Weak solutions for Navier–Stokes equations with initial data in weighted L^2 spaces.

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Abstract

We show the existence of global weak solutions of the 3D Navier-Stokes equations with initial velocity in the weighted spaces $L^2_{w_\gamma}$, where $w_\gamma(x) = (1 + |x|)^{-\gamma}$ and $0 < \gamma \leq 2$, using new energy controls. As application we give a new proof of the existence of global weak discretely self-similar solutions of the 3D Navier–Stokes equations for discretely self-similar initial velocities which are locally square integrable.

Keywords : Navier–Stokes equations, weighted spaces, discretely self-similar solutions, energy controls
AMS classification : 35Q30, 76D05.

1 Introduction.

Infinite-energy weak Leray solutions to the Navier–Stokes equations were introduced by Lemarié-Rieusset in 1999 [8] (they are presented more completely in [9] and [10]). This has allowed to show the existence of local weak solutions for a uniformly locally square integrable initial data.

Other constructions of infinite-energy solutions for locally uniformly square integrable initial data were given in 2006 by Basson [1] and in 2007 by Kikuchi and Seregin [7]. These solutions allowed Jia and Sverak [6] to construct in 2014 the self-similar solutions for large (homogeneous of degree -1) smooth data. Their result has been extended in 2016 by Lemarié-Rieusset [10] to

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solutions for rough locally square integrable data. We remark that an homogeneous (of degree -1) and locally square integrable data is automatically uniformly locally L^2.

Recently, Bradshaw and Tsai [2] and Chae and Wolf [3] considered the case of solutions which are self-similar according to a discrete subgroup of dilations. Those solutions are related to an initial data which is self-similar only for a discrete group of dilations; in contrast to the case of self-similar solutions for all dilations, such an initial data, when locally L^2, is not necessarily uniformly locally L^2, therefore their results are no consequence of constructions described by Lemarié-Rieusset in [10].

In this paper, we construct an alternative theory to obtain infinite-energy global weak solutions for large initial data, which include the discretely self-similar locally square integrable data. More specifically, we consider the weights

$$w_\gamma(x) = \frac{1}{(1+|x|)^\gamma}$$

with $0 < \gamma$, and the spaces

$$L^2_{w_\gamma} = L^2(w_\gamma\, dx).$$

Our main theorem is the following one :

Theorem 1 *Let $0 < \gamma \leq 2$. If $\mathbf{u}_0$ is a divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ and if $\mathbb{F}$ is a tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1\leq i,j\leq 3}$ such that $\mathbb{F} \in L^2((0,+\infty), L^2_{w_\gamma})$, then the Navier–Stokes equations with initial value $\mathbf{u}_0$*

$$(NS) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \nabla\cdot\mathbb{F} \\ \\ \nabla\cdot\mathbf{u} = 0, \qquad\qquad \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

has a global weak solution $\mathbf{u}$ such that :

- *for every $0 < T < +\infty$, $\mathbf{u}$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{u}$ belongs to $L^2((0,T), L^2_{w_\gamma})$*
- *the pressure p is related to $\mathbf{u}$ and $\mathbb{F}$ through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula*

$$p = \sum_{i=1}^{3}\sum_{j=1}^{3} R_iR_j(u_iu_j - F_{i,j})$$

where, for every $0 < T < +\infty$, $\sum_{i=1}^3\sum_{j=1}^3 R_iR_j(u_iu_j)$ belongs to $L^4((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $\sum_{i=1}^3\sum_{j=1}^3 R_iR_jF_{i,j}$ belongs to $L^2((0,T), L^2_{w_\gamma})$

