Multiplicity of solutions for a p-Schrödinger-Kirchhoff-type integro-differential equation

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Abstract. We consider the integro-differential problem (P):

$$-\left(a+b\left(\int_{\mathbb{R}^N}|\nabla u|^p dx\right)^{p-1}\right)\Delta_p u + V(x)|u|^{p-2}u = f(x,u), \quad x\in\mathbb{R}^N,$$

with $|u(x)|\longrightarrow 0$, as $|x|\longrightarrow +\infty$. We assume that $a,b>0$, $N\geq 2$, $1<p<N<+\infty$, $V\in \mathrm{C}(\mathbb{R}^N)$ with $\inf(V)>0$, and that $f:\mathbb{R}^N\times\mathbb{R}\longrightarrow\mathbb{R}$ verifies conditions introduced by Duan and Huang. We prove the existence of a non-trivial ground state solution and, by a Ljusternik-Schnirelman scheme, the existence of infinitely many non-trivial solutions.

Mathematics Subject Classification (2010). Primary 45K05; Secondary 35J60.

Keywords. p-Schrödinger-Kirchhoff-type equation, Ljusternik-Schnirelman theory, critical point theory.

1. Introduction

We consider the following Schrödinger-Kirchhoff-type integro-differential problem

$$\begin{cases}-\left(a+b\left(\int_{\mathbb{R}^N}|\nabla u|^p dx\right)^{p-1}\right)\Delta_p u + V(x)|u|^{p-2}u = f(x,u), \quad x\in\mathbb{R}^N,\\ |u(x)|\longrightarrow 0, \quad \text{as } |x|\longrightarrow +\infty,\end{cases} \tag{P}$$

where $a,b>0$, $1<p<N<+\infty$ and $N\geq 2$.

Non-local problems like (P) with $p=2$ have been used to model physical and biological phenomena where the density $u(x)$ at the point x is affected by the average of u on its whole domain (see e.g. [1], [7], [10] and [16] and the references therein). In this context, problem (P) considers a more complicated situation where the nonlinear diffusion process is also governed by the p-Laplace operator,

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

which coincides with the Laplace operator Δ when $p = 2$.

Equations containing the p-Laplace operator, $p > 2$, are helpful to study drift-diffusion models for the electro-thermal behavior of organic semiconductor devices (see e.g. [8] and [9] and the references therein).

Problem (P) is also a generalization of the stationary version of both

a) the nonlinear Schrödinger equation,

$$i\hbar\, u_t + \frac{\hbar^2}{2}\Delta u - V_0(x)\, u + f(x,u) = 0,$$

which appears in natural way e.g. when studying the evolution of Bose-Einstein condensates (see e.g.[13]) and the propagation of light in nonlinear optical materials, (see e.g. [14] and [6]), and

b) the Kirchhoff equation, [11],

$$u_{tt} - \left(a + b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\right)\Delta u = f(x,u),$$

which is a wave equation that considers the lenght changes of a string that are produced by transverse vibrations.

Grossly speaking, in this paper we extend, for $p > 1$, the results obtained in [7] for the case $p = 2$, that is, we prove - see Theorems 1.1 and 1.2 below - the existence of a non-trivial ground state solution for (P) as well as the existence of infinitely many solutions.

We assume that that the potential $V : \mathbb{R}^N \longrightarrow \mathbb{R}$ verifies

(V) $V \in \mathrm{C}(\mathbb{R}^N)$ and $\theta = \inf_{x\in\mathbb{R}^N} V(x) > 0$;

and that the nonlinear function $f : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ verifies

(F1) there are positive functions $\beta_1 \in \mathrm{L}^{p/(p-r_1)}(\mathbb{R}^N)$ and $\beta_2 \in \mathrm{L}^{p/(p-r_2)}(\mathbb{R}^N)$, such that

$$\forall (x,t) \in \mathbb{R}^N \times \mathbb{R} : \quad |f(x,t)| \le r_1\beta_1(x)|t|^{r_1-1} + r_2\beta_2(x)|t|^{r_2-1},$$

for some $1 < r_1 < r_2 < p$; and,

(F2) there exist $\Omega \subseteq \mathbb{R}^N$ open bounded and constants $\delta, \eta > 0$ and $r_3 \in]1,p[$ such that

$$\forall (x,t) \in \Omega \times [-\delta,\delta] : \quad F(x,t) \ge \eta |t|^{r_3},$$

where $F(x,t) = \int_0^t f(x,s)ds$.

Let's state our main results.

Theorem 1.1. *Assume that conditions (V), (F1) and (F2) hold. Then problem* (P) *has a non-trivial ground state solution.*

Let's observe that condition (F1) implies that (P) has the trivial solution, $u \equiv 0$. However, the trivial solution is not a ground state solution, i.e., a weak solution of (P) that minimizes the associated energy functional, given in (2.3) as

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N} (a|\nabla u|^p + V(x)|u|^p)\, dx + \frac{b}{p^2}\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^p - \int_{\mathbb{R}^N} F(x,u(x))dx,$$

where u varies on E^p, the space of functions $u \in \mathrm{W}^{1,p}(\mathbb{R}^N)$ such that $V^{1/p}\cdot u \in \mathrm{L}^p(\mathbb{R}^N)$.

In the statement of our second main result, a multiplicity one, we shall use the following condition.

(F3) f is odd in the second variable, i.e.,

$$\forall (x,t) \in \mathbb{R}^N \times \mathbb{R}: \quad f(x,t) = -f(x,-t).$$

Theorem 1.2. *Assume the conditions (V), (F1), (F2) and (F3). Then problem* (P) *has infinitely many non-trivial solutions.*

As it was already mentioned, Theorems 1.1 and 1.2 extend, for a general value $p > 1$, the results obtained by Duan and Huang, [7], for the case $p = 2$. We prove Theorem 1.1 by a direct method of the calculus of variations, Theorem 3.1. We prove Theorem 1.2 by a Ljusternik-Schnirelman scheme for even functionals, see Theorem 4.2. To this purpose, we need to show that the functional associated to (P) verifies the Palais-Smale condition; and, for this, the main problem yields in the fact that the Sobolev space $\mathrm{W}^{1,p}(\mathbb{R}^N)$ is not compactly embedded into the Lebesgue spaces $\mathrm{L}^\alpha(\mathbb{R}^N)$, $\alpha \in [p, pN/(N-p)[$. To handle this difficulty, it is usual to require a coercivity property on the potential V like

$$V(x) \longrightarrow +\infty, \quad \text{as } |x| \longrightarrow +\infty, \tag{1.1}$$

or the weaker one

$$\forall K > 0: \quad \text{meas}\left(V^{-1}(]-\infty, K])\right) < +\infty, \tag{1.2}$$

because this kind of conditions imply that E^p is compactly contained in $\mathrm{L}^\alpha(\mathbb{R}^N)$. We produce the proofs of our results without relying on (1.1), (1.2) or on any other coercitivity condition.

