Few remarks on the regularity of weak solutions for two elliptic coupled systems of fluid dynamics

Oscar Jarrín *[1]

[1]Escuela de Ciencias Físicas y Matemáticas, Universidad de Las Américas, Vía a Nayón, C.P.170124, Quito, Ecuador.

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Abstract

Abstract. We consider two elliptic coupled systems of relevance in the fluid dynamics. These systems are posed on the whole space $\mathbb{R}^3$ and they consider the action of external forces. The first system deals with the simplified Ericksen-Leslie (SEL) system, which describes the dynamics of liquid crystal flows. The second system is the time-independent magneto-hydrodynamic (MHD) equations.

For the (SEL) system, we obtain a new criterion to improve the regularity of weak solutions, provided some a priori decaying properties given in the fairly general setting of the homogeneous Morrey spaces. As a bi-product, we also prove some new regularity criterion for the steady state Navier-Stokes equations and for a non-linear harmonic map flow.

This new regularity criterion also holds true for the (MHD) equations. Furthermore, for this last system we are able to use the Gevrey class to prove that all finite energy weak solutions are analytic functions, provided the external forces belong to the Gevrey class.

Keywords: Coupled systems in fluid mechanics; simplified Ericksen-Leslie system; Magneto-hydrodynamic system; Morrey spaces; the Gevrey class.

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1 Introduction

This note deals with two elliptic coupled systems arising from the fluid dynamics. The first system concerns the study of the dynamics in liquid crystal flows. This system is posed on the whole space $\mathbb{R}^3$ and strongly couples the incompressible and time-independent Navier-Stokes equations with a non-linear harmonic map flow as follows:

$$\begin{cases} -\Delta \vec{U} + \text{div}(\vec{U}\otimes\vec{U}) + \text{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V}) + \vec{\nabla}P = \text{div}(\mathbb{F}), \\ -\Delta\vec{V} + \text{div}(\vec{V}\otimes\vec{U}) - |\vec{\nabla}\otimes\vec{V}|^2\,\vec{V} = \text{div}(\mathbb{G}). \\ \text{div}(\vec{U}) = 0. \end{cases} \tag{1}$$

Here, $\vec{\nabla}\otimes\vec{V} = (\partial_i V_j)_{1\leq i,j\leq 3}$, denotes the deformation tensor of the vector field $\vec{V}$ and moreover, for $i = 1,2,3$, each component of the vector field $\text{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V})$ explicitly writes down as:

$$\Big[\text{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V})\Big]_i = \sum_{j=1}^{3}\sum_{k=1}^{3}\partial_j(\partial_i V_k\,\partial_j V_k). \tag{2}$$

*corresponding author: oscar.jarrin@udla.edu.ec

The velocity of the fluid $\vec{U}:\mathbb{R}^3\to\mathbb{R}^3$ and the pressure $P:\mathbb{R}^3\to\mathbb{R}$ are the classical unknowns of the fluid mechanics. Moreover, this system also considers a third unknown $\vec{V}:\mathbb{R}^3\to\mathbb{S}^2$, where $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$. The *unit vector field* $\vec{V}$ represents the macroscopic orientation of the nematic liquid crystal molecules [20]. Moreover, the physical model [20] also considers the action of external forces, which can be written as the divergence of the tensors $\mathbb{F}=(F_{i,j})_{1\leq i,j\leq 3}$ and $\mathbb{G}=(G_{i,j})_{1\leq i,j\leq 3}$, with $F_{i,j},G_{i,j}:\mathbb{R}^3\to\mathbb{R}$. Finally, the equation $\text{div}(\vec{U})=0$ always represents the fluid's incompressibility.

The elliptic system (1) is the time-independent counterpart of the following parabolic (time-dependent) system:

$$\begin{cases} \partial_t\vec{u}-\Delta\vec{u}+\text{div}(\vec{u}\otimes\vec{u})+\text{div}(\nabla\otimes\vec{v}\odot\nabla\otimes\vec{v})+\nabla p=\text{div}(\mathbb{F}), \quad \text{div}(\vec{u})=0, \\ \partial_t\vec{v}-\Delta\vec{v}+\text{div}(\vec{v}\otimes\vec{u})-|\nabla\otimes\vec{v}|^2\,\vec{v}=\text{div}(\mathbb{G}), \end{cases} \tag{3}$$

also known as the *simplified Ericksen-Leslie system*. This parabolic system was proposed by H.F. Lin in [17] as a simplification of the general *Ericksen-Leslie system* which models the hydrodynamic flow of nematic liquid crystal material [3], [20]. The simplified Ericksen-Leslie system, has been successful to model various dynamical behavior for nematic liquid crystals. More precisely, it provides a good macroscopic description of the evolution of the material under the influence of fluid velocity field and the macroscopic description of the microscopic orientation of fluid velocity of rod-like liquid crystals. See the book [8] for more details.

The system (3) has recently attired a lot of interest in the research community of mathematical fluid dynamics. It is worth mentioning one of the major challenges is due, on the one hand, to the presence of the trilinear term $|\nabla\otimes\vec{v}|^2\,\vec{v}$ and, on the other hand, to the presence of the super-critical nonlinear term $\text{div}(\nabla\otimes\vec{v}\odot\nabla\otimes\vec{v})$ defined in (2). Precisely, the double derivatives in this last term make it more delicate to treat than the classical nonlinear transport term: $\text{div}(\vec{u}\otimes\vec{u})$. These facts make challenging the study of both (1) and (3). See, for instance, the articles [10, 11, 14, 18, 19, 22] and the references therein.

Some previous works in the homogeneous case. When $\mathbb{F}=0$ and $\mathbb{G}=0$, the first works on the mathematically study of the system (3) were devoted to the existence of global in time weak solutions [14, 19]. Thereafter, in the spirit of the celebrated result by H. Koch & D. Tataru [13], the global well-posedness of small solutions in the space $BMO^{-1}(\mathbb{R}^3)$ was proven in [22].

Concerning the regularity issues of solutions, T. Huang proved in [6] an $\varepsilon-$ regularity criterion in the framework of the Lebesgue spaces. This result allows him to establish a sufficient condition on the *weak solutions* to improve their regularity in both the temporal and the spatial variable. This results also holds true for the system (1) in the case $\mathbb{F}=\mathbb{G}=0$. Indeed, first we consider a weak solution of (1) as a couple $(\vec{u},\vec{v})$ where $(\vec{U},\vec{\nabla}\otimes\vec{V})\in H^1(\mathbb{R}^3)$. Thereafter, we obtain $\vec{U}\in\mathcal{C}^\infty(\mathbb{R}^3)$ and $\vec{V}\in\mathcal{C}^\infty(\mathbb{R}^3)$, provided that $\vec{U}\in L^p(\mathbb{R}^3)$ and $\vec{\nabla}\otimes\vec{V}\in L^p(\mathbb{R}^3)$, with $p>3$.

Regularity of weak solutions is one of the key assumptions when studying another relevant problem for the system (1) in the homogeneous case, which is known as the *Liouville-type problem*. We look for some functional spaces in which the uniqueness of the trivial solution $(\vec{U},\vec{V},P)=(0,0,0)$ hold. This problem was recently studied in [11], where the main interest is the use of more general spaces than the L^p ones, for instance, the Lorentz and the Morrey spaces. However, to the best of our knowledge, the regularity in these spaces was not studied before and it must be assumed.

A new regularity criterion in the non-homogeneous case. Motivated by this last questions, in this note we study some new *a priori* conditions in the setting of the Morrey spaces to improve the regularity of weak solutions of the system (1). Moreover, we also consider the more general case under the action of the external forces $\text{div}(\mathbb{F})$ and $\text{div}(\mathbb{G})$.

We shall consider here a weaker notion of weak solutions, which is given in the following:

Definition 1.1 *Let $\mathbb{F},\mathbb{G}\in\mathcal{D}'(\mathbb{R}^3)$. A very weak solution of the coupled system (1) is a triplet $(\vec{U},P,\vec{V})$, where: $\vec{U}\in L^2_{loc}(\mathbb{R}^3)$, $P\in\mathcal{D}'(\mathbb{R}^3)$, $|\vec{V}(x)|=1$ for almost all $x\in\mathbb{R}^3$ and $\vec{\nabla}\otimes\vec{V}\in L^2_{loc}(\mathbb{R}^3)$, such that it verifies (1) in the distributional sense.*

It is worth observing we use *minimal conditions* on the functions $\vec{U}, \vec{V}$ and P to ensure that all the terms in (1) are well defined as distributions. Moreover, we let the pressure P to be a very general object since we only have $P \in \mathcal{D}'(\mathbb{R}^3)$.

As $\vec{U}$ and $\vec{\nabla} \otimes \vec{V}$ are locally square integrable functions, in order to improve their regularity we look for some natural conditions on the local quantities $\displaystyle\int_{B(x_0,R)} |\vec{U}(x)|^2\, dx$ and $\displaystyle\int_{B(x_0,R)} |\vec{\nabla} \otimes \vec{V}(x)|^2\, dx$, where $B(x_0, R)$ denotes the ball of center $x_0 \in \mathbb{R}^3$ and radious $R > 0$. Thus, the Morrey spaces appear naturally. We recall that for a parameter $2 < p < +\infty$, the homogeneous Morrey space $\dot{M}^{2,p}(\mathbb{R}^3)$ is the Banach space of functions $f \in L^2_{loc}(\mathbb{R}^3)$ such that:

$$\|f\|_{\dot{M}^{2,p}} = \sup_{R>0,\ x_0\in\mathbb{R}^n} R^{\frac{3}{p}} \left(\frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} |f(x)|^2 dx \right)^{\frac{1}{2}} < +\infty. \tag{4}$$

The parameter p measures the decaying rate of the (local) mean quantity $\displaystyle\left(\frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} |f(x)|^2 dx \right)^{\frac{1}{2}}$ as R goes to infinity. This is a homogeneous space of degree $-\frac{3}{p}$, and moreover, we have the following chain of continuous embeddings $L^p(\mathbb{R}^3) \subset L^{p,q}(\mathbb{R}^3) \subset \dot{M}^{2,p}(\mathbb{R}^3)$. Here, $L^{p,q}(\mathbb{R}^3)$ (with $p < q \leq +\infty$) denotes a Lorentz space which describes the decaying properties of functions in a different setting. See the book [1] for a detailed study of these spaces.

