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SEMICLASSICAL ASYMPTOTICS OF INFINITELY MANY SOLUTIONS FOR THE INFINITE CASE OF A NONLINEAR SCHRÖDINGER EQUATION WITH CRITICAL FREQUENCY

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Abstract. We consider a nonlinear Schrödinger equation with critical frequency, (P_ε) : $\varepsilon^2\ \Delta v(x) - V(x)\ v(x) + |v(x)|^{p-1}\ v(x) = 0$, $x \in \mathbb{R}^N$, and $v(x) \to 0$ as $|x| \to +\infty$, for the *infinite case* as described by Byeon and Wang. *Critical* means that $0 \leq V \in \mathrm{C}(\mathbb{R}^N)$ verifies $\mathcal{Z} = \{V = 0\} \neq \emptyset$. *Infinite* means that $\mathcal{Z} = \{x_0\}$ and that, grossly speaking, the potential V decays at an exponential rate as $x \to x_0$. For the semiclassical limit, $\varepsilon \to 0$, the infinite case has a characteristic limit problem, $(\mathrm{P}_{\mathrm{inf}})$: $\Delta u(x) - P(x)\, u(x) + |u(x)|^{p-1}\, u(x) = 0$, $x \in \Omega$, with $u(x) = 0$ as $x \in \Omega$, where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded strictly star-shaped region related to the potential V. We prove the existence of an infinite number of solutions for both the original and the limit problem via a Ljusternik-Schnirelman scheme for even functionals. Fixed a topological level k we show that $v_{k,\varepsilon}$, a solution of (P_ε), subconverges, up to a scaling, to a corresponding solution of $(\mathrm{P}_{\mathrm{inf}})$, and that $v_{k,\varepsilon}$ exponentially decays out of Ω. Finally, uniform estimates on $\partial\Omega$ for scaled solutions of (P_ε) are obtained.

1. Introduction

Semiclassical mechanics is an asymptotic method which has helped to study a number of quantum mechanics situations, e.g., phenomena involving atomic and mollecular collisions, by transforming them into classical mechanics objects which are mathematically easier to deal with. This is done by passing to the limit when the reduced Planck constant is allowed to tend to zero and, frequently, the accuracy is good enough, [12].

The time-dependent nonlinear Schrödinger equation,

$$(1)\qquad i\hbar\, \Psi_t(x,t) + \frac{\hbar^2}{2}\Delta\Psi(x,t) - V_0(x)\Psi(x,t) + |\Psi(x,t)|^{p-1}\Psi(x,t) = 0,$$

arises naturally when studying the evolution of Bose-Einstein condensates, [13], and it's also the model for the propagation of light in some nonlinear optical materials, [14]. Now assume that $\hbar$ is no longer a constant but a small positive

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parameter which will decrease to zero. Then a semi-classical state of (1) is a standing-wave having the form $\Psi(x,t) = v(x)\exp(-iEt/\hbar)$, where v, the time-independent component, verifies

$$\varepsilon^2 \Delta v(x) - V(x)\, v(x) + |v(x)|^{p-1} v(x) = 0, \tag{2}$$

with $\varepsilon^2 = \hbar^2/2$ and $V(x) = V_0(x) - E$.

Let's assume that $\mathcal{Z} = \{x \in \mathbb{R}^N \,/\, V(x) = \inf(V)\} \neq \emptyset$. The case $\inf(V) > 0$ is referred to as non-critical and the critical frequency situation corresponds to $\inf(V) = 0$. We shall see that the term *critical* is justified as the qualitative behaviour of the solutions of (2) changes notably.

For the non-critical framework there is a number of works (see e.g. [1], [6], [8], [10], [15], [17] and [19]) based on the variational method or on the Lyapunov-Schmidt reduction. The following are common features:

(N1) a solution of (2), say $\check{v}_\varepsilon$, is bounded away from zero, i.e.,

$$\liminf_{\varepsilon \to 0} \max_x |\check{v}_\varepsilon(x)| > 0; \tag{3}$$

(N2) $\check{v}_\varepsilon$ concentrates around some critical points of V;
(N3) $\check{v}_\varepsilon$ exponentially decays to zero away from such critical points; and,
(N4) there is a unique limit problem and, therefore, a unique profile, as $\varepsilon \to 0$.

The present work helps to complete the study of asymptotic profiles and concentration phenomena for the critical case that was initiated in [5], and elaborated afterwards in [7] for infinitely many solutions whenever $\mathrm{int}(\mathcal{Z}) \neq \emptyset$.

Concretely, we will be concerned with

$$\begin{cases} \varepsilon^2 \Delta v(x) - V(x)\, v(x) + |v(x)|^{p-1} v(x) = 0, & x \in \mathbb{R}^N, \\ v(x) \to 0, & \text{as } |x| \to +\infty, \end{cases} \tag{P$_\varepsilon$}$$

where $p > 1$ if $N = 1, 2$, and

$$2 < 1 + p < 2^* = \frac{2N}{N-2}, \qquad \text{if } N \geq 3. \tag{4}$$

In [5] it's shown the existence of v_ε, a positive standing wave, a least energy solution, for which:

(C1) (3) no longer holds and, instead, the following behaviour is verified:

$$\lim_{\varepsilon \to 0} \|v_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R}^N)} = 0, \tag{5}$$

$$\liminf_{\varepsilon \to 0} \frac{\|v_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R}^N)}}{\varepsilon^{2/(p-1)}} > 0; \tag{6}$$

(C2) v_ε concentrates around an isolated component of $\mathcal{Z} = \{V = 0\}$;
(C3) v_ε exponentially decays outside $\mathcal{Z}$; and,
(C4) there is not a unique limit problem so that there is no unique profile; it actually depends on how the potential V behaves nearby $\mathcal{Z}$.

In [5] three cases were considered: *Flat*, where $\operatorname{int}(\mathcal{Z}) \neq \emptyset$ is bounded; *Finite*, where $\mathcal{Z}$ is finite and V vanishes like a polynomial around it; and, *Infinite*, where $\mathcal{Z}$ is finite and V vanishes like an exponential function around it. The limit problem for the finite case is defined on the whole space, meanwhile the flat and infinite cases have their limit problems defined on appropriate subregions of $\mathbb{R}^N$. In addition, for the three cases, it was proved that

- (C5) a scaling of v_ε converges to u, a positive least-energy solution of the corresponding limit problem;
- (C6) the energy of v_ε converges to the energy of u.

The work [7] deals with the flat case assuming that the potential satisfies the following conditions:

- (V1) $V \in \mathrm{C}(\mathbb{R}^N)$ is non-negative;
- (V2) $V(x) \to +\infty$, as $|x| \to +\infty$;
- ($\mathrm{V}_{\mathrm{flat}}$) $\operatorname{int}(\mathcal{Z}) \neq \emptyset$ is connected and smooth.

The corresponding limit problem is

$$(\mathrm{P}_{\mathrm{flat}}) \qquad \begin{cases} \Delta u(x) + |u(x)|^{p-1}u(x) = 0, & x \in \mathcal{Z}, \\ u(x) = 0, & x \in \partial\mathcal{Z}. \end{cases}$$

In [7] it was applied a Ljusternik-Schnirelman machinery to natural even funcionals I_ε and I, and was showed the existence of sequences of solutions, $(v_{k,\varepsilon})_{k\in\mathbb{N}}$ and $(u_k)_{k\in\mathbb{N}}$, for (P_ε) and ($\mathrm{P}_{\mathrm{flat}}$), respectively. Fixed k, the authors proved that, as $\varepsilon \to 0$, the solution $v_{k,\varepsilon}$, not necessarily positive, behaves like v_ε, the positive solution found in [5]: (C1)-(C3) and (C6) hold. Point (C5) holds as well: a scaling of $v_{k,\varepsilon}$ subconverges to w_k a solution of ($\mathrm{P}_{\mathrm{flat}}$) sharing the energy level of u_k, $I(w_k) = I(u_k)$. In [7] further asymptotic estimates on the boundary of $\mathcal{Z}$ were obtained.

Remark 1.1. Condition (V2) is more restrictive than the one considered in [5] were it's required $\liminf_{|x|\to+\infty} V(x) > 2\beta$, for some $\beta > 0$. In other hand, condition (V2) allows us to use Theorem 2.11, a Sobolev-like embedding, and, as a consequence, be able to apply Theorem 3.3, our multiplicity tool.

In short, we show in this paper that the type of results of [7] hold for the infinite case. The document is organized in the following way

- In Section 2.1 we state, in a precise way, the general setting which characterize the infinite case: conditions (V3) and ($\mathrm{V}_{\mathrm{inf}}$). This allows us to introduce the main results in Section 2.2.
- In Section 2.3 we present a number of properties of the potential V derived from (V3) and ($\mathrm{V}_{\mathrm{inf}}$). This is also the place for the important Theorem 2.11 which states that a kind of Sobolev space, where the solutions of (P_ε) are found, is compactly contained in a range of L^q-spaces.

- In Section 3 a Ljusternik-Schnirelman scheme for even functionals is set up to obtain the multiplicity result, Theorem 2.3.
- In Section 4 we prove Theorem 2.4 which provides energy asymptotics, i.e., that the energy of each solution $v_{k,\varepsilon}$ of (P_ε) converges, as $\varepsilon \to 0$, to the energy of a corresponding solution of the limit problem, $(\mathrm{P}_{\mathrm{inf}})$. In the context of the Ljusternik-Schnirelman theory, the index k represents the topological characteristic of the level set, so that this result implies that the k-th level sets of appropriate functionals associated to (P_ε) and $(\mathrm{P}_{\mathrm{inf}})$ are topologically equivalent.
- Section 5 is devoted to the proof of Theorem 2.5, i.e., the asymptotic profiles of the solutions of (P_ε) inside $\Omega \subseteq \mathbb{R}^N$, a smooth bounded stricly star-shaped domain related to the potential V.
- Section 6 is dedicated to the proof of Theorem 2.6, an asymptotic concentration phenomena, the exponential decay of the solutions of (P_ε) out of Ω.
- Finally, Section 7 is the place for the proof of Theorem 2.7, which is a type of uniform estimate on $\partial\Omega$ for scaled solutions of (P_ε).