Another situation appears when we deal with the Palais-Smale condition. In [7] the authors worked on the Hilbert space, E^2, which is automaticaly reflexive and, therefore, allows to extract a weakly converging subsequence from any bounded sequence. In our case we prove that the Banach space E^p is actually reflexive (see Lemma 2.2 below).

Remark 1.3. As it will be seen in our arguments, condition (F1) can be immediately replaced by the following one.

(F1') For each $k = 1, ..., l$, there is a positive function $\beta_k \in \mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)$ such that

$$\forall (x,t) \in \mathbb{R}^N \times \mathbb{R}: \quad |f(x,t)| \leq \sum_{k=1}^{l} r_k \beta_k(x)|t|^{r_k - 1},$$

for some $1 < r_1 < r_2 < ... < r_l < p$.

It's also clear that the following condition implies (F2).

(F2') There exist a bounded open set $\Omega \subseteq \mathbb{R}^N$ and constants $\delta, \eta > 0$ and $r_3 \in]1, p[$ such that

$$\forall (x,t) \in \Omega \times [-\delta, \delta] : \quad f(x,t) \cdot t \geq r_3 \eta |t|^{r_3 - 1}.$$

The paper is organized in the following way. In Section 2 we introduce the functional setting and some preliminaries. In Sections 3 and 4 we prove Theorems 1.1 and 1.2, respectively.

2. General setting

Let E^p be the Banach space that results from completing $C_0^\infty(\mathbb{R}^N)$ in the norm given by

$$\|u\|_p = \left(\int_{\mathbb{R}^N} [|\nabla u(x)|^p + V(x)|u(x)|^p] \, dx \right)^{1/p},$$

so that E^p is formed by all the functions $u \in W^{1,p}(\mathbb{R}^N)$ such that $V^{1/p}u \in L^p(\mathbb{R}^N)$. We denote by $\|\cdot\|_{p'}$ the norm on the dual space $(E^p)'$.

Remark 2.1. Let's denote by $p^* = pN/(N-p)$ the critical Sobolev value from the Sobolev-Gagliardo-Niremberg theorem (see e.g. [5, Th.9.9]). It's well-known that the embedding $E^p \subseteq L$ is

a) continuous for $L = L^q(\mathbb{R}^N)$ with $p \leq q \leq p^*$:

$$\exists C_q > 0, \forall u \in E^p : \quad \|u\|_{L^q(\mathbb{R}^N)} \leq C_q \|u\|_p; \tag{2.1}$$

b) compact for $L = L^q_{\text{loc}}(\mathbb{R}^N)$ with $p \leq q < p^*$;

c) continuous for $L = W^{1,p}(\mathbb{R}^N)$:

$$\forall u \in E^p : \quad \|u\|_{W^{1,p}(\mathbb{R}^N)} \leq \max\{1, \theta^{-1}\}^{1/p} \|u\|_p. \tag{2.2}$$

Lemma 2.2. *The space E^p is reflexive.*

Proof. Let's consider the Banach space $X = L^p_V(\mathbb{R}^N) \times \left[L^p(\mathbb{R}^N)\right]^N$, where

$$\|(u,w)\|_X = \left(\|u\|^p_{L^p_V(\mathbb{R}^N)} + \|w\|_{[L^p(\mathbb{R}^N)]^N} \right)^{1/p},$$

$$\|u\|_{L^p_V(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^p d\mu \right)^{1/p}, \qquad d\mu = V(x)dx,$$

$$\|w\|_{[L^p(\mathbb{R}^N)]^N} = \|(w_1, \ldots, w_N)\|_{[L^p(\mathbb{R}^N)]^N} = \left(\int_{\mathbb{R}^N} |w|^p dx \right)^{1/p}.$$

Since $L^p(\mathbb{R}^N)$ and $L^p_V(\mathbb{R}^N)$ are reflexive (see e.g. [4, Th. 4.7.15 and Cor. 4.7.16]), it follows that X is reflexive. The operator $T : E^p \longrightarrow X$, given by $T(u) = (u, \nabla u)$, is an isometry. Since E^p is a Banach space, it follows that $T(E^p)$ is a closed subspace of X and, therefore, by [5, Prop.3.20], $T(E^p)$ is also reflexive. With the last, we conclude that E^p is reflexive. ◻

Associated to problem (P) is the functional $I : E^p \longrightarrow \mathbb{R}$, given by

$$I(u) = J(u) + G(u) + H(u) - W(u), \tag{2.3}$$

where

$$J(u) = \frac{a}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx, \quad G(u) = \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx$$

$$H(u) = \frac{b}{p^2}\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^p, \quad W(u) = \int_{\mathbb{R}^N} F(x, u(x))dx.$$

In fact, as a consequence of Theorem 2.3 below, the critical points of I are weak solutions of (P).

Theorem 2.3. *The functional I is of class C^1. For every for $u, h \in E^p$, we have*

$$\begin{aligned}\langle I'(u), h\rangle =& a\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla h\, dx + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uh\, dx \\ &+ b\left(\int_{\mathbb{R}^N} |\nabla u|^p\, dx\right)^{p-1}\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla h\, dx - \int_{\mathbb{R}^N} f(x, u)\, h\, dx.\end{aligned}$$

The proof of Theorem 2.3 is given in the following lemmas.

