad c_{k,\varepsilon} \leq c_k + \delta, \tag{37}$$

   whence,

   $$\forall \varepsilon \in ]0, \bar{\varepsilon}_\delta[: \quad \|\hat{w}_{k,\varepsilon}\|_P^2 \leq \|\hat{w}_{k,\varepsilon}\|_\varepsilon^2 = 2c_{k,\varepsilon} \leq 2(c_k + \delta),$$

   so that $(\hat{w}_{k,\varepsilon})_{\varepsilon \in (0, \bar{\varepsilon}_\delta)}$ is bounded in $H_P$. By [15, Th.3.18], $\hat{w}_{k,\varepsilon}$ weakly subconverges to some $\hat{u}_k \in H_P$, as $\varepsilon \to 0$. By Remarks 2.3 and 2.4 and point (29) we have that $w_{k,\varepsilon} = (2c_k)^{1/(p-1)} \hat{w}_{k,\varepsilon}$ weakly subconverges to $u_k$, given by (36), as $\varepsilon \to 0$.
2. Let us prove that $u_k$ is a weak solution of $(P_{\text{fin}})$. Point 1 implies that $\hat{w}_{k,\varepsilon}$ subconverges to $\hat{u}_k$ point-wise almost everywhere. From Theorem 2.8 and Remark 2.3, we have for $\phi \in \mathrm{C}_0^\infty(\mathbb{R}^N)$ that

   $$\int_{\mathbb{R}^N} (\nabla \hat{w}_{k,\varepsilon} \cdot \nabla \phi + V_\varepsilon \hat{w}_{k,\varepsilon} \phi \, dx) = 2c_{k,\varepsilon} \int_{\mathbb{R}^N} |\hat{w}_{k,\varepsilon}|^{p-1} \hat{w}_{k,\varepsilon} \phi \, dx. \tag{38}$$

   Since $Q$ is $o(h^m)$, it immediately follows, for all $\phi \in \mathrm{C}_0^\infty(\mathbb{R}^N)$, that

   $$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \varepsilon^{\frac{-2m}{m+2}} Q\left(\varepsilon^{\frac{2}{m+2}} x\right) \hat{w}_{k,\varepsilon}(x) \phi(x) \, dx = \\ &\quad = \lim_{\varepsilon \to 0} \int_{\text{supp}(\phi)} \varepsilon^{\frac{-2m}{m+2}} Q\left(\varepsilon^{\frac{2}{m+2}} x\right) \hat{w}_{k,\varepsilon}(x) \phi(x) \, dx = 0. \end{aligned} \tag{39}$$

Therefore, by passing to the limit when $\varepsilon \to 0$ in (38), we have by (29) and (39), that

$$\int_{\mathbb{R}^N} (\nabla \hat{u}_k \cdot \nabla \phi + P\hat{u}_k \phi \, dx) = 2c_k \int_{\mathbb{R}^N} |\hat{u}_k|^{p-1} \hat{u}_k \phi \, dx, \tag{40}$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$, i.e., $u_k$ is a weak solution of $(P_{\text{fin}})$. By a density argument, (40) holds for all $\phi \in \mathrm{H}_P$. Therefore, by taking $\phi = \hat{u}_k$ in (40), we get that $J(\hat{u}_k) = c_k$.

3. By Proposition 4.3 and the non-negativeness of $Q$, we get that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \|\hat{w}_{k,\varepsilon}\|_P^2 &\leq \limsup_{\varepsilon\to 0} \|\hat{w}_{k,\varepsilon}\|_\varepsilon^2 \leq 2 \limsup_{\varepsilon\to 0} c_{k,\varepsilon} \\ &\leq 2\, c_k = \|\hat{u}_k\|_P^2. \end{aligned} \tag{41}$$

Since $H_P$ is a Hilbert space, it is also a uniformly convex Banach space. This, together with (41) and point 1 provide, by [15, Prop.3.32], the subconvergence of $w_{k,\varepsilon}$ to $u_k$ in the norm $H_P$, as $\varepsilon \to 0$.

□

For the last part of this paper, devoted to the proof of Theorem 2.11, let's strengthen the assumption (V1) by requiring that

(V1$_\eta$) For some $\eta > 0$, $V \in C^\eta(\mathbb{R}^N)$.

Then, by using standard regularity arguments, it follows that $v_{k,\varepsilon}$ and $w_k$ belong to $C^{2,\eta}(\mathbb{R}^N)$ and that they are classical solutions of $(P_\varepsilon)$ and $(P_{\text{fin}})$, respectively.

We shall use the following result.

**Proposition 5.1.** *Let $D$ be an open and connected subset of $\mathbb{R}^N$. If $w \in H_0^1(D)$ is a classical subsolution of the elliptic problem*

$$\begin{cases} \Delta w - f(w) \geq 0, & \text{in } D, \\ w > 0, & \text{in } D, \\ w = 0, & \text{on } \partial D, \end{cases}$$

*where $N \geq 3$, $p+1 \in ]2, 2^*[$ and for all $t \in \mathbb{R}^+$*

$$t f(t) \leq c t^{p+1},$$

*for some $c > 0$, there exists $C = C(c, p, N) > 0$ such that*

$$\|w\|_{L^\infty(D)} \leq C \|w\|_{L^{2^*}(D)}^{4/[N+2-p(N-2)]}.$$

A proof of Proposition 5.1 is provided in [13] under the conditions that $A$ is smooth and bounded. Nevertheless, as it's mentioned in [14], the proof can be modified to release the constraints of boundedness and regularity of the domain.