2. Main results and preliminaries

2.1. Infinite case setting

We study the problem (P_ε) where, in addition to properties (V1) and (V2), we shall assume that V verifies two more conditions which replace $(\mathrm{V}_{\mathrm{flat}})$. One of them is

(V3) $\mathcal{Z} = \{0\}$.

The second condition, $(\mathrm{V}_{\mathrm{inf}})$ below, differentiates our situation with that of the finite case. For its statement we need a couple of concepts.

Let $\Omega \subseteq \mathbb{R}^N$ be a smooth bounded stricly star-shaped domain, i.e., there exists a ball $B \subseteq \Omega$ such that given any $x \in B$ and any $y \in \Omega$, $[x, y] \subseteq \Omega$. It's well known, [18], that for $q \geq 1$, Ω is a q-Poincaré domain, i.e., there exists $M_q > 0$ such that for every $u \in \mathrm{C}^1(\Omega)$, $\|u - u_\Omega\|_{\mathrm{L}^q(\Omega)} \leq M_q \left(\int_\Omega |\nabla u(x)|^q \, dx\right)^{1/q}$, where u_Ω denotes the average of u over Ω. We assume that Ω is generated by a positive function $r \in \mathrm{C}(\mathbb{R}^N \setminus \{0\})$ that verifies

$$
\begin{aligned}
t = r(x) \ &\Rightarrow\ x/t \in \partial\Omega, \\
t > r(x) \ &\Rightarrow\ x/t \in \Omega, \\
t < r(x) \ &\Rightarrow\ x/t \in \mathbb{R}^N \setminus \Omega.
\end{aligned} \tag{7}
$$

Point (7) implies that every non-zero point is well determined by a point in the boundary of Ω, i.e., given $x \in \mathbb{R}^N \setminus \{0\}$ there exists a unique $s(x) \in \partial\Omega$ such that $x = r(x)\,s(x)$. Also observe that

$$r(x) = 1 \;\Leftrightarrow\; x \in \partial\Omega, \qquad r(x) > 1 \;\Leftrightarrow\; x \in \mathbb{R}^N \setminus \overline{\Omega}, \qquad r(x) < 1 \;\Leftrightarrow\; x \in \Omega, \tag{8}$$

and $r\,(x/t) = r(x)/t$, for every $x \in \mathbb{R}^N \setminus \{0\}$ and every $t > 0$.

Let's consider $b \in \mathrm{C}(\mathbb{R}^N)$, an Ω-*quasi homogeneous function.* This means that there exists a function $\beta : [0, +\infty[\to \mathbb{R}$ such that

b1) $b(x) = \beta(r(x))$, for every $x \in \mathbb{R}^N$;
b2) β is non-negative and strictly-increasing;
b3) given $L = \lim_{r\to 0} \beta(cr)/\beta(r)$, it holds $L < 1$ if $c < 1$ and $L > 1$ if $c > 1$.

We also consider $a \in \mathrm{C}(\mathbb{R}^N)$, an *asymptotically* (Ω, b)-*quasihomogeneous function*, i.e., a is positive and $a(x)/b(x) \to 1$, as $|x| \to 0$. Now we can write the condition for the potential that characterizes the infinite case:

($\mathrm{V}_{\mathrm{inf}}$) $V(x) = \exp\left(-1/a(x)\right)$ if $|x| \le 1$.

Let's introduce some more objects to state our main results in the following section. Given $\varepsilon > 0$ and $x \in \mathbb{R}^N$ we put

$$g(\varepsilon) = \frac{1}{b^{-1}\left(\dfrac{-1}{\ln(\varepsilon^2)}\right)} \quad \text{and} \quad V_\varepsilon(x) = \frac{1}{[\varepsilon\, g(\varepsilon)]^2}\, V\left(\frac{x}{g(\varepsilon)}\right). \tag{9}$$

Remark 2.1. The following problems are closely related to (P_ε):

$$(\mathrm{P}'_\varepsilon) \qquad \begin{cases} \Delta w(x) - V_\varepsilon(x)\, w(x) + |w(x)|^{p-1} w(x) = 0, & x \in \mathbb{R}^N, \\ w(x) \to 0, & \text{as } |x| \to +\infty, \end{cases}$$

$$(\hat{\mathrm{P}}_\varepsilon) \qquad \begin{cases} \Delta \hat{w}(x) - V_\varepsilon(x)\, \hat{w}(x) + 2\Theta\, |\hat{w}(x)|^{p-1} \hat{w}(x) = 0, & x \in \mathbb{R}^N, \\ \hat{w}(x) \to 0, & \text{as } |x| \to +\infty, \end{cases}$$

where $\Theta = (\hat{w}, \hat{w})_\varepsilon/2$, and

$$(u, v)_\varepsilon = \int_{\mathbb{R}^N} \left[\nabla u(x) \nabla v(x) + V_\varepsilon(x) u(x) v(x)\right] dx. \tag{10}$$

In fact, if $\hat{w}$ is a solution of ($\hat{\mathrm{P}}_\varepsilon$), then

$$w(x) = (2\Theta)^{1/(p-1)} \hat{w}(x), \quad x \in \mathbb{R}^N,$$

is a solution of (P'_ε), and

$$v(x) = [\varepsilon\, g(\varepsilon)]^{2/(p-1)}\, w(g(\varepsilon)\, x) = \left[2\Theta \cdot (\varepsilon\, g(\varepsilon))^2\right]^{1/(p-1)} \hat{w}(g(\varepsilon)\, x),$$

$x \in \mathbb{R}^N$, is a solution of (P_ε).

Remark 2.2. Under (V1), (V2), (V3) and ($\mathrm{V_{inf}}$), the limit problem of (P_ε) is

$$(\mathrm{P_{inf}}) \qquad \begin{cases} \Delta w(x) + |w(x)|^{p-1}\, w(x) = 0, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega. \end{cases}$$

Related to ($\mathrm{P_{inf}}$) is the problem

$$(\hat{\mathrm{P}}_{\mathrm{inf}}) \qquad \begin{cases} \Delta \hat{w}(x) + 2\Upsilon\, |\hat{w}(x)|^{p-1}\, \hat{w}(x) = 0, & x \in \Omega, \\ \hat{w}(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Upsilon = (\hat{w}, \hat{w})_{\mathrm{H}_0^1(\Omega)}/2$. In fact, if $\hat{w}$ is a solution of ($\hat{\mathrm{P}}_{\mathrm{inf}}$), then

$$w(x) = (2\Upsilon)^{1/(p-1)}\hat{w}(x), \quad x \in \Omega,$$

is a solution of ($\mathrm{P_{inf}}$). Here, as usual, the Sobolev space $\mathrm{H}_0^1(\Omega)$ is equipped with the inner-product given by

$$(u, v)_{\mathrm{H}_0^1(\Omega)} = \int_\Omega \nabla u(x) \nabla v(x)\, dx.$$

2.2. Main results

Let H_ε be the completion of $\mathrm{C}_0^\infty(\mathbb{R}^N)$ with the inner-product given by (10). The corresponding norm is denoted by $\|\cdot\|_\varepsilon$. We consider the functionals $J_\varepsilon : \mathcal{M}_\varepsilon \subseteq \mathrm{H}_\varepsilon \to \mathbb{R}$ and $J : \mathcal{M} \subseteq \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ given by

$$J_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla u(x)|^2 + V_\varepsilon(x)|u(x)|^2\right] dx, \tag{11}$$

$$J(u) = \frac{1}{2}\|u\|_{\mathrm{H}_0^1(\Omega)}^2 = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx,$$

working on the Nehari manifolds $\mathcal{M}_\varepsilon = \{w \in \mathrm{H}_\varepsilon \,/\, \|w\|_{\mathrm{L}^{p+1}(\mathbb{R}^N)} = 1\}$ and $\mathcal{M} = \{w \in \mathrm{H}_0^1(\Omega) \,/\, \|w\|_{\mathrm{L}^{p+1}(\mathbb{R}^N)} = 1\}$, respectively.

Now we present our main results. We shall always assume that (V1)-(V3) and ($\mathrm{V_{inf}}$) hold. We start with the multiplicity result.

Theorem 2.3. *The following points are true.*

i) *Given $\varepsilon > 0$, the functional J_ε has a sequence of different critical points $(\hat{w}_{k,\varepsilon})_{k\in\mathbb{N}} \subseteq \mathcal{M}_\varepsilon$. For each $k \in \mathbb{N}$ the function given by*

$$v_{k,\varepsilon}(x) = \left[2c_{k,\varepsilon}\,(\varepsilon\, g(\varepsilon))^2\right]^{1/(p-1)} \hat{w}_{k,\varepsilon}\left(\frac{x}{g(\varepsilon)}\right), \quad x \in \mathbb{R}^N, \tag{12}$$

where $c_{k,\varepsilon} = J_\varepsilon(\hat{w}_{k,\varepsilon})$, is a weak solution of (P_ε).

ii) *The functional J has a sequence of different critical points $(\hat{w}_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$. For each $k \in \mathbb{N}$ function given by*

$$w_k(x) = (2c_k)^{1/(p-1)}\, \hat{w}_k(x), \quad x \in \overline{\Omega},$$

where $c_k = J(\hat{w}_k)$ is a weak solution of ($\mathrm{P_{inf}}$).

We apply a Ljusternik-Schnirelman scheme to prove Theorem 2.3. In this context, the index k of a critical value represents the topological characteristic of the level set, as captured by Krasnoselskii's genus. Consequently, the convergence of energies which we are about to state implies that the critical values of J and J_ε are topologically equivalent.