Finally, for the parameter $p > 2$ given above, and for the regularity parameter $k \geq 0$, we introduce now the Sobolev-Morrey space

$$\mathcal{W}^{k,p}(\mathbb{R}^3) = \left\{ f \in \dot{M}^{2,p}(\mathbb{R}^3) :\ \partial^\alpha f \in \dot{M}^{2,p}(\mathbb{R}^3),\ \text{for all multi-indice } |\alpha| \leq k \right\}.$$

Then, our first result reads as follows:

Theorem 1.1 *Let $(\vec{U}, P, \vec{V})$ be a very weak solution of the coupled system (1) given in Definition 1.1. We assume $\vec{U} \in \dot{M}^{2,p}(\mathbb{R}^3)$ and $\vec{\nabla} \otimes \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^3)$, with $p > 3$. Then, if for $k \geq 0$:*

$$\mathbb{F},\, \mathbb{G} \in \mathcal{W}^{k+1,p}(\mathbb{R}^3) \cap W^{k+1,\infty}(\mathbb{R}^3), \tag{5}$$

it follows that $\vec{U} \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$, $P \in \mathcal{W}^{k+1,p}(\mathbb{R}^3)$ and $\vec{V} \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$. Moreover, for all multi-indice $|\alpha| \leq k+1$, the functions $\partial^\alpha \vec{U}$ and $\partial^\alpha \vec{V}$ are Hölder continuous with exponent $0 < 1 - 3/p < 1$, while for $|\alpha| \leq k$ the function $\partial^\alpha P$ is also Hölder continuous with the same exponent.

Remark 1.1 *Recall that the external forces acting on (1) are given by $div(\mathbb{F})$ and $div(\mathbb{G})$. Then, by (5) we have $div(\mathbb{F})$, $div(\mathbb{G}) \in \mathcal{W}^{k,p}(\mathbb{R}^3)$ which yields a gain of solutions of the order $k+2$. This expected maximum gain of regularity is given by the effects of the Laplacian operator in the system (1). In this sense, the main contribution of this result is the study of the regularity in a general enough framework.*

Remark 1.2 *For the particular homogeneous case when $\mathbb{F} = \mathbb{G} = 0$, we obtain that a very solution $(\vec{U}, P, \vec{V})$ of the system (1) verify $(\vec{U}, P, \vec{V}) \in \mathcal{C}^\infty(\mathbb{R}^3)$, provided that $\vec{U} \in \dot{M}^{2,p}(\mathbb{R}^3)$ and $\vec{\nabla} \otimes \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^3)$, with $p > 3$. As explained, this particular result is of interest in connection to the Liouville-type problem for (1) in the Morrey spaces [11].*

Mathematically, the coupled system (1) is also of interests as it contains two relevant equations. On the one hand, by setting $\vec{V}$ a constant unitary vector we get the time-independent and forced Navier-Stokes equations:

$$-\Delta \vec{U} + \text{div}(\vec{U} \otimes \vec{U}) + \vec{\nabla} P = \text{div}(\mathbb{F}), \qquad \text{div}(\vec{U}) = 0. \tag{6}$$

On the other hand, by setting now $\vec{U}=0$ we obtain the following harmonic map flow:

$$-\Delta\vec{V} - |\vec{\nabla}\otimes\vec{V}|^2\,\vec{V} = \text{div}(\mathbb{G}). \tag{7}$$

As a direct consequence of this result we obtain a new regularity criterion for these equations:

Corollary 1.1

1. *Let $(\vec{U},P)\in L^2_{loc}(\mathbb{R}^3)\times\mathcal{D}'(\mathbb{R}^3)$ be a weak solution of the equation (6). If $\vec{U}\in\dot{M}^{2,p}(\mathbb{R}^3)$, with $p>3$, and $\mathbb{F}$ verifies (5) then we have $\vec{U}\in\mathcal{W}^{k+2,p}(\mathbb{R}^3)$ and $P\in\mathcal{W}^{k+1,p}(\mathbb{R}^3)$.*

2. *Let $\vec{V}(x)\in\mathcal{S}^2$ with $\vec{\nabla}\otimes\vec{V}\in L^2_{loc}(\mathbb{R}^3)$ be a weak solution of the equation (7). If $\vec{\nabla}\otimes\vec{V}\in\dot{M}^{2,p}(\mathbb{R}^3)$, $p>3$ and $\mathbb{G}$ verifies (5) then we have $\vec{V}\in\mathcal{W}^{k+2,p}(\mathbb{R}^3)$.*

Let us briefly explain the general strategy of the proof of Theorem 1.1. The proof bases on two key ideas. First, by the information $\vec{U},\vec{\nabla}\otimes\vec{V}\in\dot{M}^{2,p}(\mathbb{R}^3)$ and by using the framework of an auxiliary parabolic system (9), we prove that the $\vec{U}$ and $\vec{\nabla}\otimes\vec{V}$ are bounded functions on $\mathbb{R}^3$. Thereafter, we use a bootstrap argument to show that $\vec{U} in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$ and $\vec{V}\in\mathcal{W}^{k+2,p}(\mathbb{R}^3)$.

These ideas can also be applied to other relevant coupled system of the fluid dynamics. This system is the time-independent magneto-hydrodynamic equations which describe the steady state of the magnetic properties of electrically conducting fluids, including plasma and liquid metals [21]:

$$\begin{cases} -\Delta\vec{U} + \text{div}(\vec{U}\otimes\vec{U}) - \text{div}(\vec{B}\otimes\vec{B}) + \vec{\nabla}P = \text{div}(\mathbb{F}), & \text{div}(\vec{U})=0, \\ -\Delta\vec{B} + \text{div}(\vec{B}\otimes\vec{U}) - \text{div}(\vec{U}\otimes\vec{B}) = \text{div}(\mathbb{G}), & \text{div}(\vec{B})=0, \end{cases} \tag{8}$$

Here, $\vec{U}:\mathbb{R}^3\to\mathbb{R}^3$ and $P:\mathbb{R}^3\to\mathbb{R}$ always denote the fluid velocity and the pressure respectively; and $\vec{B}:\mathbb{R}^3\to\mathbb{R}^3$ is the magnetic field. Moreover, $\text{div}(\mathbb{F})$ and $\text{div}(\mathbb{G})$ are the external forces acting on this system.

Our second result states as follows:

Theorem 1.2 *Let $\vec{U}\in L^2_{loc}(\mathbb{R}^3)$, $\vec{B}\in L^2_{loc}(\mathbb{R}^3)$, $P\in\mathcal{D}'(\mathbb{R}^3)$ be a very weak solution of the system (8). We assume $\vec{U},\vec{B}\in\dot{M}^{2,p}(\mathbb{R}^3)$ with $p>3$. If for $k\geq 0$ the functions $\mathbb{F}$ and $\mathbb{G}$ verify (5) then we have $\vec{U}\in\mathcal{W}^{k+2,p}(\mathbb{R}^3)$, $\vec{B}\in\mathcal{W}^{k+2,p}(\mathbb{R}^3)$ and $P\in\mathcal{W}^{k+1,p}(\mathbb{R}^3)$. Moreover, for $|\alpha|\leq k+1$, $\partial^\alpha\vec{U}$ and $\partial^\alpha\vec{B}$ are Hölder continuous functions with exponent $1-3/p$, while this fact holds true for $\partial^\alpha P$ with $|\alpha|\leq k$.*

The system (8) has a simpler structure than the system (1) as the four nonlinear terms have the same writing. Consequently, they are treated similarly provided that $\vec{U}$ and $\vec{B}$ have the same properties. Thus, we are able to adapt the ideas above to obtain this new criterion for very weak solutions in the setting of the Morrey spaces.

When $\mathbb{F},\mathbb{G}\in L^2(\mathbb{R}^3)$, and consequently $\text{div}(\mathbb{F}),\text{div}(\mathbb{F})\in\dot{H}^{-1}(\mathbb{R}^3)$, it is well-known that the system (8) has *finite energy* weak solutions $\vec{U}\,\vec{B}\in\dot{H}^1(\mathbb{R}^3)$ such that $\|U\|^2_{\dot{H}^1}+\|\vec{B}\|^2_{\dot{H}^1}\leq c\|\text{div}(\mathbb{F})\|^2_{\dot{H}^{-1}}+c\|\text{div}(\mathbb{G})\|^2_{\dot{H}^{-1}}$. By recalling that we have the following embedding: $\dot{H}^1(\mathbb{R}^3)\subset L^6(\mathbb{R}^3)\subset\dot{M}^{2,6}(\mathbb{R}^3)$, the following corollary gives us regularity criterion for the finite energy weak solutions of (8).

Corollary 1.2 *Let $\mathbb{F},\mathbb{G}\in L^2(\mathbb{R}^3)$ and let $(\vec{U},\vec{B})\in\dot{H}^1(\mathbb{R}^3)$ be a weak solution of the system (8). If for $k\geq 0$ the functions $\mathbb{F}$ and $\mathbb{G}$ verify (5) then we have $\vec{U}\in\mathcal{W}^{k+2,6}(\mathbb{R}^3)$, $\vec{B}\in\mathcal{W}^{k+2,6}(\mathbb{R}^3)$ and $P\in\mathcal{W}^{k+1,6}(\mathbb{R}^3)$. Moreover, if $\mathbb{F}=\mathbb{G}=0$ we have $(\vec{U},\vec{B},P)\in\mathcal{C}^\infty(\mathbb{R}^3)$.*

For the finite energy weak solutions of (8), we are able to go further in the study of their regularity. We recall that for a parameter $b > 0$ we define the weighted exponential operator $e^{b\sqrt{-\Delta}}$ as $\mathcal{F}\Big(e^{b\sqrt{-\Delta}}f\Big)(\xi) = e^{b|\xi|}\widehat{f}(\xi)$. Thereafter, for a parameter $s \in \mathbb{R}$ we define the Gevrey class

$$G^s_b(\mathbb{R}^3) = \left\{ f \in \dot{H}^s(\mathbb{R}^3) : \, e^{b\sqrt{-\Delta}} f \in \dot{H}^s(\mathbb{R}^3) \right\}.$$

For $|s| < 3/2$, $G^s_b(\mathbb{R}^3)$ is a Banach space with the norm $\|e^{b\sqrt{-\Delta}}(\cdot)\|_{\dot{H}^s}$. Moreover, for $s \geq 0$ the functions belonging to $G^s_b(\mathbb{R}^3)$ are analytic. Thus, our third result writes down as follows:

Theorem 1.3 *Let $\mathbb{F}, \mathbb{G} \in L^2(\mathbb{R}^3)$. For $b > 0$ we assume $div(\mathbb{F}), div(\mathbb{F}) \in G^{-1}_b(\mathbb{R}^3)$. Then, there exists $b_1 > 0$ such that all the finite energy weak solutions of the system (8) verify $\vec{U} \in G^1_{b_1}(\mathbb{R}^3)$, $\vec{B} \in G^1_{b1}(\mathbb{R}^3)$ and $P \in G^{1/2}_{b1}(\mathbb{R}^3)$.*

Consequently, we obtain that $\vec{U}, P$ and $\vec{B}$ are analytic functions and they admit holomorphic extensions to the strip $\big\{(x+iz) \in \mathbb{C}^3 : \, |z| < b_1\big\}$. In addition, in the homogeneous case when $\mathbb{F} = \mathbb{G} = 0$, we get that all finite energy solution of (8) belong to the Gevrey class $G^1_{b1}(\mathbb{R}^3)$, for any $b_1 > 0$.

2 Some well-known results

For the reader's convenience, we summarize here some well-known results which will be useful in the sequel. For $1 < r < p$ and $1 < p < +\infty$, we consider the homogeneous Morrey space $\dot{M}^{r,p}(\mathbb{R}^3)$, which is defined as in (4) with r instead of 2.