Lemma 2.4. *The functional I is well defined.*

Proof. By the definitions of E^p and I, we just have to show that $W(u) \in \mathbb{R}$, for every $u \in E$. By condition (F1) we have, for every $x \in \mathbb{R}^N$ and every $t \in \mathbb{R}$, that

$$\begin{aligned}|F(x,t)| &\le \int_0^t |f(x,s)|ds \le \int_0^t \left[r_1\beta_1(x)|s|^{r_1-1} + r_2\beta_2(x)|s|^{r_2-1}\right] ds \\ &\le \beta_1(x)|t|^{r_1} + \beta_2(x)|t|^{r_2},\end{aligned}$$

so that $|W(u)| \le \mathcal{M}_1(u) + \mathcal{M}_2(u)$, where

$$\mathcal{M}_k(u) = \int_{\mathbb{R}^N} \beta_k(x)|u|^{r_k} dx, \quad k = 1, 2.$$

Then, using (2.1) and Hölder's inequality for $k = 1, 2$, $P = p/(p - r_k)$ and $P' = p/r_k$, we get

$$\begin{aligned}\mathcal{M}_k(u) &\le \|\beta_k\|_{\mathrm{L}^P(\mathbb{R}^N)} \left\||u|^{r_k}\right\|_{\mathrm{L}^{P'}(\mathbb{R}^N)} \\ &= \left(\int_{\mathbb{R}^N} \beta_k^{p/(p-r_k)}(x)dx\right)^{(p-r_k)/p}\left(\int_{\mathbb{R}^N} |u|^p dx\right)^{r_k/p} \\ &= \|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \cdot \|u\|^{r_k}_{\mathrm{L}^p(\mathbb{R}^N)} \\ &\le C_p^{r_k}\|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \cdot \|u\|_p^{r_k} < +\infty.\end{aligned}$$

and we are done. □

Remark 2.5. From the previous proof, it follows that, for every $u \in E^p$,

$$|W(u)| \le \sum_{k=1}^{2} \theta^{-r_k/p}\|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \cdot \|u\|_p^{r_k}. \tag{2.4}$$

Remark 2.6. Let's write a couple of inequalities, taken from [12], that will be useful. Given $x, y \in \mathbb{R}^m$,

$$2^{2-p}|y-x|^{p-1} \geq \left||y|^{p-2}y - |x|^{p-2}x\right|, \qquad \text{if } 1 \leq p \leq 2; \quad (2.5)$$

$$\left\langle |y|^{p-2}y - |x|^{p-2}x, y-x \right\rangle \geq 2^{2-p}|y-x|^p, \qquad \text{if } 1 \leq p \leq 2; \quad (2.6)$$

$$\left\langle |y|^{p-2}y - |x|^{p-2}x, y-x \right\rangle \geq (p-1)\left(1+|y|^2+|x|^2\right)^{\frac{p-2}{2}} |y-x|^2, \quad \text{if } p \geq 2. \quad (2.7)$$

Lemma 2.7. *The functionals J, G and H are of class* C^1.

Proof. This proof is standard so that we omit most of its details.

i) The functionals J and G are Fréchet-differentiable and, for every $u, h \in E^p$,

$$\langle J'(u), h\rangle = a\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla h\, dx, \qquad \langle G'(u), h\rangle = \int_{\mathbb{R}^N} V(x)|u|^{p-2}uh\, dx.$$

We shall show that J is of class C^1. G is treated in a similar way. Since

$$H(u) = \frac{b}{p^2}\left[\frac{p}{a}J(u)\right]^p,$$

it follows, by the chain rule, that H is also of class C^1 and

$$\langle H'(u), h\rangle = b\left(\int_{\mathbb{R}^N} |\nabla u|^p\, dx\right)^{p-1} \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla h\, dx.$$

ii) Let's show that J is of class C^1, i.e., that $J' : E^p \longrightarrow (E^p)'$ is continuous. Let $u_0 \in E^p$. We have to prove that for any $\tau > 0$, there exists $\delta > 0$ such that $\|u-u_0\|_p < \delta$ implies that

$$\forall v \in E^p : \quad |\langle J'(u) - J'(u_0), v\rangle| \leq \tau\|v\|_p. \quad (2.8)$$

Assume that $1 < p \leq 2$. Let $\tau > 0$. Let's pick $\delta \in]0, (2^{p-2}\tau)^{1/(p-1)}[$. For $u, v \in E^p$ with $\|u-u_0\|_p < \delta$, we get, by using (2.5) and Hölder's inequality, that

$$\begin{aligned}
|\langle J'(u_0) - J'(u), v\rangle| &\leq \int_{\mathbb{R}^N} \left||\nabla u_0|^{p-2}\nabla u_0 - |\nabla u|^{p-2}\nabla u\right| |\nabla v|\, dx \\
&\leq 2^{2-p}\int_{\mathbb{R}^N} |\nabla u_0 - \nabla u|^{p-1}|\nabla v| dx \\
&\leq 2^{2-p}\left(\int_{\mathbb{R}^N} |\nabla u_0 - \nabla u|^p\right)^{(p-1)/p} \|v\|_{\mathrm{L}^p(\mathbb{R}^N)} \\
&\leq 2^{2-p}\|u_0 - u\|_p^{p-1}\|v\|_p \\
&\leq 2^{2-p}\delta^{p-1}\|v\|_p \leq \tau\|v\|_p.
\end{aligned}$$

The case of $p > 2$ is dealt with in a similar way.

□

Remark 2.8. It's clear that the functional $\mathcal{N}_p : E^p \longrightarrow \mathbb{R}$, given by

$$\mathcal{N}_p(u) = [J(u) + G(u)]^{1/p},$$

is a norm equivalent to $\|\cdot\|_p$.

Remark 2.9. Let's recall that for $t, s \geq 0$,

$$\begin{aligned} (t+s)^m &\leq t^m + s^m, && m \in]0,1[;\\ (t+s)^m &\leq 2^m(t^m + s^m), && m \in [1,+\infty[;\\ t^m + s^m &\leq (t+s)^m, && m \in]0,+\infty[. \end{aligned}$$

Remark 2.10. Let X and Y be Banach spaces, $\mathcal{O} \subseteq X$ open and $T : \mathcal{O} \subseteq X \to Y$ Gateaux differentiable at $u_0 \in \mathcal{O}$. It's well-known (see e.g. [2]) that if T'_G is continuous at u_0, then T is Fréchet differentiable at u_0 and $T'(u_0) = T'_G(u_0)$.

Lemma 2.11. *The functional W is of class C^1.*

Proof. Let $\mu, t \in]0,1[$ and $u, h \in E$.

i) We have, by (F1) and Remark 2.9, that

$$\begin{aligned} &\int_{\mathbb{R}^N} |f(x,u(x)+t\mu h(x))\, h(x)|dx \leq \int_{\mathbb{R}^N} \max_{t\in[0,1]} |f(x,u(x)+t\mu h(x))|\,|h(x)|dx\\ &\leq \int_{\mathbb{R}^N} \max_{t\in[0,1]} \left[r_1\beta_1(x)|u+\mu t\, h|^{r_1-1} + r_2\beta_2(x)|u+\mu t\, h|^{r_2-1}\right]|h|dx.\\ &\leq \sum_{k=1}^{2} r_k \int_{\mathbb{R}^N} \beta_k(x)\left(|u|+|h|\right)^{r_k-1}|h|dx\\ &\leq \sum_{k=1}^{2} 2^{r_k-1} r_k \int_{\mathbb{R}^N} \beta_k(x)\left(|u|^{r_k-1}+|h|^{r_k-1}\right)|h|dx \qquad (2.9) \end{aligned}$$