Theorem 2.4. *Let $k \in \mathbb{N}$. Then, $c_{k,\varepsilon} \to c_k$, as $\varepsilon \to 0$.*

In the following result we provide the asymptotic profiles of the solutions of (P_ε) that we found in Theorem 2.3. To state it, we need the concept of subconvergence as introduced in [5]. A family of functions $(f_\varepsilon)_{\varepsilon>0}$ is said to subconverge in a space X, as $\varepsilon \to 0$, iff every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to zero, has a subsequence $(\varepsilon_{n_i})_{i\in\mathbb{N}}$ such that $\left(f_{\varepsilon_{n_i}}\right)_{i\in\mathbb{N}}$ converges in X, as $i \to +\infty$.

Theorem 2.5. *Let $k \in \mathbb{N}$. As $\varepsilon \to 0$, $(w_{k,\varepsilon})_{\varepsilon>0}$ subconverges in $\mathrm{H}^1(\mathbb{R}^N)$ to some $u_k \in \mathrm{H}^1(\mathbb{R}^N)$ such that its restriction to Ω is a solution of $(\mathrm{P}_{\inf})$ and verifies $J(\hat{u}_k|_\Omega) = c_k$, where $\hat{u}_k = (2c_k)^{1/1-p} u_k$.*

Next, we have the result concernig the exponential decay of the solutions of (P_ε) out of Ω. Given $h > 0$, let's denote

$$\Omega^h = \{x \in \mathbb{R}^N \,/\, \mathrm{dist}(x,\Omega) < h\}. \tag{13}$$

Theorem 2.6. *Let $k \in \mathbb{N}$ and $\delta > 0$. Then there exist $\varepsilon_\delta > 0$ and $C = C(N,k,p,\delta) > 0$ such that*

$$\forall \varepsilon \in]0,\varepsilon_\delta[, \forall x \in \mathbb{R}^N : \quad |\hat{w}_{k,\varepsilon}(x)| < \frac{C}{(2c_k)^{1/(p-1)}} \exp\left(\gamma_{\delta,\varepsilon}\,\mathrm{dist}(x,\Omega^\delta)\right),$$

where $\gamma_{\delta,\varepsilon} = \gamma_{\delta,\varepsilon}(N,k,p) \to -\infty$, as $\varepsilon \to 0$.

Let's remark that in the path of proving Theorem 2.6 we show, for each $k \in \mathbb{N}$, that $v_{k,\varepsilon}$ verifies (5) and

$$\liminf_{\varepsilon\to 0} \frac{\|v_{k,\varepsilon}\|_{\mathrm{L}^\infty(\mathbb{R}^N)}}{[2c_{k,\varepsilon}(\varepsilon g(\varepsilon)^2)]^{1/(p-1)}} > 0,$$

which is qualitatively analogous to (6).

To finish this section let's present a type of uniform estimate on $\partial\Omega$ that was first found in [7] for the solutions of $(\mathrm{P}'_\varepsilon)$ in the flat case.

Theorem 2.7. *Let $k \in \mathbb{N}$. Then, $\max_{x\in\partial\Omega} |w_{k,\varepsilon}(x)| = 0$, as $\varepsilon \to 0$.*

2.3. Preliminary results

Let's first state some useful properties that involve the functions g, V_ε, b and β which come from points (7), b1), b2) and b3).

The functions g and V_ε are given in (9) so that, by $(\mathrm{V}_{\inf})$,

$$V_\varepsilon(x) = \frac{1}{[\varepsilon\, g(\varepsilon)]^2} \exp\left(-\frac{1}{a\,(x/g(\varepsilon))}\right), \qquad |x| \le g(\varepsilon).$$

As it's stated in [5], we have that

$$\lim_{\varepsilon\to 0} g(\varepsilon) = +\infty;$$

$$\exists \alpha > 0 : \quad \lim_{r\to 0} \frac{\beta(r)}{r^\alpha} = 0 \ \wedge \ \lim_{\varepsilon\to 0} \frac{g(\varepsilon)}{|\ln(\varepsilon)|^{1/\alpha}} = 0;$$

$$\forall c > 0 : \quad \lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \exp\left(\frac{c}{b\,(1/g(\varepsilon))}\right) = \lim_{\varepsilon\to 0} \frac{1}{[\varepsilon^c g(\varepsilon)]^2} = +\infty. \tag{14}$$

The following property is an easy consequence of point (22) in [5].

Proposition 2.8. *For every measurable* $B \subseteq \Omega$, $\|V_\varepsilon\|_{\mathrm{L}^\infty(B)} \to 0$, *as* $\varepsilon \to 0$.

Now we rewrite point (23) of [5] for our context.

Proposition 2.9. *There exists* $D \in]0,1[$ *such that for all* $d > 1$,

$$\lim_{\varepsilon\to 0} \min_{x\in R_{\varepsilon,D,d}} V_\varepsilon(x) = +\infty, \tag{15}$$

where $R_{\varepsilon,D,d} = \{x \in \mathbb{R}^N \,/\, |x| \le D\, g(\varepsilon) \ \wedge \ r(x) \ge d\}$.

Remark 2.10. By (8), the region $R_{\varepsilon,D,d}$ is the set of points in the closed ball centered at zero and of radius $Dg(\varepsilon)$ which are out of the expanded star $\Omega^{h_0} = \{x \in \mathbb{R}^N \,/\, r(x) < d\}$, where $h_0 > 0$ is the value compatible with (13).

In short, the following result states that a weighted Sobolev space such that the weight function verifies (V1)-(V2) is compactly contained in a range of L^q spaces.

Theorem 2.11. *Let* $W \in \mathrm{C}(\mathbb{R}^N)$ *be non-negative and such that* $W(x) \to +\infty$ *as* $|x| \to +\infty$ *and* H_W *the Hilbert space that results of completing* $\mathrm{C}_0^\infty(\mathbb{R}^N)$ *with the inner-product given by*

$$(v,w)_W = \int_{\mathbb{R}^N} \left[\nabla v(x)\nabla w(x) + U(x)\, v(x)w(x)\right]\, dx.$$

Then, the embedding $\mathrm{H}_W \subseteq \mathrm{L}^q(\mathbb{R}^N)$ *is compact for all* $q \in [2, Q[$, *where* $Q = 2^*$ *if* $N \ge 3$ *and* $Q = +\infty$ *if* $N = 1, 2$.

Theorem 2.11 is well known and has been extended to the context of $\mathrm{W}^{1,p}(\mathbb{R}^N)$ spaces, [2, Lemma 2.4]. It can be obtained by an application of Fréchet-Kolmogorov compactness criteria, [3, Cor. 4.26 & 4.27], by compensating the non-boundedness of the domain with the property of W exploding at infinity.

Given an open set $U \subseteq \mathbb{R}^N$, we shall always identify a function $u \in \mathrm{H}_0^1(U)$ with its extension by zero, $\overline{u}$,

$$\overline{u}(x) = \begin{cases} u(x), & \text{if } x \in U; \\ 0, & \text{if } x \in \mathbb{R}^N \setminus U. \end{cases}$$

With this identification, it makes sense the following result.

Proposition 2.12. *Let $\varepsilon > 0$. Then the embedding $\mathrm{H}_0^1(\Omega) \subseteq \mathrm{H}_\varepsilon$ is continuous. On $\mathrm{H}_0^1(\Omega)$ the norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_{\mathrm{H}_0^1(\Omega)}$ are equivalent.*

Proof. By a direct computation, we obtain

$$\forall u \in \mathrm{H}_0^1(\Omega): \quad \|u\|_{\mathrm{H}_0^1(\Omega)} \leq \|u\|_\varepsilon \leq C_{\Omega,\varepsilon} \, \|u\|_{\mathrm{H}_0^1(\Omega)}\,, \tag{16}$$

where

$$C_{\Omega,\varepsilon} = \left(1 + C_\Omega^2 \, \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)}\right)^{1/2} > 0, \tag{17}$$

with C_Ω the constant appearing in Poincaré's inequality, [3, Cor.9.19]. □

To finish this section, let's recall that, by Hölder inequality, for a measurable set $\Lambda \subseteq \mathbb{R}^N$ such that $|\Lambda| < +\infty$, it holds

$$\forall w \in \mathrm{L}^{p+1}(\Lambda): \quad \|w\|_{\mathrm{L}^2(\Lambda)} \leq |\Lambda|^{(p-1)/2(p+1)} \, \|w\|_{\mathrm{L}^{p+1}(\Lambda)}\,. \tag{18}$$

3. Multiplicity

In this section we show how the Ljusternik-Schnirelman theory provides our multiplicity result, Theorem 2.3, in a very straightforward way. Starting in this section we focus on the case $N \geq 3$ as the alternatives $N = 1, 2$ are easier to deal with.

Let E be a Banach space. We write

$$\Sigma_E = \left\{A \subseteq E \,/\, A = \overline{A},\, A = -A,\, 0 \notin A\right\}.$$

The genus of $A \in \Sigma_E$, denoted by $\gamma(A)$ is the least natural number k for which there exists an odd function $f \in \mathrm{C}\left(A, \mathbb{R}^k \backslash \{0\}\right)$. If there is not such k, then $\gamma(A) = +\infty$; and, by definition, $\gamma(\emptyset) = 0$. The concept of genus, introduced by Krasnoselskii, generalizes the notion of dimension: $\gamma(\mathbb{S}^{m-1}) = m$ and $\gamma(\mathbb{S}_Y^\infty) = +\infty$, where $\mathbb{S}^{m-1}$ and $\mathbb{S}_Y^\infty$ are the unit-spheres of $\mathbb{R}^m$ and Y, an infinite-dimensional Banach space, respectively. A proof of the following properties can be found in [16].

Proposition 3.1. *Let $A, B \in \Sigma_E$. Then*

$$\begin{aligned} f \in \mathrm{C}(A,B) \text{ odd} &\Rightarrow \gamma(A) \leq \gamma(B); \\ A \subseteq B &\Rightarrow \gamma(A) \leq \gamma(B); \\ A \text{ compact} &\Rightarrow \gamma(A) < +\infty. \end{aligned} \tag{19}$$

Remark 3.2. Let M be a C^1 manifold in the Banach space X and $\phi \in \mathrm{C}^1(M)$. Recall that $(y_n)_{n\in\mathbb{N}} \subseteq M$ is a Palais-Smale (PS) sequence iff $(\phi(y_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $\|\phi'(y_n)\|_{X^*} \to 0$, as $n \to +\infty$. We say that (M, ϕ) verifies (PS) condition iff every (PS) sequence has a converging subsequence.