Lemma 2.1 (Page 169 of [16]) *The space $\dot{M}^{r,p}(\mathbb{R}^3)$ is stable under convolution with functions in the space $L^1(\mathbb{R}^3)$ and we have we have $\|g * f\|_{\dot{M}^{r,p}} \leq c\|g\|_{L^1}\|f\|_{\dot{M}^{r,p}}$.*

Lemma 2.2 (Page 171 of [16]) *For all $t > 0$ we have $t^{\frac{3}{2p}}\|h_t * f\|_{L^\infty} \leq c\|f\|_{\dot{M}^{r,p}}$.*

This estimate is a direct consequence of the continuous embedding $\dot{M}^{r,p}(\mathbb{R}^3) \subset \dot{B}^{-\frac{3}{2},\infty}_{\infty}(\mathbb{R}^3)$. We recall that the homogeneous Besov space $\dot{B}^{-\frac{3}{2},\infty}_{\infty}(\mathbb{R}^3)$ can be characterized as the space of temperate distributions $f \in \mathcal{S}'(\mathbb{R}^3)$ such that $\sup_{t>0} t^{\frac{3}{2p}}\|h_t * f\|_{L^\infty} < +\infty$, where h_t denotes the heat kernel.

Lemma 2.3 (Lemme 4.2 of [12]) *For $i = 1, 2, 3$ let $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ be the Riesz transform. Then, for $i, j = 1, 2, 3$ the operator $\mathcal{R}_i\mathcal{R}_j$ is continuous in the space $\dot{M}^{r,p}(\mathbb{R}^3)$ and we have $\|\mathcal{R}_i\mathcal{R}_j(f)\|_{\dot{M}^{r,p}} \leq c\|f\|_{\dot{M}^{r,p}}$.*

Finally, we shall use the following result linking the Morrey spaces and the Hölder regularity of functions.

Lemma 2.4 (Proposition 3.4 of [9]) *Let $f \in \mathcal{S}'(\mathbb{R}^3)$ such that $\vec{\nabla} f \in \dot{M}^{1,p}(\mathbb{R}^3)$, with $p > 3$. There exists a constant $C > 0$ such that for all for all $x, y \in \mathbb{R}^3$ we have $|f(x) - f(y)| \leq C\,\|\vec{\nabla} f\|_{\dot{M}^{1,p}}\,|x - y|^{1-3/p}$.*

Recall that the Morrey space $\dot{M}^{1,p}(\mathbb{R}^3)$ is defined as the space of locally finite Borel measures $d\mu$ such that

$$\sup_{x_0 \in \mathbb{R}^3,\, R>0} R^{\frac{3}{p}} \left(\frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} d|\mu|(x) \right) < +\infty.$$

3 Proof of the Theorem 1.1

For the sake of clearness, we shall divide the proof of this theorem in three main steps.

Step 1. The auxiliary parabolic system. Our starting point is the study of the following auxiliary parabolic system. Let $\vec{V}:\mathbb{R}^3\to\mathbb{S}^2$ be the vector field given in Definition 1.1. Moreover, $\mathbb{P}$ stands for the the Leray's projector. We consider the initial value problem for the parabolic coupled system:

$$\begin{cases} \partial_t\vec{u}-\Delta\vec{u}+\mathbb{P}(\text{div}(\vec{u}\otimes\vec{u}))+\mathbb{P}(\text{div}(\mathbf{V}\odot\mathbf{V}))=\mathbb{P}\left(\text{div}(\mathbb{F})\right), \quad \text{div}(\vec{u})=0,\\ \partial_t\mathbf{V}-\Delta\mathbf{V}+\vec{\nabla}\otimes(\vec{u}\,\mathbf{V})-\vec{\nabla}\otimes(|\mathbf{V}|^2\,\vec{V})=\vec{\nabla}\otimes(\text{div}(\mathbb{G})),\\ \vec{u}(0,\cdot)=\vec{u}_0, \quad \mathbf{V}(0,\cdot)=\mathbf{V}_0, \end{cases} \tag{9}$$

where, the vector field $\vec{u}=(u_1,u_2,u_3)$ and the matrix $\mathbf{V}=(v_{i,j})_{1\leq i,j\leq 3}$ are the unknowns. For a time $0<T<+\infty$, we denote $\mathcal{C}_*([0,T],\dot{M}^{2,p}(\mathbb{R}^3))$ the functional space of bounded and weak$-*$ continuous functions from $[0,T]$ with values in the Morrey space $\dot{M}^{2,p}(\mathbb{R}^3)$. We prove now the following:

Proposition 3.1 *Consider the initial value problem (9) where $\mathbb{F}$ and $\mathbb{G}$ verify (5). If $\vec{u}_0\in\dot{M}^{2,p}(\mathbb{R}^3)$ and $\boldsymbol{V}_0\in\dot{M}^{2,p}(\mathbb{R}^3)$, with $p>3$, then there exists a time $T_0>0$, depending on $\vec{u}_0$, $\boldsymbol{V}_0$, $\mathbb{F}$ and $\mathbb{G}$; and there exist $(\vec{u},\boldsymbol{V})\in\mathcal{C}_*([0,T_0],\dot{M}^{2,p}(\mathbb{R}^3))$, which is the unique solution of (9). Moreover this solution verifies*

$$\sup_{0<t<T_0} t^{\frac{3}{2p}}\Big(\|\vec{u}(t,\cdot)\|_{L^\infty}+\|\boldsymbol{V}(t,\cdot)\|_{L^\infty}\Big)<+\infty. \tag{10}$$

Proof. Mild solutions of the system (9) write down as the integral formulation:

$$\begin{aligned}\vec{u}(t,\cdot)=e^{t\Delta}\vec{u}_0+\int_0^t e^{(t-s)\Delta}\,\mathbb{P}\left(\text{div}(\mathbb{F})\right)ds+\underbrace{\int_0^t e^{(t-s)\Delta}\mathbb{P}(div(\vec{u}\otimes\vec{u}))(s,\cdot)ds}_{B_1(\vec{u},\vec{u})}\\ +\underbrace{\int_0^t e^{(t-s)\Delta}\mathbb{P}(\text{div}(\mathbf{V}\odot\mathbf{V}))(s,\cdot)ds}_{B_2(\mathbf{V},\mathbf{V})},\end{aligned} \tag{11}$$

and

$$\begin{aligned}\mathbf{V}(t,\cdot)=e^{t\Delta}\mathbf{V}_0+\int_0^t e^{(t-s)\Delta}\,\vec{\nabla}\otimes(\text{div}(\mathbb{G}))\,ds+\underbrace{\int_0^t e^{(t-s)\Delta}\vec{\nabla}\otimes(\vec{u}\,\mathbf{V})(s,\cdot)ds}_{B_3(\vec{u},\mathbf{V})}\\ -\underbrace{\int_0^t e^{(t-s)\Delta}\vec{\nabla}\otimes(|\mathbf{V}|^2\,\vec{V})(s,\cdot)ds}_{B_4(\mathbf{V},\mathbf{V})}.\end{aligned} \tag{12}$$

By the Picard's fixed point argument, we will solve both problems (11) and (12) in the Banach space

$$E_T=\left\{f\in\mathcal{C}_*([0,T],\dot{M}^{2,p}(\mathbb{R}^3)):\ \sup_{0<t<T} t^{\frac{3}{2p}}\|f(t,\cdot)\|_{L^\infty}<+\infty\right\},$$

with the norm

$$\|f\|_{E_T}=\sup_{0\leq t\leq T}\|f(t,\cdot)\|_{\dot{M}^{2,p}}+\sup_{0<t<T} t^{\frac{3}{2p}}\|f(t,\cdot)\|_{L^\infty}.$$

Let us mention that for $f_1,f_2\in E_T$, for the sake of simplicity, we shall write $\|(f_1,f_2)\|_{E_T}=\|f_1\|_{E_T}+\|f_2\|_{E_T}$.

We start by studying the linear terms in (11) and (12). As $\vec{u}_0\in\dot{M}^{2,p}(\mathbb{R}^3)$ and $\mathbf{V}_0\in\dot{M}^{2,p}(\mathbb{R}^3)$ by Lemma 2.1 we have $\|(e^{t\Delta}\vec{u}_0,e^{t\Delta}\mathbf{V}_0)\|_{\dot{M}^{2,p}}\leq c\|(\vec{u}_0,\mathbf{V}_0)\|_{\dot{M}^{2,p}}$, hence we obtain $e^{t\Delta}\vec{u}_0\in\mathcal{C}_*([0,T],\dot{M}^{2,p}(\mathbb{R}^3))$

and $e^{t\Delta}\mathbf{V}_0 \in \mathcal{C}_*([0,T], \dot{M}^{2,p}(\mathbb{R}^3))$. By Lemma 2.2 we get $\left\| \left(e^{t\Delta}\vec{u}_0, e^{t\Delta}\mathbf{V}_0\right)\right\|_{L^\infty} \leq c\, t^{-\frac{3}{2p}} \left\| (\vec{u}_0, \mathbf{V}_0)\right\|_{\dot{M}^{2,p}}$, hence $\sup\limits_{0<t<T} t^{\frac{3}{2p}} \left\| \left(e^{t\Delta}\vec{u}_0, e^{t\Delta}\mathbf{V}_0\right)\right\|_{L^\infty} \leq c\|(\vec{u}_0, \mathbf{V}_0)\|_{\dot{M}^{2,p}}$. Thus, we have $e^{t\Delta}\vec{u}_0 \in E_T$ and $e^{t\Delta}\mathbf{V}_0 \in E_T$, and moreover, the following estimate holds:

$$\left\| \left(e^{t\Delta}\vec{u}_0, e^{t\Delta}\mathbf{V}_0\right)\right\|_{E_T} \leq c\|(\vec{u}_0, \mathbf{V}_0)\|_{\dot{M}^{2,p}}. \tag{13}$$

On the other hand, as $\mathbb{F}, \mathbb{G}$ are time independent tensors, and moreover, as we assume (5), we write:

$$\begin{aligned}\left\| \int_0^t e^{(t-s)\Delta} \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right) ds \right\|_{\dot{M}^{2,p}} &\leq \int_0^t \left\| e^{(t-s)\Delta} \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right)\right\|_{\dot{M}^{2,p}} ds \\ &\leq c \left\| \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right)\right\|_{\dot{M}^{2,p}} \left(\int_0^t ds \right),\end{aligned}$$

to get

$$\sup_{0\leq t\leq T} \left\| \int_0^t e^{(t-s)\Delta} \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right) ds \right\|_{\dot{M}^{2,p}} \leq c\, T\, \left\| \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right)\right\|_{\dot{M}^{2,p}}. \tag{14}$$