- *the map* $t \in [0,+\infty) \mapsto \mathbf{u}(t,.)$ *is weakly continuous from* $[0,+\infty)$ *to* $L^2_{w_\gamma}$*, and is strongly continuous at* $t=0$ *:*

$$\lim_{t\to 0}\|\mathbf{u}(t,.)-\mathbf{u}_0\|_{L^2_{w_\gamma}}=0.$$

- *the solution* $\mathbf{u}$ *is suitable : there exists a non-negative locally finite measure* μ *on* $(0,+\infty)\times\mathbb{R}^3$ *such that*

$$\partial_t(\frac{|\mathbf{u}|^2}{2})=\Delta(\frac{|\mathbf{u}|^2}{2})-|\nabla\mathbf{u}|^2-\nabla\cdot\left((\frac{|\mathbf{u}|^2}{2}+p)\mathbf{u}\right)+\mathbf{u}\cdot(\nabla\cdot\mathbb{F})-\mu.$$

In particular, we have the energy controls

$$\begin{aligned}\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}}&+2\int_0^t\|\nabla\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}}\,ds\\ \leq&\|\mathbf{u}_0\|^2_{L^2_{w_\gamma}}-\int_0^t\int\nabla|\mathbf{u}|^2\cdot\nabla w_\gamma\,dx\,ds+\int_0^t\int(|\mathbf{u}|^2+2p)\mathbf{u}\cdot\nabla(w_\gamma)\,dx\,ds\\ &-2\sum_{i=1}^3\sum_{j=1}^3\int_0^t\int F_{i,j}(\partial_i u_j)w_\gamma+F_{i,j}u_i\partial_j(w_\gamma)\,dx\,ds\end{aligned}$$

and

$$\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}}\leq\|\mathbf{u}_0\|^2_{L^2_{w_\gamma}}+C_\gamma\int_0^t\|\mathbb{F}(s,.)\|^2_{L^2_{w_\gamma}}\,ds+C_\gamma\int_0^t\|\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}}+\|\mathbf{u}(s,.)\|^6_{L^2_{w_\gamma}}\,ds$$

A key tool for proving Theorem 1 and for applying it to the study of discretely self-similar solutions is given by the following a priori estimates for an advection-diffusion problem :

Theorem 2 *Let* $0<\gamma\leq 2$*. Let* $0<T<+\infty$*. Let* $\mathbf{u}_0$ *be a divergence-free vector field such that* $\mathbf{u}_0\in L^2_{w_\gamma}(\mathbb{R}^3)$ *and* $\mathbb{F}$ *be a tensor* $\mathbb{F}(t,x)=(F_{i,j}(t,x))_{1\leq i,j\leq 3}$ *such that* $\mathbb{F}\in L^2((0,T),L^2_{w_\gamma})$*. Let* $\mathbf{b}$ *be a time-dependent divergence free vector-field (*$\nabla\cdot\mathbf{b}=0$*) such that* $\mathbf{b}\in L^3((0,T),L^3_{w_{3\gamma/2}})$*.*

Let $\mathbf{u}$ *be a solution of the following advection-diffusion problem*

$$(AD)\begin{cases}\partial_t\mathbf{u}=\Delta\mathbf{u}-(\mathbf{b}\cdot\nabla)\mathbf{u}-\nabla p+\nabla\cdot\mathbb{F}\\ \nabla\cdot\mathbf{u}=0, \qquad\qquad \mathbf{u}(0,.)=\mathbf{u}_0\end{cases}$$

be such that :

- $\mathbf{u}$ *belongs to* $L^\infty((0,T),L^2_{w_\gamma})$ *and* $\nabla\mathbf{u}$ *belongs to* $L^2((0,T),L^2_{w_\gamma})$

- *the pressure* p *is related to* $\mathbf{u}$, $\mathbf{b}$ *and* $\mathbb{F}$ *through the Riesz transforms* $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ *by the formula*

$$p = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (b_i u_j - F_{i,j})$$

where $\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (b_i u_j)$ *belongs to* $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ *and* $\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j F_{i,j}$ *belongs to* $L^2((0,T), L^2_{w_\gamma})$

- *the map* $t \in [0,T) \mapsto \mathbf{u}(t,.)$ *is weakly continuous from* $[0,T)$ *to* $L^2_{w_\gamma}$, *and is strongly continuous at* $t = 0$ *:*

$$\lim_{t \to 0} \|\mathbf{u}(t,.) - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0.$$