The following theorem, stated and proved in [16], is our main tool.

Theorem 3.3. *Let $M \in \Sigma_E$ be a C^1 manifold of E and let $f \in \mathrm{C}^1(E)$ be even. Suppose that (M, f) satisfy the Palais-Smale (PS) condition and let*

$$C_k(f) = \inf_{A \in \mathcal{A}_k(M)} \max_{u \in A} f(u),$$

where $\mathcal{A}_k(M) = \{A \in \Sigma_E \cap M \,/\, \gamma(A) \geq k\}$. Let's denote by K_c the set of critical points of f corresponding to the value c. Then

- i) *$\gamma(\mathcal{M}) \leq \sum_{c \in \mathbb{R}} \gamma(K_c)$ so that $f|_{\mathcal{M}}$ has at least $\gamma(\mathcal{M})$ pairs of critical points;*
- ii) *if $C_k(f) \in \mathbb{R}$, then $C_k(f)$ is a critical value of f. Moreover, if $c = C_k(f) = \cdots = C_{k+m}(f)$, then $\gamma(K_c) \geq m+1$. In particular, if $m > 1$, then K_c contains infinitely many elements.*

The potentials V and V_ε verify the conditions of Theorem 2.11 so that, in particular, the result holds for $\mathrm{H}_\varepsilon = \mathrm{H}_{V_\varepsilon}$. With the last, it is proved that the functional J_ε is of class C^1 and satisfies the Palais-Smale condition on $\mathcal{M}_\varepsilon$. Then, in the context of Theorem 3.3, we write, for $k \in \mathbb{N}$ and $\varepsilon > 0$,

$$\Sigma_\varepsilon = \Sigma_{\mathrm{H}_\varepsilon}, \quad \mathcal{A}_{k,\varepsilon} = \mathcal{A}_k(\mathcal{M}_\varepsilon), \quad c_{k,\varepsilon} = C_k(J_\varepsilon) = J(\hat{w}_{k,\varepsilon}) \in]0, +\infty[.$$

It's clear that the functional J also satisfies the hypothesis of Theorem 3.3. For $k \in \mathbb{N}$, we write

$$\Sigma = \Sigma_{\mathrm{H}_0^1(\Omega)}, \quad \mathcal{A}_k = \mathcal{A}_k(\mathcal{M}), \quad c_k = C_k(J) = J(\hat{w}_k) \in]0, +\infty[.$$

The proof of Theorem 2.3 is then completed by the changes of variables presented in Remarks 2.1 and 2.2.

Remark 3.4. In the coming sections, the following intermediate problem will be useful:

$$(\mathrm{P}^\delta_{\inf}) \qquad \begin{cases} \Delta u(x) + |u(x)|^{p-1} u(x), & x \in \Omega^\delta, \\ u(x) = 0, & x \in \partial\Omega^\delta. \end{cases}$$

We put $\mathcal{M}^\delta = \{u \in \mathrm{H}_0^1(\Omega^\delta) \,/\, \|u\|_{\mathrm{L}^p(\Omega^\delta)} = 1\}$. The functional $J^\delta : \mathcal{M}^\delta \to \mathbb{R}$, given by $J^\delta(u) = \|u\|^2_{\mathrm{H}_0^1(\Omega^\delta)}/2$ also satisfies the conditions of Theorem 3.3 so that we write, for $k \in \mathbb{N}$,

$$\Sigma^\delta = \Sigma_{\mathrm{H}_0^1(\Omega^\delta)}, \quad \mathcal{A}_k^\delta = \mathcal{A}_k(\mathcal{M}^\delta), \quad c_k^\delta = C_k(J^\delta) = J^\delta(\hat{w}_k^\delta) \in]0, +\infty[.$$

The function $w_k^\delta = \left(2c_k^\delta\right)^{1/(p-1)} \hat{w}_k^\delta$ is a solution of $(\mathrm{P}^\delta_{\inf})$.

4. Energy asymptotics

This section is devoted to the proof of Theorem 2.4, i.e., the energy asymptotics $c_{k,\varepsilon} \to c_k$, as $\varepsilon \to 0$, for every $k \in \mathbb{N}$, where

$$c_{k,\varepsilon} = \inf_{A \in \mathcal{A}_{k,\varepsilon}} \max_{u \in A} J_\varepsilon(u), \tag{20}$$

$$c_k = \inf_{A \in \mathcal{A}_k} \max_{u \in A} J(u). \tag{21}$$

By the Ljusternik-Schnirelman scheme that was used in Section 3, this means that the k-th level sets of J_ε and J are topologically equivalent. Also recall that k represents the topological characteristic of the level set, as captured by Krasnoselskii's genus.

Proposition 4.1. *Let $k \in \mathbb{N}$. Then the following points hold*

$$\forall \varepsilon > 0 : \quad \mathcal{A}_k \subseteq \mathcal{A}_{k,\varepsilon} \ \wedge \ c_{k,\varepsilon} \le c_k\, C_{\Omega,\varepsilon}, \tag{22}$$

$$\limsup_{\varepsilon \to 0} c_{k,\varepsilon} \le c_k, \tag{23}$$

where $C_{\Omega,\varepsilon} > 0$ is given in (17).

Proof. Let $\varepsilon > 0$. By Proposition 2.12, the norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_{\mathrm{H}^1_0(\Omega)}$ induce the same topology. This immediately implies that $\mathcal{A}_k \subseteq \mathcal{A}_{k,\varepsilon}$. Moreover, by (20), (21) and (16),

$$\begin{aligned} c_{k,\varepsilon} &= \inf_{A \in \mathcal{A}_{k,\varepsilon}} \max_{u \in A} J_\varepsilon(u) = \inf_{A \in \mathcal{A}_k} \max_{u \in A} J_\varepsilon(u) \\ &\le C_{\Omega,\varepsilon} \inf_{A \in \mathcal{A}_k} \max_{u \in A} J(u) = C_{\Omega,\varepsilon}\, c_k. \end{aligned} \tag{24}$$

By Proposition 2.8 we have that $\|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)} \to 0$, as $\varepsilon \to 0$, so that (16) and (24) imply (23). □

For the next step let's observe that for $k \in \mathbb{N}$ and $\delta > 0$,

$$c_k^\delta = \inf_{A \in \mathcal{A}_k^\delta} \max_{u \in A} J^\delta(u).$$

Proposition 4.2. *Let $k \in \mathbb{N}$ and $\sigma > 0$. There exist $\delta_0, \varepsilon_2 > 0$ such that*

$$\forall \delta \in]0, \delta_0[, \ \forall \varepsilon \in]0, \varepsilon_2[: \quad c_k^\delta \le c_{k,\varepsilon} + \sigma. \tag{25}$$

Proof. This is the longest proof of this paper so that we shall divide it in several steps.

i) Let $\varepsilon > 0$ and $\delta \in]0,1[$. By (20) there exists $A_\sigma(\varepsilon) \in \mathcal{A}_{k,\varepsilon}$ such that

$$\max_{u \in A_\sigma(\varepsilon)} J_\varepsilon(u) \le c_{k,\varepsilon} + \frac{\sigma}{4}. \tag{26}$$

By Proposition 2.8 we have that

$$\forall \mu > 0, \exists \check{\varepsilon} = \check{\varepsilon}(\mu) > 0 : \quad \varepsilon \in]0, \check{\varepsilon}[\ \Rightarrow \ \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)} < \mu. \tag{27}$$

We choose

$$\mu = \frac{8\sigma c_k + \sigma^2}{16 C_\Omega^2\, c_k^2}, \qquad \varepsilon_0 = \check{\varepsilon}(\mu) = \varepsilon_0(\sigma, k). \tag{28}$$

From now on we assume that $\varepsilon \in]0, \varepsilon_0[$. Then, by (22), (28), (27) and (17), we get

$$\begin{aligned} c_{k,\varepsilon}^2 &\le c_k^2 + C_\Omega^2 \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)}\, c_k^2 \\ &\le c_k^2 + \frac{\sigma}{2} c_k + \frac{\sigma^2}{16} = \left(c_k + \frac{\sigma}{4}\right)^2. \end{aligned} \tag{29}$$

Let's denote $b_{k,\sigma} = c_k + \sigma/4$. Then, by (26) and (29), it follows that

$$\forall v \in A_\sigma(\varepsilon): \quad J_\varepsilon(v) \leq b_{k,\sigma}, \tag{30}$$

which, by (11), implies that

$$\forall v \in A_\sigma(\varepsilon): \quad \int_{\mathbb{R}^N} |\nabla v(x)|^2\, dx \leq 2b_{k,\sigma}, \tag{31}$$

$$\forall v \in A_\sigma(\varepsilon): \quad \int_{\mathbb{R}^N} V_\varepsilon(x) \cdot |v(x)|^2\, dx \leq 2b_{k,\sigma}. \tag{32}$$

For $\rho > 0$, let's denote

$$V_{\rho,\varepsilon} = \inf_{x \in \mathbb{R}^N \setminus \Omega^\rho} V_\varepsilon(x). \tag{33}$$

Points (15) and (V2) imply that

$$\lim_{\varepsilon \to 0} V_{\delta,\varepsilon} = +\infty. \tag{34}$$

ii) From (32) we get

$$\forall v \in A_\sigma(\varepsilon): \quad \|v\|^2_{\mathrm{L}^2(\mathbb{R}^N \setminus \Omega^\delta)} = \int_{\mathbb{R}^N \setminus \Omega^\delta} |v(x)|^2 dx \leq \frac{2b_{k,\sigma}}{V_{\delta,\varepsilon}}. \tag{35}$$