Moreover, by Lemma 2.2 we write

$$\begin{aligned} t^{\frac{3}{2p}} \left\| \int_0^t e^{(t-s)\Delta} \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right) ds \right\|_{L^\infty} &\leq t^{\frac{3}{2p}} \int_0^t \left\| e^{(t-s)\Delta} \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right)\right\|_{L^\infty} ds \\ &\leq c\, t^{\frac{3}{2p}} \int_0^t (t-s)^{-\frac{3}{2p}} \left\| \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right)\right\|_{\dot{M}^{2,p}} ds \\ &\leq c\, t^{\frac{3}{2p}} \left\| \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right)\right\|_{\dot{M}^{2,p}} \left(\int_0^t (t-s)^{-\frac{3}{2p}} ds \right) \\ &\leq c\, t\, \left\| \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right)\right\|_{\dot{M}^{2,p}}. \end{aligned}$$

Then, we obtain

$$\sup_{0<t<T} t^{\frac{3}{2p}} \left\| \int_0^t e^{(t-s)\Delta} \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right) ds \right\|_{L^\infty} \leq c\, T\, \left\| \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right)\right\|_{\dot{M}^{2,p}}. \tag{15}$$

By the estimates (14) ans (15) we get

$$\left\| \int_0^t e^{(t-s)\Delta} \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right) ds \right\|_{E_T} \leq c\, T\, \left\| \left(\operatorname{div}(\mathbb{F}), \operatorname{div}(\mathbb{G})\right)\right\|_{\dot{M}^{2,p}}. \tag{16}$$

We study now the bi-linear terms in (11) and (12). First, the terms $B_1(\vec{u}, \vec{u})$ and $B_2(\mathbf{V}, \mathbf{V})$ in (11) are estimated as follows:

$$\sup_{0\leq t\leq T} \|B_1(\vec{u}, \vec{u}) + B_2(\mathbf{V}, \mathbf{V})\|_{\dot{M}^{2,p}} \leq c\, T^{\frac{1}{2}-\frac{3}{2p}}\, \|(\vec{u}, \mathbf{V})\|^2_{E_T} \tag{17}$$

where, as $p>3$ then we have $\frac{1}{2}-\frac{3}{2p}>0$. Indeed, by Lemma 2.3 and the well-known estimate on the heat kernel: $\|\vec{\nabla} h_{(t-s)}(\cdot)\|_{L^1}\leq \dfrac{c}{(t-s)^{1/2}}$, for the first term in the norm $\|\cdot\|_{E_T}$ we have

$$
\begin{aligned}
&\sup_{0\leq t\leq T}\|B_1(\vec{u},\vec{u})+B_2(\mathbf{V},\mathbf{V})\|_{\dot{M}^{2,p}}\\
=&\sup_{0\leq t\leq T}\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}(div(\vec{u}\otimes\vec{u}))(s,\cdot)ds+\int_0^t e^{(t-s)\Delta}\mathbb{P}(\text{div}(\mathbf{V}\odot\mathbf{V}))(s,\cdot)ds\right\|_{\dot{M}^{2,p}}\\
\leq& c\sup_{0\leq t\leq T}\int_0^t\left\|e^{(t-s)\Delta}(div(\vec{u}\otimes\vec{u}))(s,\cdot)+e^{(t-s)\Delta}(\text{div}(\mathbf{V}\odot\mathbf{V}))(s,\cdot)\right\|_{\dot{M}^{2,p}}ds\\
\leq& c\sup_{0\leq t\leq T}\int_0^t\frac{1}{(t-s)^{1/2}}\left(\|\vec{u}(s,\cdot)\otimes\vec{u}(s,\cdot)\|_{\dot{M}^{2,p}}+\|\mathbf{V}(s,\cdot)\odot\mathbf{V}(s,\cdot)\|_{\dot{M}^{2,p}}\right)ds\\
\leq& c\sup_{0\leq t\leq T}\int_0^t\frac{1}{(t-s)^{\frac{1}{2}}s^{\frac{3}{2p}}}\left((s^{\frac{3}{2p}}\|\vec{u}(s,\cdot)\|_{L^\infty})\|\vec{u}(s,\cdot)\|_{\dot{M}^{2,p}}+(s^{\frac{3}{2p}}\|\mathbf{V}(s,\cdot)\|_{L^\infty})\|\mathbf{V}(s,\cdot)\|_{\dot{M}^{2,p}}\right)ds\\
\leq& cT^{\frac{1}{2}-\frac{3}{2p}}\|(\vec{u},\mathbf{V})\|^2_{E_T}.
\end{aligned}
\tag{18}
$$

We will prove now the following estimate:

$$
\sup_{0\leq t\leq T}t^{\frac{3}{2p}}\|B_1(\vec{u},\vec{u})+B_2(\mathbf{V},\mathbf{V})\|_{L^\infty}\leq cT^{\frac{1}{2}-\frac{3}{2p}}\|(\vec{u},\mathbf{V})\|^2_{E_T}. \tag{19}
$$

We write:

$$
\begin{aligned}
&\sup_{0\leq t\leq T}t^{\frac{3}{2p}}\|B_1(\vec{u},\vec{u})+B_2(\mathbf{V},\mathbf{V})\|_{L^\infty}\\
=&\sup_{0<t<T}t^{\frac{3}{2p}}\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}(div(\vec{u}\otimes\vec{u}))(s,\cdot)ds+\int_0^t e^{(t-s)\Delta}\mathbb{P}(\text{div}(\mathbf{V}\odot\mathbf{V}))(s,\cdot)ds\right\|_{L^\infty}\\
\leq&\sup_{0<t<T}t^{\frac{3}{2p}}\int_0^t\left\|e^{(t-s)\Delta}\mathbb{P}(div(\vec{u}\otimes\vec{u}))(s,\cdot)ds+e^{(t-s)\Delta}\mathbb{P}(\text{div}(\mathbf{V}\odot\mathbf{V}))(s,\cdot)\right\|_{L^\infty}ds=(a),
\end{aligned}
$$

The operator $e^{(t-s)\Delta}\mathbb{P}(div(\cdot))$ is a matrix of convolutions operators (in the spatial variable) whose kernels $K_{i,j}$ verify $|K_{i,j}(t-s,x)|\leq\dfrac{c}{((t-s)^{1/2}+|x|)^4}$, see Proposition 11.1 of [15]. Then, $\|K_{i,j}(t-s,\cdot)\|_{L^1}\leq\dfrac{c}{(t-s)^{1/2}}$; and we have:

$$
\begin{aligned}
(a)\leq& c\sup_{0\leq t\leq T}t^{\frac{3}{2p}}\int_0^t\frac{1}{(t-s)^{1/2}}\left(\|\vec{u}(s,\cdot)\otimes\vec{u}(s,\cdot)\|_{L^\infty}+\|\mathbf{V}(s,\cdot)\odot\mathbf{V}(s,\cdot)\|_{L^\infty}\right)ds\\
\leq& c\sup_{0\leq t\leq T}t^{\frac{3}{2p}}\int_0^t\frac{ds}{(t-s)^{1/2}s^{\frac{3}{p}}}\left(\left(s^{\frac{3}{2p}}\|\vec{u}(s,\cdot)\|_{L^\infty}\right)^2+\left(s^{\frac{3}{2p}}\|\mathbf{V}(s,\cdot)\|_{L^\infty}\right)^2\right)ds\\
\leq& c\left(\sup_{0\leq t\leq T}t^{\frac{3}{2p}}\int_0^t\frac{ds}{(t-s)^{1/2}s^{\frac{n}{p}}}\right)\|(\vec{u},\mathbf{V})\|^2_{E_T}.\\
\leq&\left(c\sup_{0\leq t\leq T}\left[t^{\frac{3}{2p}}\int_0^{t/2}\frac{ds}{(t-s)^{1/2}s^{\frac{3}{p}}}+t^{\frac{3}{2p}}\int_{t/2}^t\frac{ds}{(t-s)^{1/2}s^{\frac{3}{p}}}\right]\right)\|(\vec{u},\mathbf{V})\|^2_{E_T}\\
\leq& c\left(\sup_{0\leq t\leq T}\left[t^{\frac{3}{2p}-\frac{1}{2}}\int_0^{t/2}\frac{ds}{s^{3/p}}+t^{\frac{3}{2p}-\frac{3}{p}}\int_{t/2}^t\frac{ds}{(t-s)^{1/2}}\right]\right)\|(\vec{u},\mathbf{V})\|^2_{E_T}\\
\leq& cT^{\frac{1}{2}-\frac{3}{2p}}\|(\vec{u},\mathbf{V})\|^2_{E_T}.
\end{aligned}
$$

For the terms $B_3(\vec{u},\mathbf{V})$ and $B_4(\mathbf{V},\mathbf{V})$ in (12), the following estimates hold:

$$
\sup_{0\leq t\leq T}\|B_3(\vec{u},\mathbf{V})+B_4(\mathbf{V},\mathbf{V})\|_{\dot{M}^{2,p}}\leq cT^{\frac{1}{2}-\frac{3}{2p}}\|(\vec{u},\mathbf{V})\|^2_{E_T}, \tag{20}
$$

and

$$\sup_{0\leq t\leq T} t^{\frac{3}{2p}} \|B_3(\vec{u},\mathbf{V})+B_4(\mathbf{V},\mathbf{V})\|_{L^\infty}\, c\,T^{\frac{1}{2}-\frac{3}{2p}}\, \|(\vec{u},\mathbf{V})\|^2_{E_T}. \tag{21}$$

Indeed, recalling that by the physical model we have $|\vec{V}(x)|=1$, and then $\|\vec{V}\|_{L^\infty}=1$, for the first in the norm $\|\cdot\|_{E_T}$ we have:

$$\begin{aligned}
&\sup_{0\leq t\leq T}\|B_3(\vec{u},\mathbf{V})+B_4(\mathbf{V},\mathbf{V})\|_{\dot{M}^{2,p}}\\
=&\sup_{0\leq t\leq T}\left\|\int_0^t e^{(t-s)\Delta}\vec{\nabla}\otimes(\vec{u}\,\mathbf{V})(s,\cdot)ds-\int_0^t e^{(t-s)\Delta}\vec{\nabla}\otimes(|\mathbf{V}|^2\,\vec{V})(s,\cdot)ds\right\|_{\dot{M}^{2,p}}\\
\leq& c\sup_{0\leq t\leq T}\int_0^t\frac{1}{(t-s)^{1/2}}\left(\|\vec{u}\,\mathbf{V}(s,\cdot)\|_{\dot{M}^{2,p}}+\||\mathbf{V}(s,\cdot)|^2\vec{V}\|_{\dot{M}^{2,p}}\right)ds\\
\leq& c\sup_{0\leq t\leq T}\int_0^t\frac{1}{(t-s)^{1/2}}\left(\|\vec{u}(s,\cdot)\|_{\dot{M}^{2,p}}\|\mathbf{V}(s,\cdot)\|_{L^\infty}+\||\mathbf{V}(s,\cdot)|^2\|_{\dot{M}^{2,p}}\|\vec{V}\|_{L^\infty}\right)ds\\
\leq& c\sup_{0\leq t\leq T}\int_0^t\frac{1}{(t-s)^{1/2}}\left(\|\vec{u}(s,\cdot)\|_{\dot{M}^{2,p}}\|\mathbf{V}(s,\cdot)\|_{L^\infty}+\||\mathbf{V}(s,\cdot)|^2\|_{\dot{M}^{2,p}}\right)ds\\
\leq& c\sup_{0\leq t\leq T}\int_0^t\frac{1}{(t-s)^{1/2}}\left(\|\vec{u}(s,\cdot)\|_{\dot{M}^{2,p}}\|\mathbf{V}(s,\cdot)\|_{L^\infty}+\|\mathbf{V}(s,\cdot)\|_{\dot{M}^{2,p}}\|\mathbf{V}(s,\cdot)\|_{L^\infty}\right)ds\\
\leq& c\left[\sup_{0\leq t\leq T}\int_0^t\frac{ds}{(t-s)^{1/2}s^{3/2p}}\right]\|(\vec{u},\mathbf{V})\|^2_{E_T}\leq c\,T^{\frac{1}{2}-\frac{3}{2p}}\,\|(\vec{u},\mathbf{V})\|^2_{E_T}.
\end{aligned}\tag{22}$$