By Hölder's inequality with

$$p_1 = \frac{p}{p-r_k},\ p_2 = \frac{p}{r_k-1},\ p_3 = p, \qquad \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} = 1,$$

we get, for $k = 1,2$,

$$\begin{aligned} \int_{\mathbb{R}^N} \beta_k(x)|u|^{r_k-1}|h|dx &\leq \|\beta_k\|_{\mathrm{L}^{p_1}(\mathbb{R}^N)} \left\||u|^{r_k-1}\right\|_{\mathrm{L}^{p_2}(\mathbb{R}^N)} \|h\|_{\mathrm{L}^{p_3}(\mathbb{R}^N)}\\ &\leq \|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \|u\|^{r_k-1}_{\mathrm{L}^p(\mathbb{R}^N)} \|h\|_{\mathrm{L}^p(\mathbb{R}^N)}\\ &\leq \theta^{-r_k/p} \|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \left\|V^{1/p}u\right\|^{r_k-1}_{\mathrm{L}^p(\mathbb{R}^N)} \left\|V^{1/p}h\right\|_{\mathrm{L}^p(\mathbb{R}^N)}\\ &\leq \theta^{-r_k/p} \|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \|u\|_p^{r_k-1}\|h\|_p. \qquad (2.10) \end{aligned}$$

By Hölder's inequality with $P = p/(p-r_k)$ and $P' = p/r_k$, we get

$$\begin{aligned} \int_{\mathbb{R}^N} \beta_k(x)|h|^{r_k}dx &\leq \|\beta_k\|_{\mathrm{L}^P(\mathbb{R}^N)} \||h|^{r_k}\|_{\mathrm{L}^{P'}(\mathbb{R}^N)}\\ &\leq \|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \|h\|^{r_k}_{\mathrm{L}^p(\mathbb{R}^N)}\\ &\leq \theta^{-r_k/p} \|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \|h\|_p^{r_k}. \qquad (2.11) \end{aligned}$$

By (2.9), (2.10) and (2.11), we get

$$\begin{aligned} &\int_{\mathbb{R}^N} |f(x,u(x)+t\mu h(x))\, h(x)|dx \leq\\ &\qquad \leq \sum_{k=1}^{2} 2^{r_k-1} r_k \theta^{-r_k/p} \|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \left(\|u\|_p^{r_k-1} + \|h\|_p^{r_k-1}\right)\|h\|_p, \end{aligned}$$

which, together with the dominated convergence theorem (see e.g. [5, Th.4.2]), provides the Gateaux differentiability of W at u:

$$\begin{aligned}\langle W_G'(u), h\rangle &= \lim_{t\downarrow 0}\frac{1}{t}\int_{\mathbb{R}^N}[F(x,u(x)+th(x))-F(x,u(x))]dx\\ &= \lim_{t\downarrow 0}\frac{1}{t}\int_{\mathbb{R}^N} f(x,u(x)+t\mu h(x))\,h(x)\,dx\\ &= \int_{\mathbb{R}^N} f(x,u(x))\,h(x)\,dx.\end{aligned}$$

By Remark 2.10 and the arbitrariness of u, it remains to show that W_G' is continuous at u.

ii) Let $u_0, v \in E$. By Hölder inequality and working as in the line before to (2.10), we get

$$\begin{aligned}|\langle W'(u)-W'(u_0), v\rangle| &= \left|\int_{\mathbb{R}^N}[f(x,u(x))-f(x,u_0(x))]\,v(x)dx\right|\\ &\le \left(\int_{\mathbb{R}^N}|f(x,u_0(x))-f(x,u(x))|^{p/(p-1)}\,dx\right)^{(p-1)/p}\|v\|_{\mathrm{L}^p(\mathbb{R}^N)}\\ &\le \theta^{-1/p}\left(\int_{\mathbb{R}^N}|f(x,u_0(x))-f(x,u(x))|^{p/(p-1)}\,dx\right)^{(p-1)/p}\|v\|_p,\end{aligned}$$

so that, by the arbitrariness of v,

$$\|W'(u)-W'(u_0)\|_{p'} \le \theta^{-1/p}\left(\int_{\mathbb{R}^N}|f(x,u_0)-f(x,u)|^{p/(p-1)}\,dx\right)^{(p-1)/p}.$$

iii) Let $(u_n)_{n\in\mathbb{N}} \subseteq E$ such that

$$\|u_n-u\|_p \longrightarrow 0, \quad \text{as } n\longrightarrow +\infty. \tag{2.12}$$

By point ii), to show that W' is continuous at u, it's enough to show that

$$\int_{\mathbb{R}^N}\phi_n(x)dx \longrightarrow 0, \quad \text{as } n\longrightarrow +\infty, \tag{2.13}$$

where $\phi_n(x) = |f(x,u_n(x))-f(x,u(x))|^{p/(p-1)}$. By (2.12) and (2.1),

$$\|u_n-u\|_{\mathrm{L}^p(\mathbb{R}^N)} \longrightarrow 0, \quad \text{as } n\longrightarrow +\infty,$$

so that, by [5, Th.4.9], up to a subsequence $(u_m)_{m\in\mathbb{N}} = (u_{n_m})_{m\in\mathbb{N}}$,

$$u_m(x) \longrightarrow u(x), \quad \text{for a.e. } x\in\mathbb{R}^N.$$

Clearly we can also assume that

$$\sum_{m=1}^{+\infty}\|u_m-u\|^p_{\mathrm{L}^p(\mathbb{R}^N)} < +\infty.$$

Thefore, $w\in \mathrm{L}^p(\mathbb{R}^N)$, where

$$w(x) = \sum_{m=1}^{+\infty}|u_m(x)-u(x)|, \qquad x\in\mathbb{R}^N. \tag{2.14}$$