**Proposition 5.2.** *Let $k \in \mathbb{N}$, $\delta > 0$ and $\bar{\varepsilon}_\delta > 0$ as in* (37). *Then there exists $K_\delta > 0$ such that*

$$\forall \varepsilon \in (0, \bar{\varepsilon}_\delta): \quad \left\| w_{k,\varepsilon} \right\|_{\mathrm{L}^\infty\left(\mathbb{R}^N\right)} \leq K_\delta. \tag{42}$$

*Proof.* Let's assume that $N \geq 3$ as the cases $N = 1, 2$ are easier. Let $\varepsilon \in (0, \bar{\varepsilon}_\delta)$ and $D_\varepsilon^+$ a connected component of $W_{k,\varepsilon}^+ = \{x \in \mathbb{R}^N \ / \ w_{k,\varepsilon} > 0\}$. Then, since $w_{k,\varepsilon}$ is a solution of $(P'_\varepsilon)$, we have that

$$\begin{cases} \Delta w_{k,\varepsilon} + w_{k,\varepsilon}^p \geq 0, & \text{in } D_\varepsilon^+, \\ w_{k,\varepsilon} > 0, & \text{in } D_\varepsilon^+, \\ w_{k,\varepsilon} = 0, & \text{on } \partial D_\varepsilon^+. \end{cases}$$

Therefore, by Proposition 5.1, we get

$$\| w_{k,\varepsilon} \|_{L^\infty(D_\varepsilon^+)} \leq C \| w_{k,\varepsilon} \|_{L^{2^*}(D_\varepsilon^+)}^{4/[N+2-p(N-2)]}. \tag{43}$$

On the other hand, by Theorem 2.6 and (37), we have that

$$\frac{1}{2} \left\| \hat{w}_{k,\varepsilon} \right\|_{\mathrm{L}^{2^*}\left(D_\varepsilon^+\right)}^2 \leq \left\| \hat{w}_{k,\varepsilon} \right\|_{\mathrm{L}^{2^*}\left(\mathbb{R}^N\right)}^2 \leq \frac{K_1}{2} \| \hat{w}_{k,\varepsilon} \|_\varepsilon^2 = K_1 c_{k,\varepsilon} \leq K_1 (c_k + \delta).$$

From this and (43), there is $K_\delta > 0$ such that

$$\forall \varepsilon \in (0, \bar{\varepsilon}_\delta): \quad \left\| w_{k,\varepsilon} \right\|_{\mathrm{L}^\infty\left(W_{k,\varepsilon}^+\right)} \leq K_\delta,$$

because $D_\varepsilon^+$ was chosen arbitrarily. The same result can be worked out for $W_{k,\varepsilon}^- = \{x \in \mathbb{R}^N \ / \ w_{k,\varepsilon} < 0\}$ . ☐

**Remark 5.3.** By the definition of $v_{k,\varepsilon}$, (25), we see that Proposition 5.2 immediately implies that

$$\lim_{\varepsilon \to 0} \left\| v_{k,\varepsilon} \right\|_{\mathrm{L}^\infty\left(\mathbb{R}^N\right)} = 0.$$

Moreover, since for all $k \in \mathbb{N}$ and all $\varepsilon > 0$, $\left\| \hat{w}_{k,\varepsilon} \right\|_{\mathrm{L}^{p+1}\left(\mathbb{R}^N\right)} = 1$, it's possible to find $\eta_k > 0$ such that

$$\liminf_{\varepsilon \to 0} \frac{\left\| v_{k,\varepsilon} \right\|_{\mathrm{L}^\infty\left(\mathbb{R}^N\right)}}{\varepsilon^{2m/[(p-1)(m+2)]}} \geq \eta_k > 0.$$

**Remark 5.4.** To prove Theorem 2.11, the exponential decay of $w_{k,\varepsilon}$, out of $\mathcal{Z}$ we shall use the following comparison result. Given $a, b, d > 0$ and $D \subseteq \mathbb{R}^N$ bounded, let $U$ be a positive solution of the problem

$$\begin{cases} \Delta U - 2\, b\, U = 0, & x \in \mathbb{R}^N \setminus D^d, \\ U = a, & x \in \partial D^d, \\ \lim\limits_{|x| \to +\infty} U(x) = 0. \end{cases} \tag{44}$$