In other hand, by (31) and Sobolev-Gagliardo-Niremberg inequality, [3, Th. 9.9], it follows that

$$\forall v \in A_\sigma(\varepsilon): \quad \|v\|_{\mathrm{L}^{2^*}(\mathbb{R}^N)} \leq \theta\, \|\nabla v\|_{\mathrm{L}^2(\mathbb{R}^N)} \leq \theta(2b_{k,\sigma})^{1/2},$$

where $\theta = \theta_N > 0$. Therefore,

$$\forall v \in A_\sigma(\varepsilon): \quad \|v\|_{\mathrm{L}^{2^*}(\mathbb{R}^N \setminus \Omega^\delta)} \leq \theta(2b_{k,\sigma})^{1/2}. \tag{36}$$

Now, by (4), we choose $\alpha \in]0,1[$ such that $1/(p+1) = (1-\alpha)/2 + \alpha/2^*$. Then, by (35), (36) and the interpolation inequality for L^q-spaces, [3, pg.93], it follows, for $v \in A_\sigma(\varepsilon)$, that

$$\begin{aligned}\|v\|_{\mathrm{L}^{p+1}(\mathbb{R}^N \setminus \Omega^\delta)} &\leq \|v\|^{1-\alpha}_{\mathrm{L}^2(\mathbb{R}^N \setminus \Omega^\delta)}\ \|v\|^{\alpha}_{\mathrm{L}^{2^*}(\mathbb{R}^N \setminus \Omega^\delta)} \\ &\leq \left(\frac{2b_{k,\sigma}}{V_{\delta,\varepsilon}}\right)^{(1-\alpha)/2} \theta^\alpha (2b_{k,\sigma})^{\alpha/2} = \frac{\theta^\alpha (2b_{k,\sigma})^{1/2}}{V_{\delta,\varepsilon}^{(1-\alpha)/2}},\end{aligned}$$

which, by (34), implies that $\max_{v \in A_\sigma(\varepsilon)} \|v\|_{\mathrm{L}^{p+1}(\mathbb{R}^N \setminus \Omega^\delta)} \to 0$, as $\varepsilon \to 0$. Therefore, given $s > 0$, there exists $\varepsilon_1 = \varepsilon_1(\delta, s;\, \sigma, k) \in]0, \varepsilon_0[$ such that

$$\forall \varepsilon \in]0, \varepsilon_1[: \quad \max_{v \in A_\sigma(\varepsilon)} \|v\|_{\mathrm{L}^{p+1}(\mathbb{R}^N \setminus \Omega^\delta)} < \delta^s. \tag{37}$$

In particular, for $s = 1$ and $\hat{\varepsilon}_1 = \varepsilon_1(\delta, 1; \sigma, k) \in]0, \varepsilon_0[$, we get

$$\forall \varepsilon \in]0, \hat{\varepsilon}_1[,\ \forall v \in A_\sigma(\varepsilon): \quad \|v\|_{\mathrm{L}^{p+1}(\Omega^\delta)} \geq 1 - \delta. \tag{38}$$

iii) Now let's pick a cut-off function $\phi_\delta \in \mathrm{C}_0^\infty(\mathbb{R}^N)$ such that

$$\begin{aligned} &\forall x \in \Omega^{\delta/2} : \quad \phi_\delta(x) = 1; \\ &\forall x \in \mathbb{R}^N \setminus \Omega^\delta : \quad \phi_\delta(x) = 0; \\ &\forall x \in G^\delta : \quad 0 < \phi_\delta(x) < 1; \\ &\forall x \in G^\delta : \quad |\nabla \phi_\delta(x)| \le \frac{1}{\delta^r}, \end{aligned} \tag{39}$$

for some $r > 1$ and

$$G^\delta = \Omega^\delta \setminus \overline{\Omega^{\delta/2}}. \tag{40}$$

Let's prove that $\Phi_\delta : A_\sigma(\varepsilon) \subseteq \mathcal{M}_\varepsilon \to \mathcal{M}^\delta$, given by

$$\Phi_\delta[u] = \frac{u\,\phi_\delta}{\|u\,\phi_\delta\|_{\mathrm{L}^{p+1}(\Omega^\delta)}}, \tag{41}$$

is well defined and Lipschitz continuous. From now on we assume that $\varepsilon \in]0, \tilde{\varepsilon}_1[$, where $\tilde{\varepsilon}_1 = \min\{\hat{\varepsilon}_1,\ \varepsilon_1(\delta/2, 1; \sigma, k)\}$.

(a) By (38) we have, for $v \in A_\sigma(\varepsilon)$, that

$$\begin{aligned} 1 \ge \|v\phi_\delta\|^{p+1}_{\mathrm{L}^{p+1}(\Omega^\delta)} &= \int_{\Omega^{\delta/2}} |v(x)|^{p+1} dx + \int_{G^\delta} |\phi_\delta(x) v(x)|^{p+1} dx \\ &\ge \int_{\Omega^{\delta/2}} |v(x)|^{p+1} dx \ge \left(1 - \frac{\delta}{2}\right)^{p+1} \ge (1-\delta)^{p+1}, \end{aligned} \tag{42}$$

so that Φ_δ is well defined.

(b) Let $u, v \in A_\sigma(\varepsilon) \subseteq \mathcal{M}_\varepsilon$. Then, by (42),

$$\begin{aligned} \|\Phi_\delta[u] - \Phi_\delta[v]\|_{\mathrm{H}_0^1(\Omega^\delta)} &\le \frac{1}{1-\delta} \|\nabla(\phi_\delta\,(u - v))\|_{\mathrm{L}^2(\Omega^\delta)} \\ &\le \frac{1}{1-\delta}\left[\|\phi_\delta\,\nabla(u-v)\|_{\mathrm{L}^2(\Omega^\delta)} + \|(u-v)\,\nabla\phi_\delta\|_{\mathrm{L}^2(\Omega^\delta)}\right]. \end{aligned} \tag{43}$$

Let's observe that (16) is still true if we replace Ω by any $U \subseteq \mathbb{R}^N$ open and bounded. Then, by (39), we get

$$\begin{aligned} \|\phi_\delta\,\nabla(u-v)\|_{\mathrm{L}^2(\Omega^\delta)} &= \left(\int_{\Omega^\delta} \phi_\delta^2(x)\,|\nabla(u-v)(x)|^2 dx\right)^{1/2} \\ &\le \|u - v\|_{\mathrm{H}_0^1(\Omega^\delta)} \le \|u-v\|_\varepsilon\,. \end{aligned} \tag{44}$$

In other hand, by (39) and (33), we have that

$$\begin{aligned} \|(u-v)\,\nabla\phi_\delta\|_{\mathrm{L}^2(\Omega^\delta)} &= \left(\int_{G^\delta} |u(x)-v(x)|^2 |\nabla\phi_\delta(x)|^2 dx\right)^{1/2} \\ &\le \frac{1}{\delta^r \inf\limits_{y\in G^\delta} V_\varepsilon(y)} \left(\int_{G^\delta} V_\varepsilon(x)\,|u(x)-v(x)|^2 dx\right)^{1/2} \le \frac{\|u-v\|_\varepsilon}{\delta^r V_{\delta/2,\varepsilon}} \end{aligned}$$

which together with (43) and (44) imply

$$\|\Phi_\delta[u]-\Phi_\delta[v]\|_{\mathrm{H}_0^1(\Omega^\delta)} \leq \frac{1}{1-\delta}\left(1+\frac{1}{\delta^r V_{\delta/2,\varepsilon}}\right)\|u-v\|_\varepsilon .$$

Since u and v were chosen arbitrarily, we have proved that Φ_δ is Lipschitz continuous.

iv) The operator Φ_δ is odd and continuous so that, by (19), $\Phi_\delta[A_\sigma(\varepsilon)] \in \mathcal{A}_k^\delta$ and, consequently, $c_k^\delta \leq \max_{v\in\Phi_\delta[A_\sigma(\varepsilon)]} J^\delta(v)$. Then we can pick $u \in A_\sigma(\varepsilon)$ such that $\overline{v} = \Phi_\delta[u] \in \Phi_\delta[A_\sigma(\varepsilon)]$ verifies

$$c_k^\delta \leq \max_{v\in\Phi_\delta[A_\sigma(\varepsilon)]} J^\delta(v) \leq J^\delta(\overline{v}) + \frac{\sigma}{4}. \tag{45}$$

Now we claim that

$$\exists w \in A_\sigma(\varepsilon): \quad J^\delta(\overline{v}) \leq J_\varepsilon(w) + \frac{\sigma}{2}. \tag{46}$$

Then, points (46), (45) and (26) imply

$$c_k^\delta \leq J^\delta(\overline{v}) + \frac{\sigma}{4} \leq J_\varepsilon(w) + \frac{3\sigma}{4} \leq \max_{u\in A_\sigma(\varepsilon)} J_\varepsilon(u) + \frac{3\sigma}{4} \leq c_{k,\varepsilon} + \sigma.$$

v) To conclude we have to prove (46). Actually, we shall prove that choosing $w = u$ is enough. By (42) we have that

$$\begin{aligned} 2(1-\delta)^2 J^\delta(\overline{v}) &\leq 2\,\|\phi_\delta u\|^2_{\mathrm{L}^{p+1}(\Omega^\delta)}\, J^\delta(\overline{v}) \\ &= \|\phi_\delta u\|^2_{\mathrm{L}^{p+1}(\Omega^\delta)} \left\| \frac{\phi_\delta\, u}{\|\phi_\delta\, u\|_{\mathrm{L}^{p+1}(\Omega^\delta)}} \right\|^2_{\mathrm{H}_0^1(\Omega^\delta)} \\ &= \int_{\Omega^\delta} \left[u^2|\nabla\phi_\delta|^2 + 2u\phi_\delta\nabla u\nabla\phi_\delta + \phi_\delta^2|\nabla u|^2\right] dx. \end{aligned} \tag{47}$$

First, we have that

$$\int_{\Omega^\delta} \phi_\delta^2(x)|\nabla u(x)|^2 \leq \int_{\Omega^\delta} |\nabla u(x)|^2 dx \leq 2J_\varepsilon(u). \tag{48}$$