Since f is continuous, it holds that

$$\phi_m(x) \longrightarrow \phi(x), \quad \text{for a.e. } x\in\mathbb{R}^N,$$

so that, to prove (2.13) via the dominated convergence theorem, we need to find a function $\psi \in \mathrm{L}^1(\mathbb{R}^N)$ such that, for every $m \in \mathbb{N}$,

$$\phi_m(x) \leq \psi(x), \quad \text{for a.e. } x \in \mathbb{R}^N.$$

We have, by (2.14), Remark 2.9, the Lipschitz continuity of the absolute value and putting

$$\lambda_k = \frac{p(r_k - 1)}{p-1},$$

that, for every $x \in \mathbb{R}^N$ and every $m \in \mathbb{N}$,

$$\begin{aligned}
\phi_m(x) &\leq 2^{p/(p-1)} \left[|f(x, u_m(x))|^{\frac{p}{p-1}} + |f(x, u(x))|^{\frac{p}{p-1}} \right] \\
&\leq 2^{p/(p-1)} \sum_{k=1}^{2} \left[(r_k \beta_k(x) |u_m(x)|^{r_k - 1})^{p/(p-1)} + (r_k \beta_k(x) |u(x)|^{r_k-1})^{p/(p-1)} \right] \\
&= \sum_{k=1}^{2} (2r_k)^{p/(p-1)} \beta_k^{p/(p-1)}(x) \left[|u_m(x)|^{\lambda_k} + |u(x)|^{\lambda_k} \right] \\
&= \sum_{k=1}^{2} (2r_k)^{p/(p-1)} \beta_k^{p/(p-1)}(x) \left[(|u_m(x)| - |(u(x)| + |u(x)|)^{\lambda_k} + |u(x)|^{\lambda_k} \right] \\
&\leq \sum_{k=1}^{2} (2r_k)^{p/(p-1)} \beta_k^{p/(p-1)}(x) \left[2^{\lambda_k} (w^{\lambda_k}(x) + |u(x)|^{\lambda_k}) + |u(x)|^{\lambda_k} \right] \\
&= \sum_{k=1}^{2} (2r_k)^{p/(p-1)} \left[2^{\lambda_k} w^{\lambda_k}(x) + (2^{\lambda_k} + 1) |u(x)|^{\lambda_k} \right] \beta_k^{p/(p-1)}(x) \\
&= \psi(x).
\end{aligned}$$

We have that $\psi \in \mathrm{L}^1(\mathbb{R}^N)$. In fact, by using Hölder's inequality with $P = (p-1)/(r_k - 1)$ and $P' = (p-1)/(p - r_k)$, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} \psi(x) dx &\leq \sum_{k=1}^{2} (2r_k)^{p/(p-1)} \cdot \\
&\quad \cdot \left[2^{\lambda_k} \left\| w^{\lambda_k} \right\|_{\mathrm{L}^P(\mathbb{R}^N)} + (2^{\lambda_k} + 1) \left\| u^{\lambda_k} \right\|_{\mathrm{L}^P(\mathbb{R}^N)} \right] \left\| \beta_k^{p/(p-1)} \right\|_{\mathrm{L}^{P'}(\mathbb{R}^N)} \\
&= \sum_{k=1}^{2} (2r_k)^{p/(p-1)} \cdot \\
&\quad \cdot \left[2^{\lambda_k} \|w\|_{\mathrm{L}^p(\mathbb{R}^N)}^{p(r_k-1)/(p-1)} + (2^{\lambda_k} + 1) \|u\|_{\mathrm{L}^p(\mathbb{R}^N)}^{p(r_k-1)/(p-1)} \right] \|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)}^{p/(p-1)} \\
&< +\infty.
\end{aligned}$$

Therefore we have proved that the subsequence (u_{n_m}) verifies (2.13). By a contradiction argument, it can be proved that (2.13) holds also for the original sequence $(u_n)_{n \in \mathbb{N}}$.

□

Corollary 2.12. *For every $u \in E^p$, it holds*

$$\mathcal{N}_p(u) \leq \left[\frac{1}{p} \left(\langle I'(u), u \rangle + \int_{\mathbb{R}^N} f(x, u(x))\, u(x) dx \right) \right]^{1/p}. \tag{2.15}$$

Proof. By Theorem 2.3 and (2.3), we have that

$$\langle I'(u), u\rangle = p\mathcal{N}_p^p(u) + p^2 H(u) - \int_{\mathbb{R}^N} f(x, u(x))\, u(x) dx,$$

whence it immediately follows (2.15). □

3. Existence of a ground state

Let's prove Theorem 1.1. It states that, under conditions (V), (F1) and (F2), problem (P) possesses a non-trivial ground state solution. We shall apply Theorem 3.1 below, as given in [2].

Let X be a Banach space and $I \in \mathrm{C}^1(X)$. Given $c \in \mathbb{R}$ we denote

$$K_c = \{u \in X \,/\, I'(u) = 0 \,\wedge\, I(u) = c\}, \quad I^c = \{u \in X \,/\, I(u) \le c\}.$$

A sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$ is a Palais-Smale sequence for the functional I iff

a) $(I(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded, and
b) $I'(u_n) \longrightarrow 0$, as $n \longrightarrow +\infty$, in X'.

If for some $\nu \in \mathbb{R}$, it holds $I(u_n) \to \nu$, as $n \longrightarrow +\infty$, we say that $(u_n)_{n\in\mathbb{N}} \subseteq X$ is a $(\mathrm{PS})_\nu$ sequence.

We say that the functional I verifies the condition (PS) iff every Palais-Smale sequence has a converging subsequence, or the condition $(\mathrm{PS})_\nu$ iff every $(\mathrm{PS})_\nu$ sequence has a converging subsequence; in this case, the critical level K_ν is compact.

Theorem 3.1. *Assume that the functional I is bounded from below and verifies the (PS) condition. Then*

$$c = \inf_{u\in X} I(u)$$

is a critical value of I.

The proof of Theorem 1.1 is built in the following results.

Lemma 3.2. *The functional I is bounded from below, i.e., there exists $c_* \in \mathbb{R}$ such that*

$$\forall u \in E^p : \quad I(u) \ge c_*.$$

Proof. By (2.3) and (2.4), we have, for $u \in E^p$, that

$$\begin{aligned} I(u) &= J(u) + G(u) + H(u) - W(u) \\ &\ge \frac{1}{p}\min\{a, 1\}\|u\|_p^p - \sum_{k=1}^{2} \theta^{-r_k/p}\, \|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)}\, \|u\|_p^{r_k}. \end{aligned} \tag{3.1}$$

Since $1 < r_1 < r_2 < p$, the last inequality implies that

$$I(u) \longrightarrow \infty, \quad \text{as } \|u\|_p \longrightarrow +\infty, \tag{3.2}$$

so that I is bounded from below. □

Proposition 3.3. *The functional I verifies the (PS) condition.*

Proof. Let's assume that $(u_n)_{n\in\mathbb{N}} \subseteq E^p$ is such that

a) $(I(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded;
b) $I'(u_n) \longrightarrow 0$, as $n \longrightarrow +\infty$.