Moreover, following the same computations done in the estimate (19) we have:

$$\begin{aligned}
&\sup_{0\leq t\leq T} t^{\frac{3}{2p}}\|B_3(\vec{u},\mathbf{V})+B_4(\mathbf{V},\mathbf{V})\|_{L^\infty}\\
&\sup_{0\leq t\leq T} t^{\frac{3}{2p}}\left\|\int_0^t e^{(t-s)\Delta}\vec{\nabla}\otimes(\vec{u}\,\mathbf{V})(s,\cdot)ds-\int_0^t e^{(t-s)\Delta}\vec{\nabla}\otimes(|\mathbf{V}|^2\,\vec{V})(s,\cdot)ds\right\|_{L^\infty}\\
\leq& c\,T^{\frac{1}{2}-\frac{3}{2p}}\,\|(\vec{u},\mathbf{V})\|^2_{E_T}.
\end{aligned}\tag{23}$$

By gathering these estimates, we are able to write

$$\|B_1(\vec{u},\vec{u})\|_{E_T}+\|B_2(\mathbf{V},\mathbf{V})\|_{E_T}+\|B_3(\vec{u},\mathbf{V})\|_{E_T}+\|B_4(\mathbf{V},\mathbf{V})\|_{E_T}\leq c\,T^{\frac{1}{2}-\frac{n}{2p}}\,\|(\vec{u},\mathbf{V})\|^2_{E_T}. \tag{24}$$

Once we have the estimates (13) and (24), for a time $0<T_0=T_0(\vec{u}_0,\mathbf{V}_0,\text{div}(\mathbb{F}),\text{div}(\mathbb{G}))<+\infty$ small enough, the existence and uniqueness of a solution $(\vec{u},\mathbf{V})$ for the equations (11) and (12) follow from standard arguments. Proposition 3.1 is proven. ■

Step 2. The global boundness of $\vec{U}$ and $\vec{\nabla}\otimes\vec{V}$. With the help of the Proposition 3.1, we are able to prove the following:

Proposition 3.2 *Let $(\vec{U},P,\vec{V})$ be a very weak solution of (1) given in Definition 1.1. If $\vec{U}\in\dot{M}^{2,p}(\mathbb{R}^3)$ and $\vec{\nabla}\otimes\vec{V}\in\dot{M}^{2,p}(\mathbb{R}^3)$, with $p>3$, then we have $\vec{U}\in L^\infty(\mathbb{R}^3)$ and $\vec{\nabla}\otimes\vec{V}\in L^\infty(\mathbb{R}^3)$.*

Proof. In the initial value problem (9) we set the initial data $(\vec{u}_0,\vec{\nabla}\otimes\vec{v}_0)=(\vec{U},\vec{\nabla}\otimes\vec{V})$. Then, by the Proposition 3.1 there exists a time $0<T_0$, and there exists a unique solution $(\vec{u},\vec{\nabla}\otimes\vec{v})\in\mathcal{C}_*([0,T],\dot{M}^{2,p}(\mathbb{R}^3))$ of (9) arising from $(\vec{U},\vec{\nabla}\otimes\vec{V})$.

On the other hand, we have the following key remark. First, we apply the Leray's projector in the first equation of the system (1). Thereafter, we apply the operator $\vec{\nabla}\otimes(\cdot)$ in the second equation of this system. Moreover, as $\vec{U}$ and $\vec{V}$ are time-independent functions we have $\partial_t\vec{U}=0$ and $\partial_t\vec{V}=0$. Thus, the couple

$(\vec{U}, \vec{\nabla} \otimes \vec{V})$ is also a solution of the initial value problem (9) with the initial data $(\vec{u}_0, \vec{\nabla} \otimes \vec{v}_0) = (\vec{U}, \vec{\nabla} \otimes \vec{V})$, and moreover, we have $(\vec{U}, \vec{\nabla} \otimes \vec{V}) \in \mathcal{C}_*([0,T], \dot{M}^{2,p}(\mathbb{R}^3))$.

Consequently, in the space $\mathcal{C}_*([0,T], \dot{M}^{2,p}(\mathbb{R}^3))$ we have two solutions of (9) with the same initial data $(\vec{u}_0, \vec{\nabla} \otimes \vec{v}_0) = (\vec{U}, \vec{\nabla} \otimes \vec{V})$: the solution $(\vec{u}, \mathbf{V})$ given by the Proposition 3.1 and the time-independent solution $(\vec{U}, \vec{\nabla} \otimes \vec{V})$. By uniqueness we have the identity $(\vec{u}, \mathbf{V}) = (\vec{U}, \vec{\nabla} \otimes \vec{V})$ and by (10) we can write

$$\sup_{0<t<T} t^{\frac{3}{2p}} \left(\|\vec{U}\|_{L^\infty} + \|\vec{\nabla} \otimes \vec{V}\|_{L^\infty} \right) < +\infty.$$

But, as the solution $(\vec{U}, \vec{\nabla} \otimes \vec{V})$ does not depend on the time variable we finally get $\vec{U} \in L^\infty(\mathbb{R}^3)$ and $\vec{\nabla} \otimes \vec{V} \in L^\infty(\mathbb{R}^3)$. Proposition 3.2 is now proven. ■

Step 3. High derivative estimates in the Morrey spaces. The global boundness of $\vec{U}$ and $\vec{\nabla} \otimes \vec{V}$ proved in the previous step is the key tool to show the following:

Proposition 3.3 *We assume that $\mathbb{F}$ and $\mathbb{G}$ verify (5) for $k \geq 0$, and moreover, we assume $\vec{U}, \vec{\nabla} \otimes \vec{V} \in \dot{M} \in \dot{M}^{2,p}(\mathbb{R}^3)$ with $p > 3$. Then we have $\vec{U} \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$, $\vec{V} \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$ and $P \in \mathcal{W}^{k+1,p}(\mathbb{R}^3)$.*

Proof. We will study first the functions $\vec{U}$ and $\vec{V}$. For this, we (temporally) get rid of the pressure term by applying the Leray's projector $\mathbb{P}$ in the first equation in (1) and we have

$$\begin{cases} -\Delta \vec{U} + \mathbb{P}(\text{div}(\vec{U} \otimes \vec{U})) + \mathbb{P}(\text{div}(\vec{\nabla} \otimes \vec{V} \odot \vec{\nabla} \otimes \vec{V})) = \mathbb{P}\left(\text{div}(\mathbb{F})\right), \\ -\Delta \vec{V} + \text{div}(\vec{V} \otimes \vec{U}) - |\vec{\nabla} \otimes \vec{V}|^2\, \vec{V} = \text{div}(\mathbb{G}), \\ \text{div}(\vec{U}) = 0, \end{cases} \tag{25}$$

As $\vec{U}$ and $\vec{V}$ solve this system they verify the following equivalent integral formulations:

$$\vec{U} = -\frac{1}{-\Delta} \left(\mathbb{P}(\text{div}(\vec{U} \otimes \vec{U})) \right) - \frac{1}{-\Delta} \left(\mathbb{P}(\text{div}(\vec{\nabla} \otimes \vec{V} \odot \vec{\nabla} \otimes \vec{V})) \right) + \frac{1}{-\Delta} \left(\mathbb{P}\left(\text{div}(\mathbb{F})\right) \right), \tag{26}$$

$$\vec{V} = -\frac{1}{-\Delta} \left(\text{div}(\vec{V} \otimes \vec{U}) \right) + \frac{1}{-\Delta} \left(|\vec{\nabla} \otimes \vec{V}|^2\, \vec{V} \right) + \frac{1}{-\Delta} \left(\mathbb{P}\left(\text{div}(\mathbb{G})\right) \right). \tag{27}$$

By using these integral formulations, we will show that $\partial^\alpha \vec{U} \in \dot{M}^{2,p}(\mathbb{R}^3)$ and $\partial^\alpha \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^3)$ for all $|\alpha| \leq k+2$. We shall prove this fact by an iterative argument respect to the order of the multi-indice α, which we will denote as $|\alpha|$.

For the sake of clearness, we shall prove each step in the iterative argument by separately technical lemmas.

Lemma 3.1 *Recall that by Proposition 3.2 we have $\vec{U}, \vec{\nabla} \otimes \vec{V} \in L^\infty(\mathbb{R}^3)$. Then, for $|\alpha| \leq 2$ we have $\partial^\alpha \vec{U} \in \dot{M}^{2,p}(\mathbb{R}^3)$ and $\partial^\alpha \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^3)$.*

Proof. We start by proving that $\partial^\alpha \vec{V} \in \dot{M}^{2,p}$ for all $|\alpha| \leq 2$. When $|\alpha| = 1$ we have $\partial^\alpha \vec{V} \in \dot{M}^{2\sigma, p\sigma}(\mathbb{R}^3)$ for all $1 \leq \sigma < +\infty$. Indeed, by the Proposition 3.2 we have $\vec{\nabla} \otimes \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^3)$ and $\vec{\nabla} \otimes \vec{V} \in L^\infty(\mathbb{R}^3)$; and by using the interpolation inequalities we get $\partial^\alpha \vec{V} \in \dot{M}^{2\sigma, p\sigma}(\mathbb{R}^3)$. Then, for $|\alpha| = 2$, by (27) we write

$$\partial^\alpha \vec{V} = -\frac{1}{-\Delta} \left(\partial^\alpha \text{div}(\vec{V} \otimes \vec{U}) \right) + \frac{1}{-\Delta} \left(\partial^\alpha \left(|\vec{\nabla} \otimes \vec{V}|^2\, \vec{V} \right) \right) + \frac{1}{-\Delta} \left(\mathbb{P}\left(\partial^\alpha \text{div}(\mathbb{G})\right) \right). \tag{28}$$

For the first term in the right side, we have $\text{div}(\vec{V} \otimes \vec{U}) \in \dot{M}^{2\sigma, 2p}(\mathbb{R}^3)$. Indeed, for $i, j = 1, 2, 3$ we write $\partial_i (V_i U_j) = \partial_i V_i\, U_j + V_i \partial_i U_j$. Then, recalling that $\vec{\nabla} \otimes \vec{V}, \vec{U} \in \dot{M}^{2\sigma, p\sigma}(\mathbb{R}^3)$, by the Hölder inequalities we get $\operatorname{div}(\vec{V}\otimes\vec{U})\in \dot{M}^{2\sigma,2p}(\mathbb{R}^3)$. Moreover, as $|\alpha|=2$ the operator $-\frac{1}{-\Delta}\left(\partial^\alpha(\cdot)\right)$ is continuous in the space $\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. Consequently, we have $-\frac{1}{-\Delta}\left(\partial^\alpha \operatorname{div}(\vec{V}\otimes\vec{U})\right)\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$.