Second, we get by using (18) and (40),

$$\begin{aligned} \int_{\Omega^\delta} u^2(x)|\nabla\phi_\delta(x)|^2 dx = \int_{G^\delta} u^2(x)|\nabla\phi_\delta(x)|^2 dx &\leq \frac{1}{\delta^{2r}}\int_{G^\delta} u^2(x)dx \\ &\leq \frac{1}{\delta^{2r}}|G^\delta|^{(p-1)/(p+1)}\,\|u\|^2_{\mathrm{L}^{p+1}(G^\delta)} . \end{aligned} \tag{49}$$

Third, by using (31), (18) and Cauchy-Schwartz inequalities for $\mathbb{R}^N$ and L^2, we have that

$$\begin{aligned} \int_{\Omega^\delta} 2u\phi_\delta\nabla u\nabla\phi_\delta dx \leq 2\int_{\Omega^\delta} |u|\,|\phi_\delta|\,|\nabla u|\,|\nabla\phi_\delta|\,dx &\leq \frac{2}{\delta^r}\int_{G^\delta} |u|\,|\nabla u|\;dx \\ &\leq \frac{2}{\delta^r}\left(\int_{G^\delta}|\nabla u|^2 dx\right)^{1/2}\left(\int_{G^\delta}|u|^2 dx\right)^{1/2} \end{aligned}$$

$$\leq \frac{2}{\delta^r}(2b_{k,\sigma})^{1/2}|G^\delta|^{(p-1)/2(p+1)}\,\|u\|_{\mathrm{L}^{p+1}(G^\delta)}\,. \tag{50}$$

From now on we assume that $\varepsilon\in]0,\varepsilon_2[$, where, for some $s_*>2r$, $\varepsilon_2=\min\{\tilde\varepsilon_1,\ \varepsilon_1(\delta,s_*;\sigma,k)\}$. By observing that $\delta^{2s_*}<\delta^{s_*}$ and using (37), we get from (47), (48), (49) and (50),

$$(1-\delta)^2 J^\delta(\overline{v})\leq J_\varepsilon(u)+\frac{\zeta}{2}\delta^{s_*-2r}, \tag{51}$$

where $\zeta=|G^\delta|^{(p-1)/(p+1)}+2(2b_{k,\sigma})^{1/2}|G^\delta|^{(p-1)/2(p+1)}$. Now we assume that $\delta\in]0,\delta_1[$ where $\delta_1=1-\sqrt{2}/2$. Then, by (51) and (30), we get

$$\frac{1}{2}J^\delta(\overline{v})\leq J_\varepsilon(u)+\frac{\zeta}{2}\delta^{s_*-2r}\leq b_{k,\sigma}+\frac{\zeta}{2}\delta^{s_*-2r}. \tag{52}$$

Then, by combining (51) and (52) we get

$$\begin{aligned} J^\delta(\overline{v}) &\leq J_\varepsilon(u)+\frac{\zeta}{2}\delta^{s_*-2r}+2\delta\,J^\delta(\overline{v})-\delta^2\,J^\delta(\overline{v})\\ &\leq J_\varepsilon(u)+\frac{3\zeta}{2}\delta^{s_*-2r}+4\delta b_{k,\sigma}, \end{aligned}$$

whence it's clear that we can find $\delta_0\in]0,\delta_1[$, perhaps adjusting ε_2, such that (46) holds for all $\delta\in]0,\delta_0[$ and $\varepsilon\in]0,\varepsilon_2[$.

□

Now we are in condition to prove the convergence of critical values.

Proof of Theorem 2.4. By [7, Lem. 3.3 & 3.4] we have, for $k\in\mathbb{N}$,

$$\forall\delta>0:\quad c_k^\delta\leq c_k, \tag{53}$$

$$\forall\sigma>0,\exists\delta_\sigma>0,\forall\delta\in]0,\delta_\sigma[:\quad c_k\leq c_k^\delta+\sigma. \tag{54}$$

Let $\sigma>0$ be small. We choose $\delta_\sigma>0$ from (54). Now we take $\delta_0=\delta_0(\sigma)>0$ and $\varepsilon_2=\varepsilon_2(\sigma)$ from Proposition 4.2.. Finally we put $\hat\delta_\sigma=\min\{\delta_\sigma,\ \delta_0\}$. Then, by using (54), (25) and (22), we have, for all $\delta\in]0,\hat\delta_\sigma[$ and all $\varepsilon\in]0,\varepsilon_2[$,

$$c_k\leq c_k^\delta+\sigma\leq c_{k,\varepsilon}+2\sigma\leq c_k\cdot C_{\Omega,\varepsilon}+2\sigma.$$

Since σ was arbitrary, the last shows that $c_{k,\varepsilon}\to c_k$, as $\varepsilon\to 0$. □

5. Asymptotic profiles

In this section we prove Theorem 2.5, that is we study the asymptotic behaviour of the solutions of (P_ε) inside of Ω.

Let's recall that, given $k\in\mathbb{N}$, Theorem 2.5 states that, as $\varepsilon\to 0$, the family $(w_{k,\varepsilon})_{\varepsilon>0}$ subconverges in $\mathrm{H}^1(\mathbb{R}^N)$ to some $u_k\in\mathrm{H}^1(\mathbb{R}^N)$ such that $u_k|_\Omega$ is a solution of $(\mathrm{P}_{\mathrm{inf}})$ and verifies $J(\hat u_k|_\Omega)=c_k$, where $\hat u_k=(2c_k)^{1/1-p}u_k$.

Lemma 5.1. *Let $k\in\mathbb{N}$. Then $(\hat w_{k,\varepsilon})_{\varepsilon>0}$ weakly and pointwise subconverges to some $\hat u_k\in\mathrm{H}^1(\mathbb{R}^N)$, as $\varepsilon\to 0$.*

Proof. By Theorem 2.4, given $\sigma > 0$, there exists $\varepsilon_{\sigma,1} > 0$ such that, for every $\varepsilon \in]0, \varepsilon_{\sigma,1}[$,

$$\int_{\mathbb{R}^N} \left[|\nabla \hat{w}_{k,\varepsilon}(x)|^2 + V_\varepsilon(x)|\hat{w}_{k,\varepsilon}(x)|^2\right] dx = 2c_{k,\varepsilon} \le 2c_k + \sigma \equiv B_{k,\varepsilon}. \tag{55}$$

Then, by Sobolev-Gagliardo-Niremberg theorem, [3, Th. 9.9], there is $C_N > 0$ such that, for all $\varepsilon \in]0, \varepsilon_{\sigma,1}[$,

$$\|\hat{w}_{k,\varepsilon}\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^N)} \le C_N^2 \int_{\mathbb{R}^N} |\nabla \hat{w}_{k,\varepsilon}(x)|^2 dx \le C_N^2 B_{k,\sigma}. \tag{56}$$

Let $0 < \delta < 1$. Then, by Hölder inequality and (56), for all $\varepsilon \in]0, \varepsilon_{\sigma,1}[$,

$$\begin{aligned} \|\hat{w}_{k,\varepsilon}\|^2_{\mathrm{L}^2(\Omega^\delta)} &\le |\Omega^\delta|^{2/N} \, \|\hat{w}_{k,\varepsilon}\|^2_{\mathrm{L}^{2^*}(\Omega^\delta)} \le |\Omega^\delta|^{2/N} \cdot \|\hat{w}_{k,\varepsilon}\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^N)} \\ &\le C_N^2 \, |\Omega^\delta|^{2/N} B_{k,\sigma}. \end{aligned} \tag{57}$$

In other hand, by (34), there exists $\varepsilon_{\sigma,2} \in]0, \varepsilon_{\sigma,1}[$ such that, for all $\varepsilon \in]0, \varepsilon_{\sigma,2}[$, it verifies $V_{\delta,\varepsilon}^{-1} < 1$. Then, by (55),

$$\|\hat{w}_{k,\varepsilon}\|^2_{\mathrm{L}^2(\mathbb{R}^N\setminus\Omega^\delta)} \le \int_{\mathbb{R}^N\setminus\Omega^\delta} \frac{V_\varepsilon(x)}{V_{\delta,\varepsilon}} |\hat{w}_{k,\varepsilon}(x)|^2 dx \le B_{k,\sigma}. \tag{58}$$

From (56), (57) and (58) it follows, for $\varepsilon \in]0, \varepsilon_{\sigma,2}[$, that

$$\begin{aligned} \|\hat{w}_{k,\varepsilon}\|^2_{\mathrm{H}^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \left[|\nabla \hat{w}_{k,\varepsilon}(x)|^2 + |\hat{w}_{k,\varepsilon}(x)|^2\right] dx \\ &\le B_{k,\sigma} + C_N^2 |\Omega^\delta|^{2/N} B_{k,\sigma} + B_{k,\sigma} = B_{k,\sigma}(2 + C_N^2 |\Omega^1|^{2/N}). \end{aligned} \tag{59}$$

By (59) and [3, Th. 3.18 & 4.9], there exists $\hat{u}_k \in \mathrm{H}^1(\mathbb{R}^N)$ toward which $(\hat{w}_{k,\varepsilon})_{\varepsilon>0}$ subconverges $\mathrm{H}^1(\mathbb{R}^N)$-weakly and pointwise. □

Lemma 5.2. *Let $k \in \mathbb{N}$. The function $\hat{u}_k$ is a weak solution of* ($\mathrm{P}_{\inf}$) *and verifies $J(\hat{u}_k|_\Omega) = c_k$.*

Proof. Let $\varepsilon > 0$. Since $\hat{w}_{k,\varepsilon} \in \mathcal{M}_\varepsilon$ is a critical point of J_ε we have by Remark 2.1 that, for every $\phi \in \mathrm{C}_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \left[\nabla \hat{w}_{k,\varepsilon} \nabla \phi + V_\varepsilon(x) \hat{w}_{k,\varepsilon} \phi\right] dx = 2c_{k,\varepsilon} \int_{\mathbb{R}^N} |\hat{w}_{k,\varepsilon}|^{p-1} \hat{w}_{k,\varepsilon} \phi \, dx. \tag{60}$$