We have to show that $(u_n)_{n\in\mathbb{N}}$ has a converging subsequence.

i) Let's prove that $(u_n)_{n\in\mathbb{N}} \subseteq E^p$ is bounded, i.e., that there exists $c_{**} > 0$ such that

$$\forall n \in \mathbb{N}: \quad \|u_n\|_p \le c_{**}, \tag{3.3}$$

and, therefore, we also have that

$$\forall n \in \mathbb{N}: \quad \|u_n\|_{\mathrm{L}^p(\mathbb{R}^N)} \le \theta^{-1/p}\|u_p\|_p \le \theta^{-1/p}c_{**}. \tag{3.4}$$

By a), Lemma 3.2 and (3.2), there exists $C_* > 0$ such that $c_* \le I(u_n) \le C_*$, $n \in \mathbb{N}$. Then, by (3.1), it follows that

$$\frac{1}{p}\min\{a,1\}\|u_n\|_p^p - \sum_{k=1}^{2}\theta^{-r_k/p}\,\|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)}\cdot\|u_n\|_p^{r_k} \le I(u_n) \le C_*,$$

$$\|u_n\|_p^p \le \frac{p}{\min\{a,1\}}\left[C_* + \sum_{k=1}^{2}\theta^{-r_k/p}\,\|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)}\cdot\|u_n\|_p^{r_k}\right]. \tag{3.5}$$

In case of $\|u_n\| \ge 1$ we have that $1 \le \|u_n\|^{r_1} \le \|u_n\|^{r_2}$ and so, by (3.5), it follows that

$$\|u_n\|_p^{p-r_2} \le \frac{p}{\min\{a,1\}}\left[C_* + \sum_{k=1}^{2}\theta^{-r_k/p}\,\|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)}\right].$$

Therefore, we get (3.3) with

$$c_{**} = \max\left\{1,\ \frac{p}{\min\{a,1\}}\left[C_* + \sum_{k=1}^{2}\theta^{-r_k/p}\,\|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)}\right]\right\}^{1/(p-r_2)}.$$

ii) By Lemma 2.2 and [5, Th. 3.8], there exists a subsequence $(u_{n_m})_{m\in\mathbb{N}} = (u_m)_{m\in\mathbb{N}} \subseteq E^p$ that converges weakly to some $u_0 \in E^p$, i.e.,

$$u_m \rightharpoonup u_0, \quad \text{as } m \longrightarrow +\infty. \tag{3.6}$$

Let $\epsilon > 0$. By (F1), we can choose $R_\epsilon > 0$ such that

$$\int_{B_\epsilon^c} |\beta_k(x)|^{\frac{p}{p-r_k}}\,dx < \epsilon^{p/(p-r_k)}, \quad k = 1, 2, \tag{3.7}$$

where $B_\epsilon = B(0, R_\epsilon) \subseteq \mathbb{R}^N$. By Remark 2.1, the embedding $E^p \subseteq \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^N)$ is compact and, therefore, point (3.6) implies that $u_m \longrightarrow u_0$, as $m \longrightarrow +\infty$, in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^N)$, and, consequently,

$$\lim_{m\to+\infty}\int_{B_\epsilon}|u_m(x) - u_0(x)|^p\,dx = 0.$$

Then there exists $m_0 \in \mathbb{N}$ such that

$$\int_{B_\epsilon}|u_m(x) - u_0(x)|^p\,dx \le \epsilon^p, \quad \text{for } m \ge m_0. \tag{3.8}$$

iii) Now let's show that, as $m \longrightarrow +\infty$,

$$\int_{\mathbb{R}^N} (f(x,u_m(x)) - f(x,u_0(x)))(u_m(x) - u_0(x))dx \longrightarrow 0. \tag{3.9}$$

First, let's estimate the left side of (3.9) in the ball B_ϵ. By Hölder's inequality with $P = (p-1)/(p-r_k)$ and $P' = (p-1)/(r_k-1)$, we have that

$$\begin{aligned}
&\int_{B_\epsilon} |\beta_k|^{p/(p-1)} |u_0|^{p(r_k-1)/(p-1)} dx \leq \\
&\leq \left\| |\beta_k|^{p/(p-1)} \right\|_{\mathrm{L}^P(B_\epsilon)} \left\| |u_0|^{p(r_k-1)/(p-1)} \right\|_{\mathrm{L}^{P'}(B_\epsilon)} \\
&\leq \|\beta_k\|^{p/(p-1)}_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \|u_0\|^{p(r_k-1)/(p-1)}_{\mathrm{L}^p(\mathbb{R}^N)} .
\end{aligned} \tag{3.10}$$

In the same way, using (3.4), we get

$$\begin{aligned}
&\int_{B_\epsilon} |\beta_k|^{p/(p-1)} |u_m|^{p(r_k-1)/(p-1)} dx \leq \\
&\leq \|\beta_k\|^{p/(p-1)}_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \|u_m\|^{p(r_k-1)/(p-1)}_{\mathrm{L}^p(\mathbb{R}^N)} \\
&\leq \left[\theta^{-1/p} c_{**}\right]^{p(r_k-1)/(p-1)} \|\beta_k\|^{p/(p-1)}_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} .
\end{aligned} \tag{3.11}$$

By (F1), Remark 2.9, (3.8), (3.10), (3.11) and Hölder's inequality, we have, for $m \geq m_0$, that

$$\begin{aligned}
&\int_{B_\epsilon} |f(x,u_m(x)) - f(x,u_0(x))| \cdot |u_m(x) - u_0(x)| dx \\
&\leq \left(\int_{B_\epsilon} |f(x,u_m(x)) - f(x,u_0(x))|^{p/(p-1)} dx \right)^{(p-1)/p} \|u_m - u_o\|_{\mathrm{L}^p(\mathbb{R}^N)} \\
&\leq \epsilon \left[2^{p/(p-1)} \int_{B_\epsilon} \left[|f(x,u_m(x))|^{p/(p-1)} + |f(x,u_0(x))|^{p/(p-1)} \right] dx \right]^{(p-1)/p} \\
&\leq 2\epsilon \left[\int_{B_\epsilon} \left(\left| \sum_{k=1}^{2} r_k \beta_k |u_m|^{r_k-1} \right|^{p/(p-1)} + \left| \sum_{k=1}^{2} r_k \beta_k |u_0|^{r_k-1} \right|^{p/(p-1)} \right) dx \right]^{(p-1)/p} \\
&\leq 4\epsilon \left(\sum_{k=1}^{2} r_k^{p/(p-1)} \int_{B_\epsilon} |\beta_k|^{p/(p-1)} \left[|u_m|^{p(r_k-1)/(p-1)} + |u_0|^{p(r_k-1)/(p-1)} \right] dx \right)^{(p-1)/p} \\
&\leq 4\epsilon \left\{ \sum_{k=1}^{2} r_k^{p/(p-1)} \|\beta_k\|^{p/(p-1)}_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \left[\left(\theta^{-1/p} c_{**}\right)^{\frac{p(r_k-1)}{p-1}} + \|u_0\|^{\frac{p(r_k-1)}{p-1}}_{\mathrm{L}^p(\mathbb{R}^N)} \right] \right\}^{\frac{p-1}{p}} \\
&\leq \epsilon \cdot 2^{2+2(p-1)/p} \sum_{k=1}^{2} r_k \|\beta_k\|_{\mathrm{L}^{p/(p-r_k)}(\mathbb{R}^N)} \left[\left(\theta^{-1/p} c_{**}\right)^{r_k-1} + \|u_0\|^{r_k-1}_{\mathrm{L}^p(\mathbb{R}^N)} \right]
\end{aligned} \tag{3.12}$$