For the second term in the right side, we have $|\vec{\nabla}\otimes\vec{V}|^2\,\vec{V}\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. Indeed, by Definition **??** we have $|\vec{V}(x)|=1$ *a.e.* $x\in\mathbb{R}^3$, moreover, by the Proposition 3.2 we have $\vec{\nabla}\otimes|\in \dot{M}^{2,p}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. Then, we can write

$$\||\vec{\nabla}\otimes\vec{V}|^2\,\vec{V}\|_{\dot{M}^{2\sigma,p\sigma}}\leq \|\vec{\nabla}\otimes\vec{V}\|_{\dot{M}^{2\sigma,p\sigma}}\|\vec{\nabla}\otimes\vec{V}\|_{L^\infty}\|\vec{V}\|_{L^\infty}\leq c\,\|\vec{\nabla}\otimes\|^{\theta}_{\dot{M}^{2,p}}\|\vec{\nabla}\otimes\vec{V}\|^{1-\theta}_{L^\infty}\,\|\vec{\nabla}\otimes\vec{V}\|_{L^\infty}\|\vec{V}\|_{L^\infty}.$$

Thereafter, always by the continuity of the operator $-\frac{1}{-\Delta}\left(\partial^\alpha(\cdot)\right)$ the space $\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$, we are able to write $\frac{1}{-\Delta}\left(\partial^\alpha\left(|\vec{\nabla}\otimes\vec{V}|^2\,\vec{V}\right)\right)\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$.

For the third term in the right side we also have $\frac{1}{-\Delta}\Big(\mathbb{P}\left(\partial^\alpha \operatorname{div}(\mathbb{G})\right)\Big)\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. Indeed, we can write

$$\left\|\frac{1}{-\Delta}\Big(\mathbb{P}\left(\partial^\alpha \operatorname{div}(\mathbb{G})\right)\Big)\right\|_{\dot{M}^{2\sigma,p\sigma}}\leq c\,\|\operatorname{div}(\mathbb{G})\|_{\dot{M}^{2\sigma,p\sigma}}\leq c\,\|\operatorname{div}(\mathbb{G})\|^{\theta}_{\dot{M}^{2,p}}\,\|\operatorname{div}(\mathbb{G})\|^{1-\theta}_{L^\infty},$$

where, the last expression in bounded thanks to (5).

Now, we must prove that $\partial^\alpha\vec{U}\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$ for all $|\alpha|\leq 2$. By (26) we have the identity:

$$\partial^\alpha\vec{U}=-\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \operatorname{div}(\vec{U}\otimes\vec{U}))\right)-\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \operatorname{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V}))\right)+\frac{1}{-\Delta}\Big(\mathbb{P}\left(\operatorname{div}(\partial^\alpha\mathbb{F})\right)\Big), \tag{29}$$

where, we must study each term in the right side. Let $|\alpha|=1$. For the first term above, we recall that by the Proposition 3.2 we have $\vec{U}\in \dot{M}^{2,p}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. Then, by the interpolation inequalities we get $\vec{U}\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$ for all $1\leq\sigma<+\infty$. See the Appendix B for an easy proof of this fact. Consequently, we can write $\vec{U}\in \dot{M}^{4\sigma,2p\sigma}(\mathbb{R}^3)$; and by the Hölder inequalities we have $\vec{U}\otimes\vec{U}\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$, for all $1\leq\sigma<+\infty$.

Thereafter, as $|\alpha|=1$ the operator $\dfrac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \operatorname{div}(\cdot)\right)$ writes down as a linear combination of the Riesz transforms $\mathcal{R}_i\mathcal{R}_j$ with $i,j=1,2,3$. Then, by the continuity of the operator $\mathcal{R}_i\mathcal{R}_j$ in the space $\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$ (Lemma 4.2 of [12]) we obtain $\left\|\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \operatorname{div}(\vec{U}\otimes\vec{U}))\right)\right\|_{\dot{M}^{2\sigma,p\sigma}}\leq c\,\|\vec{U}\otimes\vec{U}\|_{\dot{M}^{2\sigma,p\sigma}}$. Hence, $\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \operatorname{div}(\vec{U}\otimes\vec{U}))\right)\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$.

To study the second term in (29) we proceed similarly: as $\vec{\nabla}\otimes\vec{V}\in \dot{M}^{2,p}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ we have $\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \operatorname{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V}))\right)\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$.

We study now the third term (29). Always by the fact that the operator $\frac{1}{-\Delta}\Big(\mathbb{P}\left(\operatorname{div}(\partial^\alpha(\cdot))\right)\Big)$ is continuous in the space $\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$, we write $\left\|\frac{1}{-\Delta}\Big(\mathbb{P}\left(\operatorname{div}(\partial^\alpha\mathbb{F})\right)\Big)\right\|_{\dot{M}^{2\sigma,p\sigma}}\leq c\,\|\mathbb{F}\|_{\dot{M}^{2\sigma,p\sigma}}$. Moreover, by (5) we have $\mathbb{F}\in \dot{M}^{2,p}\cap L^\infty(\mathbb{R}^3)$, and by the interpolation inequalities (see the Appendix B) we get $\|\mathbb{F}\|_{\dot{M}^{2\sigma,p\sigma}}\leq c\,\|\mathbb{F}\|^{\theta}_{\dot{M}^{2,p}}\,\|\mathbb{F}\|^{1-\theta}_{L^\infty}$, with $\theta=$. We thus obtain $\frac{1}{-\Delta}\Big(\mathbb{P}\left(\operatorname{div}(\partial^\alpha\mathbb{F})\right)\Big)\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$.

We set now $|\alpha|=2$, and we shall prove that $\partial^\alpha\vec{U}\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. For the first term in the right side of (29) we write

$$\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \operatorname{div}(\vec{U}\otimes\vec{U}))\right)=\frac{1}{-\Delta}\left(\mathbb{P}(\partial^{\alpha_1}\operatorname{div}\,\partial^{\alpha_2}(\vec{U}\otimes\vec{U}))\right),\quad \text{where}\quad |\alpha_1|=1\quad\text{and}\quad |\alpha_2|=1.$$

Here, we must verify that $\partial^{\alpha_2}(\vec{U}\otimes\vec{U})\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. Indeed, for $i,j=1,2,3$ we write $\partial^{\alpha_2}(U_iU_j)=(\partial^{\alpha_2}U_i)U_j+U_i(\partial^{\alpha_2}U_j)$. Then, as for all multi-indice $|\beta|=1$ we have $\partial^\beta\vec{U}\in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$, and moreover, as by Proposition 3.2 we have $\vec{U}\in L^{\infty}(\mathbb{R}^3)$, we obtain $\partial^{\alpha_2}(\vec{U}\otimes\vec{U})\in\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. With this information, and as $|\alpha_1|=1$ the operator $\frac{1}{-\Delta}\left(\mathbb{P}(\partial^{\alpha_1}\text{div}(\cdot))\right)$ is continuous in the space $\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$, we finally have $\frac{1}{-\Delta}\left(\mathbb{P}(\partial^{\alpha}\text{div}(\vec{U}\otimes\vec{U}))\right)\in\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$.

The second term in the right side of (29) follows the same computations above (with $\vec{\nabla}\otimes\vec{V}$ instead of $\vec{U}$) and we have $\frac{1}{-\Delta}\left(\mathbb{P}(\partial^{\alpha}\text{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V}))\right)\in\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$.

In order to estimate the third term in the right side of (29), we also write $\alpha=\alpha_1+\alpha_2$ with $|\alpha_1|=|\alpha_2|=1$. Then, we write:

$$\begin{aligned}\left\|\frac{1}{-\Delta}\Big(\mathbb{P}\left(\text{div}(\partial^{\alpha}\mathbb{F})\right)\Big)\right\|_{\dot{M}^{2\sigma,p\sigma}}=&\left\|\frac{1}{-\Delta}\Big(\mathbb{P}\left(\partial^{\alpha_1}\text{div}(\partial^{\alpha_2}\mathbb{F})\right)\Big)\right\|_{\dot{M}^{2\sigma,p\sigma}}\leq c\,\|\partial^{\alpha_2}\mathbb{F}\|_{\dot{M}^{2\sigma,p\sigma}}\\ \leq& c\,\|\partial^{\alpha_2}\mathbb{F}\|^{\theta}_{\dot{M}^{2,p}}\|\partial^{\alpha_2}\mathbb{F}\|^{1-\theta}_{L^{\infty}},\end{aligned}\tag{30}$$

where by (5) the last expression is bounded. ■

Lemma 3.2 *For $2\leq m\leq k+1$ and for all $|\alpha|\leq m$ we assume $\partial^{\alpha}\vec{V}\in\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$, $\partial^{\alpha}\vec{U}\in\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. Then, it holds true for all $|\alpha|=m+1$.*

Proof. We start by studying the function $\partial^{\alpha}\vec{V}$ given in the identity (28), where we shall study each term in the right side of this identity. For the first term, we write

$$-\frac{1}{-\Delta}\left(\partial^{\alpha}\text{div}(\vec{V}\otimes\vec{U})\right)=-\frac{1}{-\Delta}\left(\partial^{\alpha_1}\text{div}\partial^{\alpha_2}(\vec{V}\otimes\vec{U})\right),\quad\text{with}\quad|\alpha_1|=1\quad\text{and}\quad|\alpha_2|=m.$$

In this expression, we verify now that $\partial^{\alpha_2}(\vec{V}\otimes\vec{U})\in\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. For $i,j=1,2,3$, by the Leibinz rule we write:

$$\partial^{\alpha_2}(V_iU_j)=\sum_{|\beta|\leq m}c_{\alpha_2,\beta}\,\partial^{\beta}V_i\,\partial^{\alpha_2-\beta}U_j.$$

Then, recalling that by the recurrence hypothesis we have $\partial^{\alpha_2-\beta}V_i\in\dot{M}^{4\sigma,2p\sigma}(\mathbb{R}^3)$ and $\partial^{\beta}U_j\in\dot{M}^{4\sigma,2p\sigma}(\mathbb{R}^3)$, and moreover, by applying the Hölder inequalities we get $\partial^{\alpha_2}(\vec{V}\otimes\vec{U})\in\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. Thereafter, as $|\alpha_1|=1$ the operator $-\frac{1}{-\Delta}\left(\partial^{\alpha_1}\text{div}(\cdot)\right)$ is continuous in the space $\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$; and we have $-\frac{1}{-\Delta}\left(\partial^{\alpha}\text{div}(\vec{V}\otimes\vec{U})\right)\in\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$.