Let $\phi \in \mathrm{C}_0^\infty(\Omega)$ and $\sigma > 0$. By (57) and (58), we have, for $\varepsilon \in]0, \varepsilon_{\sigma,2}[$,

$$\begin{aligned} \left| \int_{\mathbb{R}^N} V_\varepsilon(x) \hat{w}_{k,\varepsilon} \phi(x) \right| &\le \|\hat{w}_{k,\varepsilon}\|_{\mathrm{L}^2(\Omega)} \, \|\phi\|_{\mathrm{L}^2(\Omega)} \, \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)} \\ &\le \left[1 + C_N^2 |\Omega^1|^{2/N}\right]^{1/2} \|\phi\|_{\mathrm{L}^2(\Omega)} \, \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)}, \end{aligned}$$

whence, by Proposition 2.8,

$$\int_{\mathbb{R}^N} V_\varepsilon(x) \hat{w}_{k,\varepsilon} \phi(x) \to 0, \quad \text{as } \varepsilon \to 0. \tag{61}$$

By Theorem 2.11 we have that $(\hat{w}_{k,\varepsilon})_{\varepsilon>0}$ subconverges in $\mathrm{L}^{p+1}(\mathbb{R}^N)$ to $\hat{u}_k$. Therefore, by (60), (61), Theorem 2.4 and the arbitrariness of ϕ, we get

$$\forall \phi \in \mathrm{C}_0^\infty(\Omega): \quad \int_\Omega \nabla \hat{u}_k \nabla \phi \, dx = 2c_k \int_\Omega |\hat{u}_k|^{p-1} \hat{u}_k \phi \, dx. \tag{62}$$

Let's take $(\phi_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}_0^\infty(\Omega)$ that converges in $\mathrm{L}^{p+1}(\Omega)$ to $\hat{u}_k|_\Omega$. Then, by replacing $\phi = \phi_n$ in (62) and letting $n \to +\infty$, we get, by Lemma 5.1, that $c_k = J(\hat{u}_k|_\Omega)$.

For $\delta, \alpha > 0$ we write $\Gamma_{\delta,\alpha} = \{x \in \mathbb{R}^N \setminus \Omega^\delta \,/\, |\hat{u}_k(x)| \geq \alpha\}$. By contradiction it's not hard to prove that $|\Gamma_{\delta,\alpha}| = 0$, for every $\delta, \alpha > 0$, so that

$$\hat{u}_k(x) = 0, \qquad \text{for a.e. } x \in \mathbb{R}^N \setminus \Omega, \tag{63}$$

which, by [3, Prop.9.18], implies that $\hat{u}_k|_\Omega \in \mathrm{H}_0^1(\Omega)$. We conclude by this and (62). □

Proof of Theorem 2.5. Since the injection $\mathrm{H}_\varepsilon \subseteq \mathrm{L}^2(\mathbb{R}^N)$ is compact, Lemma 5.1 and point (63) imply that

$$\lim_{\varepsilon\to 0} \|\hat{w}_{k,\varepsilon}\|^2_{\mathrm{L}^2(\mathbb{R}^N)} = \|\hat{u}_k\|^2_{\mathrm{L}^2(\mathbb{R}^N)}. \tag{64}$$

By (23) and (63) we have that

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^N} |\nabla \hat{w}_{k,\varepsilon}|^2 dx \leq 2 \limsup_{\varepsilon\to 0} c_{k,\varepsilon} \leq 2c_k = \int_{\mathbb{R}^N} |\nabla \hat{u}_k|^2 dx. \tag{65}$$

Points (64) and (65) provide

$$\limsup_{\varepsilon\to 0} \|\hat{w}_{k,\varepsilon}\|_{\mathrm{H}^1(\mathbb{R}^N)} \leq \|\hat{u}_k\|_{\mathrm{H}^1(\mathbb{R}^N)},$$

so that, by [3, Prop. 3.32], we have that $(\hat{w}_{k,\varepsilon})_{\varepsilon>0}$ subconverges in $\mathrm{H}^1(\mathbb{R}^N)$ to $\hat{u}_k$, as $\varepsilon \to 0$. We conclude by Lemma 5.2. □

6. Asymptotic concentration

In this section we prove Theorem 2.6, that is, we study the asymptotic behaviour of the solutions of (P_ε) outside of Ω. Let's recall that, given $k \in \mathbb{N}$ and $\delta > 0$, Theorem 2.6 states that, for $\varepsilon > 0$ small enough and $x \in \mathbb{R}^N$, it verifies

$$|\hat{w}_{k,\varepsilon}(x)| < \frac{C}{(2c_k)^{1/(p-1)}} \exp\left(\gamma_{\delta,\varepsilon} \cdot \mathrm{dist}(x, \Omega^\delta)\right), \tag{66}$$

where $C = C(N, k, p, \delta) > 0$ and $\gamma_{\delta,\varepsilon} = \gamma_{\delta,\varepsilon}(N, k, p) \to -\infty$, as $\varepsilon \to 0$.

To this purpose let's strengthen the assumption (V1) by requiring the following condition.

$(\mathrm{V1}_\eta)$ For some $\eta > 0$, $V \in \mathrm{C}^\eta(\mathbb{R}^N)$.

Then, by using standard regularity arguments, [9], it follows that $v_{k,\varepsilon}$ and w_k belong to $\mathrm{C}^{2,\eta}(\mathbb{R}^N)$ and are classical solutions of (P_ε) and $(\mathrm{P}_{\inf})$, respectively.

The following L^∞-estimate shall be useful.

Proposition 6.1. *Let $D \subseteq \mathbb{R}^N$ be open and connected. Assume that*

$$\exists c > 0, \forall t > 0: \quad tf(t) \leq ct^{p+1},$$

and that $w \in \mathrm{H}_0^1(\Omega)$ is a classical solution of the elliptic problem

$$\begin{cases} \Delta w(x) - f(w(x)) \geq 0, & x \in D, \\ \quad w(x) > 0, & x \in D, \\ \quad w(x) = 0 & x \in \partial D, \end{cases} \tag{67}$$

where $N \geq 3$ and $p+1 \in]2, 2^*[$. Then there exists $C = C(c, p, N) > 0$ such that

$$\|w\|_{\mathrm{L}^\infty(D)} \leq C \, \|w\|_{\mathrm{L}^{2^*}(D)}^{4/[(N+2)-p(N-2)]}. \tag{68}$$

Proposition 6.1 was obtained in [7] by using the Moser iteration technique, extending a result of [4] where, in addition, it was assumed that D is smooth and bounded.

Lemma 6.2. *Let $k \in \mathbb{N}$ and $\sigma > 0$. Then there exists $\varepsilon_{\sigma,2} > 0$ and $K = K(\sigma, N, k, p) > 0$ such that*

$$\forall \varepsilon \in]0, \varepsilon_{\sigma,2}[: \quad \|w_{k,\varepsilon}\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \leq K. \tag{69}$$

The following points hold

$$\lim_{\varepsilon \to 0} \|v_{k,\varepsilon}\|_{\mathrm{L}^\infty(\mathbb{R}^N)} = 0, \quad \textit{and} \quad \liminf_{\varepsilon \to 0} \frac{\|v_{k,\varepsilon}\|_{\mathrm{L}^\infty(\mathbb{R}^N)}}{[2c_{k,\varepsilon}(\varepsilon g(\varepsilon)^2)]^{1/(p-1)}} > 0. \tag{70}$$

Proof. We pick $\varepsilon_{\sigma,2} > 0$ as in the proof of Lemma 5.1. Let $\varepsilon \in]0, \varepsilon_{\sigma,2}[$ and A_ε^+ a connected component of $W_\varepsilon^+ = \{x \in \mathbb{R}^N \,/\, w_{k,\varepsilon} > 0\}$. Since $w_{k,\varepsilon}$ is a solution of $(\mathrm{P}'_\varepsilon)$ and V_ε is non-negative, it follows that

$$\begin{cases} \Delta w_{k,\varepsilon}(x) + w_{k,\varepsilon}^p(x) \geq 0 & x \in A_\varepsilon^+, \\ w_{k,\varepsilon}(x) > 0, & x \in A_\varepsilon^+, \\ w_{k,\varepsilon}(x) = 0, & x \in \partial A_\varepsilon^+. \end{cases}$$

Then, by Proposition 6.1 and (56) we get

$$\|w_{k,\varepsilon}\|_{\mathrm{L}^\infty\left(A_\varepsilon^+\right)} \leq C \, \|w_{k,\varepsilon}\|_{\mathrm{L}^{2^*}\left(A_\varepsilon^+\right)}^{4/[(N+2)-p(N-2)]} \leq C[C_N^2 \, B_{k,\sigma}]^{2/[(N+2)-p(N-2)]} \equiv K.$$

The arbitrariness of A_ε^+ shows that $\|w_{k,\varepsilon}\|_{\mathrm{L}^\infty\left(W_\varepsilon^+\right)} \leq K$. In the same way it's proved that $\|w_{k,\varepsilon}\|_{\mathrm{L}^\infty\left(W_\varepsilon^-\right)} \leq K$, where $W_\varepsilon^- = \{x \in \mathbb{R}^N \,/\, w_{k,\varepsilon} < 0\}$. So we get (69).