Now let's estimate (3.9) out of the ball B_ϵ. By using (F1), Remark 2.9, (2.9), (3.7), (3.4) and Hölder's inequality with $P = p/(p-r_k)$ and

$P' = p/r_k$, we get

$$\begin{aligned}
&\int_{B_\epsilon^c} |f(x,u_m(x)) - f(x,u_0(x))| \cdot |u_m(x) - u_0(x)| dx \\
&\le \sum_{k=1}^{2} r_k \int_{B_\epsilon^c} \beta_k \left[|u_m|^{r_k-1} + |u_0|^{r_k-1}\right] (|u_m| + |u_0|) dx \\
&\le \sum_{k=1}^{2} r_k \int_{B_\epsilon^c} \beta_k \left[|u_m|^{r_k} + |u_0|^{r_k}\right] dx \\
&\le \sum_{k=1}^{2} \|\beta_k\|_{\mathrm{L}^P(B_\epsilon^c)} \left[\||u_m|^{r_k}\|_{\mathrm{L}^{P'}(B_\epsilon^c)} + \||u_0|^{r_k}\|_{\mathrm{L}^{P'}(B_\epsilon^c)}\right] \\
&\le \epsilon \sum_{k=1}^{2} r_k \left[\|u_m\|_{\mathrm{L}^{r_k/p}(\mathbb{R}^N)} + \|u_0\|_{\mathrm{L}^{r_k/p}(\mathbb{R}^N)}\right] \\
&\le \epsilon \sum_{k=1}^{2} r_k \left[\left(\theta^{-1/p} c_{**}\right)^{r_k/p} + \|u_0\|_{\mathrm{L}^{r_k/p}(\mathbb{R}^N)}\right].
\end{aligned} \tag{3.13}$$

By (3.12) and (3.13) and the arbitrariness of ϵ, we obtain (3.9).

iv) By (2.15), we have that

$$p\mathcal{N}_p^p(u_m - u_0) \le \langle I'(u_m - u_0), u_m - u_0\rangle + \int_{\mathbb{R}^N} f(x, u_m - u_0) \cdot (u_m - u_0)\, dx,$$

where the first term in the right-side tends to zero, as $m \longrightarrow 0$, by (3.6) and $I'(u_m - u_0) \in E^p$. By using (3.9), (2.2), (2.6) or (2.7), and using estimates like those in the proof of Lemma 2.11 we get

$$\int_{\mathbb{R}^N} f(x, u_m - u_0) \cdot (u_m - u_0)\, dx \longrightarrow 0, \quad \text{as } m \longrightarrow +\infty,$$

so that $\mathcal{N}_p(u_m - u_0) \longrightarrow 0$, as $m \longrightarrow +\infty$. We conclude by Remark 2.8.

□

Proof of Theorem 1.1. i) By Lemma 3.2, Proposition 3.3 and Theorem 3.1, $c = \inf_{u \in E} I(u)$ is a critical value of I, so that there exists $u^* \in E^p$ such that

$$I'(u_*) = 0 \quad \text{and} \quad I(u_*) = c.$$

So it remains to show that u_* is a non-trivial critical point of I.

ii) Let $u_0 \in \mathrm{C}_0^\infty(\mathbb{R}^N) \setminus \{0\}$ such that $\mathrm{supp}(u_0) \subseteq \Omega$ and $s > 0$. Then, by (F2) and (2.3), we have

$$\begin{aligned}
I(su_0) &\le \frac{s^p}{p} \max\{a,1\} \|u_0\|_E^p + \frac{bs^{p^2}}{p^2} \|u_0\|_E^{p^2} - \int_\Omega F(x, su_0(x))\, dx \\
&\le \frac{s^p}{p} \max\{a,1\} \|u_0\|_E^p + \frac{bs^{p^2}}{p^2} \|u_0\|_E^{p^2} - \eta s^{r_3} \int_J |u_0(x)|^{r_3}\, dx.
\end{aligned}$$

Since $1 < r_3 < p$, the last imples that$I(su_0) < 0$ for $s > 0$ small enough. Therefore, $I(u^*) = c \le I(su_0) < 0$, so that u^* is a nontrivial critical point of I.

□

4. Multiplicity

Our second main result, Theorem 1.2, states that, under conditions (V), $(F1)$–$(F3)$, there exist infinitely many pairs of solutions for problem (P). We achieve our goal by means of a Ljusternik-Schnirelman scheme for even functionals: we shall apply Theorem 4.2 below, as given in [15].

Let X be an infinite-dimensional Banach space and

$$\Sigma_X = \left\{ A \subseteq X \,/\, A = \overline{A},\ A = -A,\ 0 \notin A \right\}.$$

By $\gamma(A)$ we denote the genus of $A \in \Sigma_X$, that is, the least natural number n for which there exists an odd function $\varphi \in \mathrm{C}\left(A, \mathbb{R}^n \smallsetminus \{0\}\right)$. If there is not such n, then $\gamma(A) = +\infty$; and, by definition, $\gamma(\varnothing) = 0$. It's well-known that Krasnoselskii's genus generalizes the notion of dimension: $\gamma(\mathbb{S}_{\mathbb{R}^{l-1}}) = l$ and $\gamma(\mathbb{S}_X) = +\infty$, where $\mathbb{S}_{\mathbb{R}^{l-1}}$ and $\mathbb{S}_X$ denote the unit-spheres of $\mathbb{R}^l$ and X, respectively.

The following properties are useful. Their proof can be found e.g. in [3].