Then $U$ verifies

$$U(x) \leq C \cdot \exp\{-b \cdot \operatorname{dist}(x, D^d)\}, \quad x \in \mathbb{R}^N \setminus D^d,$$

where $C = C(a, d)$ and

$$D^d = \{x \in \mathbb{R}^N \,/\, \operatorname{dist}(x, D) < d\}.$$

Let's recall the statement of Theorem 2.11. Given $\mu, \delta, c > 0$, there are values $\mathring{\varepsilon}, C > 0$ such that for all $\varepsilon \in ]0, \mathring{\varepsilon}[$ and $|x| > \mu + \delta\, \varepsilon^{-2/(m+2)}$ it holds

$$|w_{k,\varepsilon}(x)| \leq C \cdot \exp\left(-c\, \varepsilon^{-m/(m+2)} \left[|x| - \mu - \delta\, \varepsilon^{-2/(m+2)}\right]\right).$$

***Proof of Theorem 2.11***. Let us consider $\bar{\varepsilon}_\delta > 0$ as in (37) and $K_\delta$ as in Proposition 5.2. Let's pick $\mathring{\varepsilon} \in ]0, \bar{\varepsilon}_\delta[$ such that

$$P_\delta = \inf_{|x|>\delta} P(x) > \left(K_\delta + 2\, c\, \mathring{\varepsilon}^{\frac{-m}{m+2}}\right) \mathring{\varepsilon}^{\frac{2m}{m+2}}. \tag{45}$$

Let $\varepsilon \in ]0, \mathring{\varepsilon}[$. By (45) and the homogeneity of $P$ it holds

$$\begin{aligned}
P_{\delta,\varepsilon} &= \inf\{P(x) \colon\ |x| > \delta\, \varepsilon^{-2/(m+2)}\} = \inf_{|y|>\delta} P\left(\varepsilon^{-2/(m+2)} y\right) \\
&= \varepsilon^{-2m/(m+2)} \inf_{|y|>\delta} P(y) = \varepsilon^{-2m/(m+2)} P_\delta \\
&> K_\delta + 2\, c\, \varepsilon^{-m/(m+2)}
\end{aligned}$$

From this and Proposition 5.2, we have, for $|x| > \delta\varepsilon^{-2/(m+2)}$, that

$$\begin{aligned}
T_{k,\varepsilon}(x) &= V_\varepsilon(x) - |w_{k,\varepsilon}|^{p-1} \\
&\geq P(x) - |w_{k,\varepsilon}|^{p-1} \\
&\geq P_{\delta,\varepsilon} - |w_{k,\varepsilon}|^{p-1} \\
&\geq P_{\delta,\varepsilon} - K_\delta \\
&> 2\, c\, \varepsilon^{-m/(m+2)}.
\end{aligned} \tag{46}$$

Let us now consider $U$, a positive solution of (44), with

$$a = K_\delta, \quad b = c\, \varepsilon^{-m/(m+2)}, \quad d = \delta\varepsilon^{-2/(m+2)},$$

and, for some $\mu > 0$

$$D = B(0, \mu), \qquad D^d = B(0, \mu + \delta\varepsilon^{-2/(m+2)}),$$

i.e.,

$$\begin{cases}
\Delta U - 2c\, \varepsilon^{-m/(m+2)} U = 0, & |x| > \mu + \delta\varepsilon^{-2/(m+2)}, \\
U = K_\delta, & |x| = \mu + \delta\varepsilon^{-2/(m+2)}, \\
\lim\limits_{|x|\to+\infty} U(x) = 0.
\end{cases}$$

Thus, by (46),

$$\begin{cases} \Delta U - T_{k,\varepsilon}(x)U \leq 0, & |x| > \mu + \delta\varepsilon^{-2/(m+2)}, \\ U = K_\delta, & |x| = \mu + \delta\varepsilon^{-2/(m+2)}, \\ \lim\limits_{|x|\to+\infty} U(x) = 0. \end{cases} \tag{47}$$

Since $w_{k,\varepsilon}$ solves $(P'_\varepsilon)$, from (47) and (42) it holds

$$\begin{cases} \Delta(U - w_{k,\varepsilon}) - T_{k,\varepsilon}(x)(U - w_{k,\varepsilon}) \leq 0, & |x| > \mu + \delta\varepsilon^{-2/(m+2)}, \\ U - w_{k,\varepsilon} > 0, & |x| = \mu + \delta\varepsilon^{-2/(m+2)}, \\ \lim\limits_{|x|\to+\infty} (U(x) - w_{k,\varepsilon}) = 0. \end{cases} \tag{48}$$

From (48), we get by the weak maximum principle (see e.g. [17]),

$$w_{k,\varepsilon}(x) \leq U(x), \quad |x| > \mu + \delta\varepsilon^{-2/(m+2)}.$$

In an analogous way it is proved that

$$-U(x) \leq -w_{k,\varepsilon}(x), \quad |x| > \mu + \delta\varepsilon^{-2/(m+2)}.$$

Therefore, by Remark 5.4, there exists $C = C(\delta, \varepsilon) > 0$ such that, for $|x| > \mu + \delta\,\varepsilon^{-2/(m+2)}$, it holds

$$|w_{k,\varepsilon}(x)| < U(x) \leq C \cdot \exp\left(-c\,\varepsilon^{-m/(m+2)}\left[|x| - \mu - \delta\,\varepsilon^{-2/(m+2)}\right]\right).$$

We conclude by the arbitrariness of $\varepsilon$. □

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