By (12), Remark 2.1 and (69) we have that

$$|v_{k,\varepsilon}(x)|^{p-1} \leq [\varepsilon g(\varepsilon)]^2 \, K^{p-1}, \qquad x \in \mathbb{R}^N,$$

whence it immediately follows the first limit of (70). The second limit of (70) is obtained having in consideration (12) and that $\|\hat{w}_{k,\varepsilon}\|_{\mathrm{L}^{p+1}(\mathbb{R}^N)} = 1$, for every $\varepsilon \in]0, \varepsilon_{\sigma,2}[$. ∎

To obtain the estimate (66) we shall apply a comparison argument as in the proof of [5, Lem. 2.7]. Given $a, b > 0$ and $\omega \subseteq \mathbb{R}^N$ smooth and bounded, it's known that there exists a positive solution of the problem

$$\begin{cases} \Delta\varphi(x) - 2b\,\varphi(x) = 0, & x \in \mathbb{R}^N \setminus \omega, \\ \varphi(x) = a, & x \in \partial\omega, \\ \varphi(x) \to 0, & \text{as } |x| \to +\infty, \end{cases} \tag{71}$$

and that, for some $C = C(a, \omega) > 0$, it verifies

$$\forall x \in \mathbb{R}^N \setminus \omega : \quad \varphi(x) \le C \exp(-b \operatorname{dist}(x, \omega)). \tag{72}$$

Proof of Theorem 2.6. Let $\sigma > 0$. By (34) and Lemma 6.2, we can pick $\varepsilon_{\sigma,3} \in]0, \varepsilon_{\sigma,2}[$ such that

$$\forall \varepsilon \in]0, \varepsilon_{\sigma,3}[: \qquad V_{\delta,\varepsilon} > K \tag{73}$$

Then, by Lemma 6.2, (73) and (33), we get

$$f_{k,\varepsilon}(x) \equiv V_\varepsilon(x) - |w_{k,\varepsilon}(x)|^{p-1} \ge V_{\delta,\varepsilon} - K > 0, \qquad x \in \mathbb{R}^N \setminus \Omega^\delta.$$

Now we take φ, a solution of (71) with $\omega = \Omega^\delta$, $a = K$ and $b = (V_{\delta,\varepsilon} - K)/2 \equiv -\gamma_{\delta,\varepsilon}$, so that, by (71),

$$\begin{cases} \Delta\varphi(x) - f_{k,\varepsilon}(x)\varphi(x) \le 0, & x \in \mathbb{R}^N \setminus \Omega^\delta, \\ \varphi(x) = K, & x \in \partial\Omega^\delta, \\ \varphi(x) \to 0, & \text{as } |x| \to +\infty, \end{cases}$$

whence, since $w_{k,\varepsilon}$ is a solution of $(\mathrm{P}'_\varepsilon)$,

$$\begin{cases} \Delta[\varphi(x) - w_{k,\varepsilon}(x)] - f_{k,\varepsilon}(x)[\varphi(x) - w_{k,\varepsilon}(x)] = 0, & x \in \mathbb{R}^N \setminus \Omega^\delta, \\ \varphi(x) - w_{k,\varepsilon}(x) > 0, & x \in \partial\Omega^\delta, \\ \varphi(x) - w_{k,\varepsilon}(x) \to 0, & \text{as } |x| \to +\infty. \end{cases}$$

Therefore, by [9, Th. 3.1], it holds $w_{k,\varepsilon} \le \varphi(x)$, for every $x \in \mathbb{R}^N \setminus \Omega^\delta$. In the same way it's proved that $-\varphi(x) \le w_{k,\varepsilon}$, for every $x \in \mathbb{R}^N \setminus \Omega^\delta$, whence, by (72), it follows that

$$\forall x \in \mathbb{R}^N \setminus \Omega^\delta : \quad |w_{k,\varepsilon}(x)| \le \varphi(x) \le \frac{C}{(2c_{k,\varepsilon})} \exp\left(\gamma_{\delta,\varepsilon} \operatorname{dist}(x, \Omega^\delta)\right), \tag{74}$$

where $C = C(K, \delta) = C(\sigma, N, k, p, \delta) > 0$. Finally, we obtain (66) by fixing $\sigma > 0$ small and enlarging $C > 0$ so to make it independent of σ and in a way that makes (74) valid for $x \in \Omega^\delta$. $\square$

7. Asymptotic behavior on the boundary

For completeness of the document, in this last part we present a scheme of the proof of Theorem 2.7:

$$\forall k \in \mathbb{N}: \quad \lim_{\varepsilon\to 0} \max_{x\in\partial\Omega} |w_{k,\varepsilon}(x)| = 0. \tag{75}$$

This is a kind of uniform estimate on $\partial\Omega$ that was first found in [7] and whose guidelines we introduce.

As in Theorem 3.3, K_{c_k} denotes the set of critical points of J that share the critical level c_k.

Proof of Theorem 2.7. Let's choose $\delta_1 > 0$ such that, for every $\delta \in]0, \delta_1[$, the sets $\Omega_\delta = \{x \in \Omega \,/\, \mathrm{dist}(x, \partial\Omega) > \delta\}$ and $U_\delta = \Omega^\delta \setminus \overline{\Omega_\delta}$ are non-empty. We also denote $U_{\delta,\pm} = \{x \in U_\delta \,/\, \pm w_{k,\varepsilon}(x) > 0\}$.

i) First we prove that for any $\mu > 0$ there exists $\delta_\mu \in]0, \delta_1[$ such that

$$\forall \delta \in]0, \delta_\mu[: \quad \|w_{k,\varepsilon}\|_{\mathrm{L}^\infty(U_\delta)} \le H_{\varepsilon,\delta} + \mu, \tag{76}$$

where $H_{\varepsilon,\delta} = \max\{|w_{k,\varepsilon}(x)| \,/\, x \in \partial U_\delta\}$.

Take $\mu > 0$ and $\delta \in]0, \delta_1[$. We have that

$$\begin{cases} \pm\Delta w_{k,\varepsilon}(x) \pm |w_{k,\varepsilon}(x)|^{p-1} w_{k,\varepsilon}(x) \ge 0, & x \in U_{\delta,\pm}, \\ w_{k,\varepsilon}(x) \ge 0, & x \in \partial U_{\delta,\pm}. \end{cases} \tag{M$^\pm$}$$

Then, by writting $\phi_{k,\varepsilon}(x) = w_{k,\varepsilon} - H_{\varepsilon,\delta}$ and $\mathcal{V}(x) = -|w_{k,\varepsilon}(x)|^{p-1}(H_{\varepsilon,\delta} + \phi_{k,\varepsilon}(x))$, we get

$$\begin{cases} \Delta\phi_{k,\varepsilon}(x) \ge \mathcal{V}(x), & x \in U_{\delta,+}, \\ \phi_{k,\varepsilon}(x) \le 0, & x \in \partial U_{\delta,+}. \end{cases}$$

Therefore, by Alexandroff's Maximum Principle (see e.g. [11]) and Hölder inequality, we get, for $c_1 = c_1(N, \mathrm{diam}(\Omega)) > 0$,

$$\begin{aligned} \|w_{k,\varepsilon}\|_{\mathrm{L}^\infty(U_{\delta,+})} &= \sup_{x\in U_{\delta,+}} w_{k,\varepsilon}(x) \\ &\le c_1 \left\|\mathcal{V}^-\right\|_{\mathrm{L}^N(U_{\delta,+})} \\ &\le c_1 |U_{\delta,+}|^{1/N} c_2^{p-1} \left[H_{\varepsilon,\delta} + \|\phi_{k,\varepsilon}\|_{\mathrm{L}^\infty(U_{\delta,+})}\right]. \end{aligned}$$

Therefore, by choosing $\delta_\mu \in]0, \delta_1[$ small enough, we get

$$\forall \delta \in]0, \delta_\mu[: \quad \|w_{k,\varepsilon}\|_{\mathrm{L}^\infty(U_{\delta,+})} \le H_{\varepsilon,\delta} + \mu. \tag{77}$$

Dealing with problem (M$^-$) in a similar way and, perhaps, adjusting δ_μ, we get

$$\forall \delta \in]0, \delta_\mu[: \quad -\|w_{k,\varepsilon}\|_{\mathrm{L}^\infty(U_{\delta,-})} = \inf_{x\in U_{\delta,-}} w_{k,\varepsilon}(x) \le -H_{\varepsilon,\delta} - \mu. \tag{78}$$

By (77) and (78) we obtain (76).

ii) Let's observe that

$$\max_{x\in\partial\Omega}|w_{k,\varepsilon}(x)|\leq H_{\varepsilon,\delta}=\max\{\hat{h}_{\varepsilon,\delta},\check{h}_{\varepsilon,\delta}\},$$

where $\hat{h}_{\varepsilon,\delta}=\max_{x\in\partial\Omega^{\delta}}|w_{k,\varepsilon}(x)|$ and $\check{h}_{\varepsilon,\delta}=\max_{x\in\partial\Omega_{\delta}}|w_{k,\varepsilon}(x)|$. By Theorem 2.6, we get that $\hat{h}_{\varepsilon,\delta}\to 0$, as $\varepsilon\to 0$. Therefore, by (76), to conclude (75) it is enough to show that

$$\check{h}_{\varepsilon,\delta}\to 0,\qquad \text{as } \varepsilon\to 0. \tag{79}$$

By Theorem 2.5, there exists $u\in K_{c_k}$ and $(\varepsilon_n)_{n\in\mathbb{N}}\subseteq]0,+\infty[$ such that $\varepsilon_n\to 0$ and $w_{k,\varepsilon_n}\to u_k$ in $\mathrm{H}^1(\mathbb{R}^N)$ and pointwise, as $n\to+\infty$.

Let $\alpha>0$ small so that $(\partial\Omega_\delta)^\alpha=\{x\in\mathbb{R}^N/\operatorname{dist}(x,\partial\Omega_\delta)<\alpha\}$ is contained in $\operatorname{int}(\Omega)$. Since the convergence of w_{k,ε_n} to u_k is uniform on each compact subregion of Ω, we can find $n_*=n_*(\alpha)\in\mathbb{N}$ such that, for $n>n_*$,

$$\|w_{k,\varepsilon_n}(x)-u_k(x)\|_{\mathrm{L}^\infty((\partial\Omega_\delta)^\alpha)}<\frac{\alpha}{2}. \tag{80}$$

Here the compactness of the critical level set K_{c_k} is a key to find n_* depending only on α. Points (63) and (80) and the arbitrariness of α imply that (79) holds for $\delta\in]0,\delta_\mu[$ small enough.

□

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