Proposition 4.1. *Let $A, B \in \Sigma_X$. Then*

$$\eta \in \mathrm{C}(A, B) \ \textit{odd} \ \Rightarrow \ \gamma(A) \le \gamma(B);$$
$$A \subseteq B \ \Rightarrow \ \gamma(A) \le \gamma(B);$$
$$A \ \textit{compact} \ \Rightarrow \ \gamma(A) < +\infty.$$

We denote, for $n \in \mathbb{N}$,

$$\mathcal{A}_n = \{ A \in \Sigma_X \,/\, \gamma(A) \ge n \}.$$

Now we can state our abstract tool.

Theorem 4.2. *Assume that $I \in \mathrm{C}^1(X)$ is even and verifies the (PS) condition. For $n \in \mathbb{N}$ we put*

$$c_n = \inf_{A \in \mathcal{A}_n} \sup_{u \in A} I(u). \tag{4.1}$$

i) *If $\mathcal{A}_n \neq \varnothing$ and $c_n \in \mathbb{R}$, then c_n is a critical value of I.*
ii) *If $I(0) \neq c_n = c_{n+1} = \ldots = c_{n+l} \in \mathbb{R}$, then $\gamma(K_c) \ge l + 1$.*

Proof of Theorem 1.2. By Theorems 2.3 and 3.1, the functional I is of class C^1, bounded from below and verifies (PS). By (2.3) and (F3), the functional I is even and $I(0) = 0$. We claim that for every $n \in \mathbb{N}$, there exists $\varepsilon > 0$ such that

$$\gamma\left(I^{-\varepsilon}\right) \ge n. \tag{4.2}$$

Then, by (4.1), it follows that $-\infty < c_n \le -\varepsilon < 0$, whence, by point i) in Theorem 4.2, for every $n \in \mathbb{N}$, c_n is a negative critical value of I.

i) Let's prove the claim. Let $n \in \mathbb{N}$. Let's pick n disjoint open sets $\Omega_i \subseteq \mathbb{R}^N$, $i = 1, \ldots, n$, such that $\bigcup_{i=1}^{n} \Omega_i \subseteq \Omega$. For each $i = 1, \ldots, n$, we take $u_i \in \mathrm{C}_0^\infty(\mathbb{R}^N)$ such that $\mathrm{supp}(u_i) \subseteq \Omega_i$ and $\|u_i\|_p = 1$. We put

$$E_n^p = \mathrm{span}\,\{u_1, u_2, \ldots, u_n\} \quad \text{and} \quad \mathbb{S}_n = \{ u \in E_n^p \,/\, \|u\|_p = 1 \}.$$

ii) Given $u \in E_n^p$, there exist $\lambda_1, ..., \lambda_n \in \mathbb{R}$ such that

$$u = \lambda_1 u_1 + ... + \lambda_n u_n. \tag{4.3}$$

Then,

$$\|u\|_{\mathrm{L}^{r_3}(\mathbb{R}^N)} = \left(\sum_{i=1}^{n} |\lambda_i|^{r_3} \int_{\Omega_i} |u|^{r_3} dx \right)^{1/r_3}, \tag{4.4}$$

and

$$\begin{aligned} \|u\|_p^p &= \sum_{j=1}^{n} \lambda_j^p \int_{J_j} \left(|\nabla u_j|^p + V(x) |u_j|^p \right) dx \\ &= \sum_{j=1}^{n} \lambda_j^p \|u_j\|_p^p = \sum_{j=1}^{n} \lambda_j^p. \end{aligned} \tag{4.5}$$

iii) Since E_n^p is finite-dimensional, all its norms are equivalent. In particular, there exists a constant $\tilde{c} > 0$ such that

$$\tilde{c}\|u\|_E \le \|u\|_{r_3}, \quad \text{for} \quad u \in E_n^p. \tag{4.6}$$

By (2.3) and (4.3)-(4.6), for $u \in \mathbb{S}_n$, we have

$$\begin{aligned} I(su) &\le \frac{s^p}{p} \max\{a,1\} \|u\|_p^p + \frac{bs^{p^2}}{p^2} \|u\|_p^{p^2} - \sum_{j=1}^{n} \int_{\Omega_j} F(x, s\lambda_j u_j(x)) \, dx \\ &\le \frac{s^p}{p} \max\{a,1\} \|u\|_p^p + \frac{bs^{p^2}}{p^2} \|u\|_p^{p^2} - \eta s^{r_3} \sum_{j=1}^{n} |\lambda_j|^{r_3} \int_{\Omega_j} |u_j(x)|^{r_3} \, dx \\ &= \frac{s^p}{p} \max\{a,1\} \|u\|_p^p + \frac{bs^{p^2}}{p^2} \|u\|_p^{p^2} - \eta s^{r_3} \|u\|_{r_3}^{r_3} \\ &\le \frac{s^p}{p} \max\{a,1\} \|u\|_p^p + \frac{bs^{p^2}}{p^2} \|u\|_p^{p^2} - \eta (\tilde{c}s)^{r_3} \|u\|_p^{r_3} \\ &= \frac{s^p}{p} \max\{a,1\} + \frac{bs^{p^2}}{p^2} - \eta (\tilde{c}s)^{r_3}, \end{aligned}$$

whence, since $1 < r_3 < p$ and u was arbitrary, it follows that for some $\epsilon, \sigma > 0$ it holds

$$\forall u \in \mathbb{S}_n : \quad I(\sigma u) < -\epsilon. \tag{4.7}$$

iv) Let $\mathbb{S}_n^\sigma = \sigma \mathbb{S}_n$ and $Q = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^N : \quad \sum_{j=1}^{n} \lambda_j^p < \sigma^p \right\}$. Then, by (4.7), it follows that $I(v) < -\epsilon$, for every $v \in \mathbb{S}_n^\sigma$, so that

$$\mathbb{S}_n^\sigma \subseteq I^{-\epsilon} \in \Sigma.$$

On the other hand, it follows from (4.3) and (4.5) that the mapping $\phi \in C(\mathbb{S}_n^\sigma, \partial Q)$, given by $\phi(u) = \sigma \cdot (\lambda_1, ..., \lambda_n)$, is an odd homeomorphism. Then, by Proposition 4.1, it follows that $\gamma(I^{-\epsilon}) \ge \gamma(\mathbb{S}_n^\sigma) = \gamma(\partial Q) = n$, and so we get (4.2).

□

Acknowledgment. The authors would like to thank the anonymous referees for their comments and suggestions that helped to clarify some point in the paper.

Funding. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Conflicts of interest/Competing interests. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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