For the second term in the right side of (28), we write now :

$$\frac{1}{-\Delta}\left(\partial^{\alpha}\left(|\vec{\nabla}\otimes\vec{V}|^2\,\vec{V}\right)\right)=\frac{1}{-\Delta}\left(\partial^{\alpha_1}\left(\partial^{\alpha_2}\left(|\vec{\nabla}\otimes\vec{V}|^2\,\vec{V}\right)\right)\right),\quad\text{with,}\quad|\alpha_1|=2\quad\textit{and}\quad|\alpha_2|=m-1,$$

where we need to verify that $\partial^{\alpha_2}\left(|\vec{\nabla}\otimes\vec{V}|^2\,\vec{V}\right)\in\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. For $i,j,k=1,2,3$, always by the Leibinz rule we have:

$$\begin{aligned}\partial^{\alpha_2}\left((\partial_iV_j)^2V_k\right)=&\sum_{|\beta|\leq m-1}\partial^{\beta}\left((\partial_iV_j)^2\right)\,\partial^{\alpha_2-\beta}V_k\\=&\sum_{|\beta|\leq m-1}\left(\sum_{|\gamma|\leq m-1}(\partial^{\gamma}\partial_iV_j)\,(\partial^{\beta-\gamma}\partial_iV_j)\right)\partial^{\alpha_2-\beta}V_k\\=&\sum_{|\beta|\leq m-1}\sum_{|\gamma|\leq m-1}\partial^{\gamma}(\partial_iV_j)\,\partial^{\beta-\gamma}(\partial_iV_j)\,(\partial^{\alpha_2-\beta}V_k).\end{aligned}$$

Then, as by the recurrence hypothesis we have $\partial^\gamma(\partial_i V_j)\,\partial^{\beta-\gamma}(\partial_i V_j)\,(\partial^{\alpha_2-\beta}V_k) \in \dot{M}^{6\sigma,3p\sigma}(\mathbb{R}^3)$ we can apply the Hölder inequalities to obtain $\partial^{\alpha_2}\left(|\vec{\nabla}\otimes\vec{V}|^2\,\vec{V}\right) \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. Moreover, as $|\alpha_1| = 2$ the operator $-\frac{1}{-\Delta}\partial^{\alpha_1}(\cdot)$ is continuous in the space $\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$ and we get $\frac{1}{-\Delta}\left(\partial^\alpha\left(|\vec{\nabla}\otimes\vec{V}|^2\,\vec{V}\right)\right) \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$.

Finally, the third term in the right side of (28) is estimated as in the equation (30), where we write $\alpha = \alpha_1 + \alpha_2$ with $|\alpha| = 1$ and $|\alpha_2| = m$.

Once we have $\partial^\alpha\vec{V} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$ for all $|\alpha| \le m+1$, we are able to prove the same property on the function $\vec{U}$. We must study each term in the right side of the equation (29), where we set $|\alpha| = m+1$. Essentially we shall follow the same arguments above, so we only detail the most important ideas.

For the first term we write $-\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \text{div}(\vec{U}\otimes\vec{U}))\right) = -\frac{1}{-\Delta}\left(\mathbb{P}(\partial^{\alpha_1}\text{div}\partial^{\alpha_2}(\vec{U}\otimes\vec{U}))\right)$ with $|\alpha_1| = 1$ and $|\alpha_2| = m$. Then, as we have $\partial^\beta\vec{U} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$ for all $|\beta| \le m$, and moreover, by using the Leibniz rule and the Hölder inequalities, we get $\partial^{\alpha_2}(\vec{U}\otimes\vec{U}) \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. Thereafter we have $-\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha\text{div}(\vec{U}\otimes\vec{U}))\right) \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$. This also holds true for the second term in the right side of (29), where we write $\vec{\nabla}\otimes\vec{V}$ instead of $\vec{U}$ and we use the fact that $\partial^\alpha\vec{V} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$ for all $|\alpha| \le m+1$. Finally, to study third term in the right side of (29), we follow the same estimates given in (30) with $|\alpha_1| = 1$ and $|\alpha_2| = m$. ■

By this iterative argument we finally get $\vec{U} \in \mathcal{W}_2^{k+2,p}(\mathbb{R}^3)$ and $\vec{V} \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$. It remains to study the pressure term in the first equation of the system (1). For this, we apply the divergence operator in the first equation of (1), to obtain that P is related to $\vec{U}$ and $\vec{\nabla}\otimes\vec{V}$ through the Riesz transforms $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the following formula

$$P = \frac{1}{-\Delta}\left(\text{div}\left(\text{div}(U_i\,U_j)\right)\right) + \frac{1}{-\Delta}\left(\text{div}\left(\text{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V})\right)\right) + \frac{1}{-\Delta}\left(\text{div}\left(\text{div}(\mathbb{F})\right)\right).$$

Lemma 3.3 *We have $P \in \mathcal{W}^{k+1,p}(\mathbb{R}^3)$.*

Proof. For the first term in the right side, as we have $\partial^\alpha\vec{U} \in \dot{M}^{2\sigma,p\sigma}$ for all $|\alpha| \le k+2$, and $\vec{U} \in L^\infty(\mathbb{R}^3)$, then we get $\frac{1}{-\Delta}\left(\text{div}\left(\text{div}(U_i\,U_j)\right)\right) \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$ for $|\alpha| \le k+2$. Then, for the second term in the right side, as we have $\partial^\alpha\vec{V} \in \dot{M}^{2\sigma,p\sigma}$ for all $|\alpha| \le k+2$, and $\vec{\nabla}\otimes \in L^\infty(\mathbb{R}^3)$, we get $\frac{1}{-\Delta}\left(\text{div}\left(\text{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V})\right)\right) \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$ for $|\alpha| \le k+1$. Finally, for the third term in the ride side, by (5) we have $\frac{1}{-\Delta}\left(\text{div}\left(\text{div}(\mathbb{F})\right)\right) \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^3)$ for $|\alpha| \le k+1$. Therefore we have $\partial^\alpha P \in \dot{M}^{2\sigma}(\mathbb{R}^3)$ for all $|\alpha| \le k+1$, hence $P \in \mathcal{W}^{k+1,p}(\mathbb{R}^3)$. ■

We are able to finish the proof of Theorem 1.1. Recall that we have the embedding $\dot{M}^{2,p}(\mathbb{R}^3) \subset \dot{M}^{1,p}(\mathbb{R}^3)$. Then, for all $|\alpha| \le k+1$, by Lemma 2.4 the functions $\partial^\alpha\vec{U}$ and $\partial^\alpha\vec{V}$ are Hölder continuous with exponent $0 < 1 - 3/p < 1$, while for $|\alpha| \le k$ the function $\partial^\alpha P$ is also Hölder continuous with the same exponent. Theorem 1.1 is now proven. ■

4 Proof of the Theorem ??

The proof of this theorem follows the same lines of the proof of the Theorem 1.1. Consequently, we will give a sketch of the main steps.

We consider first the evolution problem for the MHD equations:

$$\begin{cases} \partial_t\vec{u} - \Delta\vec{u} + \mathbb{P}(\text{div}(\vec{u}\otimes\vec{u})) - \mathbb{P}(\text{div}(\vec{b}\otimes\vec{b})) = 0, \quad \text{div}(\vec{u}) = 0, \\ \partial_t\vec{b} - \Delta\vec{b} + \text{div}(\vec{b}\otimes\vec{u}) - \text{div}(\vec{u}\otimes\vec{b}) = 0, \quad \text{div}(\vec{b}) = 0, \\ \vec{u}(0,\cdot) = \vec{u}_0 \quad \vec{b}(0,\cdot) = \vec{b}_0. \end{cases} \tag{31}$$

Then, we consider the following (equivalent) integral formulation of the system above:

$$\vec{u}(t,\cdot) = e^{t\Delta}\vec{u}_0 + \underbrace{\int_0^t e^{(t-s)\Delta}\mathbb{P}(div(\vec{u}\otimes\vec{u}))(s,\cdot)ds}_{\mathbf{B_1}(\vec{u},\vec{u})} - \underbrace{\int_0^t e^{(t-s)\Delta}\mathbb{P}(\text{div}(\vec{b}\otimes\vec{b}))(s,\cdot)ds}_{\mathbf{B_2}(\vec{b},\vec{b})}, \tag{32}$$

and

$$\vec{b}(t,\cdot) = e^{t\Delta}\vec{b}_0 + \underbrace{\int_0^t e^{(t-s)\Delta}\text{div}(\vec{b}\otimes\vec{u})(s,\cdot)ds}_{\mathbf{B_3}(\vec{b},\vec{u})} - \underbrace{\int_0^t e^{(t-s)\Delta}\text{div}(\vec{u}\otimes\vec{b})(s,\cdot)ds}_{\mathbf{B_4}(\vec{u},\vec{b})}. \tag{33}$$

Following the same computations done in the proof of the Propositions 3.1 and 3.2, for the initial data $(\vec{u}_0,\vec{b}_0)\in\dot{M}^{2,p}(\mathbb{R}^3)$, with $3<p<+\infty$, there exists a time $T>0$ and $(\vec{u},\vec{b})\in\mathcal{C}_*([0,T],\dot{M}^{2,p}(\mathbb{R}^3))$ a unique solution of (32)-(33) which verify:

$$\sup_{0<t<T} t^{\frac{3}{2p}}\Big(\|\vec{u}(t,\cdot)\|_{L^\infty}+\|\vec{b}(t,\cdot)\|_{L^\infty}\Big)<+\infty. \tag{34}$$

Now, the key idea is the fact that the time-independent functions $(\vec{U},\vec{B})\in\dot{M}^{2,p}(\mathbb{R}^3)$, which solve the system (8), also belong to the space $\mathcal{C}_*([0,T],\dot{M}^{2,p}(\mathbb{R}^3))$ and they solve the evolutionary system (31) with the initial data $\vec{u}_0=\vec{U}$ and $\vec{b}_0=\vec{B}$. Then, by the uniqueness of solutions in this space we obtain that $\vec{U}$ and $\vec{B}$ verify (34), hence, we have that $\vec{U}\in L^\infty(\mathbb{R}^3)$ and $\vec{B}\in L^\infty(\mathbb{R}^3)$.

On the other hand, as $(\vec{U},\vec{B})\in\dot{M}^{2,p}(\mathbb{R}^3)$ solve the system (8) then they also solve the following integral equations

$$\begin{aligned}
\vec{U} &= -\frac{1}{-\Delta}\Big(\mathbb{P}(\text{div}(\vec{U}\otimes\vec{U}))\Big) - \frac{1}{-\Delta}\Big(\mathbb{P}(\text{div}(\vec{B}\otimes\vec{B}))\Big),\\
\vec{B} &= -\frac{1}{-\Delta}\Big(\text{div}(\vec{B}\otimes\vec{U})\Big) - \frac{1}{-\Delta}\Big(\mathbb{P}(\text{div}(\vec{U}\otimes\vec{B}))\Big),\\
P &= \sum_{i,j=1}^{3}\mathcal{R}_i\mathcal{R}_j\Big(U_iU_j+B_iB_j\Big), \quad \mathcal{R}_i=\frac{\partial_i}{\sqrt{-\Delta}},\ i=1,2,3.
\end{aligned} \tag{35}$$

Using these equations, the information $\vec{U}\in L^\infty(\mathbb{R}^3)$ and $\vec{B}\in L^\infty(\mathbb{R}^3)$, and moreover, following the computations done in the proof of the Theorem **??** we prove that for all multi-indice $\alpha\in\mathbb{N}^n\setminus\{0\}$ we have $\partial^\alpha\vec{U}\in\dot{M}^{2,p}(\mathbb{R}^3)$, $\partial^\alpha\vec{B}\in\dot{M}^{2,p}(\mathbb{R}^3)$ and $\partial^\alpha P\in\dot{M}^{2,p}(\mathbb{R}^3)$.

Finally, with this last information and by the Proposition 3.3, we obtain the desired regularity properties stated in the Theorem **??**. ■

5 Proof of the Theorem 1.3

We consider the evolution problem for the MHD equations given in (31). Then, we have the following result.

Lemma 5.1 *Let $(\vec{u}_0,\vec{b}_0)\in\dot{H}^1(\mathbb{R}^3)$ be the divergence-free initial data. Then, there exists a time $T_0>0$, depending on $\vec{u}_0$ and $\vec{b}_0$, and there exists two functions $\vec{u},\vec{b}\in\mathcal{C}([0,T_0],\dot{H}^1(\mathbb{R}^3))$ which the couple $(\vec{u},\vec{b})$ is the unique solution of (31).*

Existence and uniqueness are classical issues. We refer to the book [16] for a proof in the case of the Navier-Stokes equations (with $\vec{b}=0$). However, this proof can be easily adapted for the MHD system (31).

We study now the belonging to the Gevrey class of the solution $(\vec{u},\vec{b})$ given in the Lemma 5.1. For this, for $\beta>0$ and for $t>0$ we recall that the weighted exponential operator $e^{\beta\, t\sqrt{-\Delta}}$ is defined in in the Fourier variable as

$$\mathcal{F}\Big(e^{\beta\, t\sqrt{-\Delta}}f(t,\cdot)\Big)(\xi)=e^{\beta\, t|\xi|}\widehat{f}(t,\xi),\quad f\in\mathcal{S}'([0,+\infty[\times\mathbb{R}^3). \tag{36}$$

Then, we have the following result.

Proposition 5.1 *Let $\beta>0$. Within the framework of the Lemma 5.1, there exists a time $0<T_1<T_0$ such that the unique solution $(\vec{u},\vec{b})\in\mathcal{C}([0,T_0],\dot{H}^1(\mathbb{R}^3))$ of the system (31) verifies:*

$$e^{\beta\, t\sqrt{-\Delta}}\vec{u}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3)),\quad \textit{and}\quad e^{\beta\, t\sqrt{-\Delta}}\vec{b}\in\mathcal{C}(]0,T_1[,\dot{H}^1(\mathbb{R}^3)). \tag{37}$$

Proof. For the time $0<T<T_0$, we consider the space

$$E=\Big\{f\in\mathcal{C}(]0,T[,\dot{H}^1(\mathbb{R}^3)):\ e^{\beta\, t\sqrt{-\Delta}}f\in\mathcal{C}(]0,T[,\dot{H}^1(\mathbb{R}^3))\Big\},$$

which is a Banach space wi the norm

$$\|\cdot\|_E=\|e^{\beta\, t\sqrt{-\Delta}}(\cdot)\|_{L^\infty_t\dot{H}^1_x}.$$

Then, for the initial data $(\vec{u}_0,\vec{b}_0)\in\dot{H}^1(\mathbb{R}^3)$, we will solve the equations (32)-(33) in the space E. The linear terms in the system (32)-(33) are easy to estimate and for a constant $C=C(\beta,T)>0$ we have

$$\Big\|\Big(e^{t\,\Delta}\vec{u}_0,e^{t\,\Delta}\vec{b}_0\Big)\Big\|_E\leq C\|(\vec{u}_0,\vec{b}_0)\|_{\dot{H}^1}. \tag{38}$$

On the other hand, as the four non-liner terms $\mathbf{B_i}(\cdot,\cdot)$ $(i=1,\cdots,4)$ in the system (32)-(33) are completely similar, it is enough to only estimate the first one.

By the Plancherel's formula and by the boundness of the Leray's projector, we can write:

$$\begin{aligned}\|\mathbf{B_1}(\vec{u},\vec{u})\|_E&=\sup_{0\leq t\leq T}\left\|e^{\beta\, t\sqrt{-\Delta}}\left(\int_0^t e^{(t-s)\Delta}\mathbb{P}(\text{div}(\vec{u}\otimes\vec{u}))(s,\cdot)ds\right)\right\|_{\dot{H}^1}\\ &\leq\sup_{0\leq t\leq T}\left\||\xi|^2\int_0^t e^{-(t-s)|\xi|^2}e^{\beta t|\xi|}\left(\widehat{\vec{u}}*\widehat{\vec{u}}\right)(s,\cdot)ds\right\|_{L^2}.\end{aligned} \tag{39}$$

We must study now the expression $e^{\beta t|\xi|}\left(\widehat{\vec{u}}*\widehat{\vec{u}}\right)(s,\xi)$. We remark that by the triangular inequality, for all $\xi,\eta\in\mathbb{R}^3$, we have $e^{\beta t|\xi|}\leq e^{\beta t|\xi-\eta|}\,e^{\beta t|\eta|}$. Then, we obtain the following pointwise estimate:

$$\left|e^{\beta t|\xi|}\left(\widehat{\vec{u}}*\widehat{\vec{u}}\right)(s,\xi)\right|\leq\left(\left(e^{\beta t|\xi|}|\widehat{\vec{u}}|\right)*\left(e^{\beta t|\xi|}|\widehat{\vec{u}}|\right)\right)(s,\xi).$$

Thus, getting back to the estimate (39) we write

$$\begin{aligned}&\sup_{0\leq t\leq T}\left\||\xi|^2\int_0^t e^{-(t-s)|\xi|^2}e^{\beta t|\xi|}\left(\widehat{\vec{u}}*\widehat{\vec{u}}\right)(s,\cdot)ds\right\|_{L^2}\\ \leq&\sup_{0\leq t\leq T}\left\||\xi|^2\int_0^t e^{-(t-s)|\xi|^2}\left(\left(e^{\beta t|\xi|}|\widehat{\vec{u}}|\right)*\left(e^{\beta t|\xi|}|\widehat{\vec{u}}|\right)\right)(s,\cdot)ds\right\|_{L^2}\\ \leq&\sup_{0\leq t\leq T}\int_0^t\left\||\xi|^{3/2}e^{-(t-s)|\xi|^2}\,|\xi|^{1/2}\left(\left(e^{\beta t|\xi|}|\widehat{\vec{u}}|\right)*\left(e^{\beta t|\xi|}|\widehat{\vec{u}}|\right)\right)(s,\cdot)ds\right\|_{L^2}=(a).\end{aligned} \tag{40}$$

Always by the Plancherel's formula, we get back to the spatial variable. Moreover, by the well-known properties of the heat and the product laws in the homogeneous Sobolev spaces, we have:

$$\begin{aligned}
(a) \leq & \sup_{0\leq t\leq T} \int_0^t \left\| (-\Delta)^{3/4} h_{(t-s)}(\cdot) * (-\Delta)^{1/2} \left(\left(e^{\beta t|\xi|} |\widehat{\vec{u}}| \right)^{\vee} * \left(e^{\beta t|\xi|} |\widehat{\vec{u}}| \right)^{\vee} \right)(s,\cdot) \right\|_{L^2} ds \\
\leq & \sup_{0\leq t\leq T} \int_0^t \left\| (-\Delta)^{3/4} h_{(t-s)}(\cdot) \right\|_{L^1} \left\| (-\Delta)^{1/2} \left(\left(e^{\beta t|\xi|} |\widehat{\vec{u}}| \right)^{\vee} * \left(e^{\beta t|\xi|} |\widehat{\vec{u}}| \right)^{\vee} \right)(s,\cdot) \right\|_{L^2} ds \\
\leq & c\, T^{1/4} \sup_{0\leq s\leq T} \left\| \left(\left(e^{\beta t|\xi|} |\widehat{\vec{u}}| \right)^{\vee} * \left(e^{\beta t|\xi|} |\widehat{\vec{u}}| \right)^{\vee} \right)(s,\cdot) \right\|_{\dot{H}^{1/2}} \\
\leq & c\, T^{1/4} \|\vec{u}\|_E \|\vec{u}\|_E.
\end{aligned}$$

The other non-linear terms are treated in the same way, and we can write

$$\|\mathbf{B_1}(\vec{u},\vec{u})\|_E + \|\mathbf{B_2}(\vec{b},\vec{b})\|_E + \|\mathbf{B_3}(\vec{b},\vec{u})\|_E + \|\mathbf{B_4}(\vec{u},\vec{b})\|_E \leq c\, T^{1/4} \|(\vec{u},\vec{b})\|_E. \tag{41}$$

Thereafter, by (38) and (41), we fix a time $0 < T_1 < T_0$ small enough such that we cal apply the Picard's iterative schema to obtain a solution $(\vec{u},\vec{b}) \in E$ of the system (32)-(33).

Finally, we remark that we have $E \subset \mathcal{C}([0,T_0], \dot{H}^1(\mathbb{R}^3))$. Then, by the uniqueness of solutions of (32)-(33) in this last space we obtain (37). ■To finish the proof of the Theorem 1.3, we observe that the time-independent solution $(\vec{U},\vec{B}) \in \dot{H}^1(\mathbb{R}^3)$ of the system (8) verifies $(\vec{U},\vec{B}) \in \mathcal{C}([0,T_1], \dot{H}^1(\mathbb{R}^3))$; and this solution also solves the evolutionary problem (31) with initial data $(\vec{u}_0,\vec{b}_0) = (\vec{U},\vec{B})$; and by the Lemma 5.1 this solution is unique. Then, by the Proposition 5.1 we have

$$e^{\beta\, t\sqrt{-\Delta}} \vec{U} \in \mathcal{C}(]0,T_1[, \dot{H}^1(\mathbb{R}^3)), \quad \text{and} \quad e^{\beta\, t\sqrt{-\Delta}} \vec{B} \in \mathcal{C}(]0,T_1[, \dot{H}^1(\mathbb{R}^3)).$$

Finally, we set the parameter $\alpha = \beta\sqrt{\frac{T_1}{2}} > 0$, hence we have $\vec{U} \in G^1_\alpha(\mathbb{R}^3)$ and $\vec{B} \in G^1_\alpha(\mathbb{R}^3)$. Moreover, by the third identity in (35) we have $P \in G^{1/2}_\alpha(\mathbb{R}^3)$. The Theorem 1.3 is proven. ■

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Data availability statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.