- *there exists a non-negative locally finite measure* μ *on* $(0,T) \times \mathbb{R}^3$ *such that*

$$\partial_t(\frac{|\mathbf{u}|^2}{2}) = \Delta(\frac{|\mathbf{u}|^2}{2}) - |\nabla \mathbf{u}|^2 - \nabla \cdot \left(\frac{|\mathbf{u}|^2}{2} \mathbf{b} \right) - \nabla \cdot (p\mathbf{u}) + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) - \mu. \quad (1)$$

Then, we have the energy controls

$$\begin{aligned} \|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}} &+ 2 \int_0^t \|\nabla \mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}} \, ds \\ \leq & \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} - \int_0^t \int \nabla |\mathbf{u}|^2 \cdot \nabla w_\gamma \, dx \, ds + \int_0^t \int |\mathbf{u}|^2 \mathbf{b} \cdot \nabla (w_\gamma) \, dx \, ds \\ &+ 2 \int_0^t \int p\mathbf{u} \cdot \nabla (w_\gamma) \, dx \, ds - 2 \sum_{i=1}^{3} \sum_{j=1}^{3} \int_0^t \int F_{i,j} (\partial_i u_j) w_\gamma + F_{i,j} u_i \partial_j (w_\gamma) \, dx \, ds \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}} &+ \int_0^t \|\nabla \mathbf{u}\|^2_{L^2_{w_\gamma}} \, ds \\ \leq & \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + C_\gamma \int_0^t \|\mathbb{F}(s,.)\|^2_{L^2_{w_\gamma}} \, ds + C_\gamma \int_0^t (1 + \|\mathbf{b}(s,.)\|^2_{L^3_{w_{3\gamma/2}}}) \|\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}} \, ds \end{aligned}$$

where C_γ *depends only on* γ *(and not on* T*, and not on* $\mathbf{b}$, $\mathbf{u}$, $\mathbf{u}_0$ *nor* $\mathbb{F}$*).*

In particular, we shall prove the following stability result :

Theorem 3 *Let $0 < \gamma \leq 2$. Let $0 < T < +\infty$. Let $\mathbf{u}_{0,n}$ be divergence-free vector fields such that $\mathbf{u}_{0,n} \in L^2_{w_\gamma}(\mathbb{R}^3)$ and $\mathbb{F}_n$ be tensors such that $\mathbb{F}_n \in L^2((0,T), L^2_{w_\gamma})$. Let $\mathbf{b}_n$ be time-dependent divergence free vector-fields such that $\mathbf{b}_n \in L^3((0,T), L^3_{w_{3\gamma/2}})$.*

Let $\mathbf{u}_n$ be solutions of the following advection-diffusion problems

$$(AD_n) \begin{cases} \partial_t \mathbf{u}_n = \Delta \mathbf{u}_n - (\mathbf{b}_n \cdot \nabla)\mathbf{u}_n - \nabla p_n + \nabla \cdot \mathbb{F}_n \\ \\ \nabla \cdot \mathbf{u}_n = 0, \qquad \qquad \mathbf{u}_n(0,.) = \mathbf{u}_{0,n} \end{cases}$$

such that :

- *$\mathbf{u}_n$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla \mathbf{u}_n$ belongs to $L^2((0,T), L^2_{w_\gamma})$*
- *the pressure p_n is related to $\mathbf{u}_n$, $\mathbf{b}_n$ and $\mathbb{F}_n$ by the formula*

$$p_n = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (b_{n,i} u_{n,j} - F_{n,i,j})$$

- *the map $t \in [0,T) \mapsto \mathbf{u}_n(t,.)$ is weakly continuous from $[0,T)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$:*

$$\lim_{t \to 0} \|\mathbf{u}_n(t,.) - \mathbf{u}_{0,n}\|_{L^2_{w_\gamma}} = 0.$$

- *there exists a non-negative locally finite measure μ_n on $(0,T) \times \mathbb{R}^3$ such that*

$$\partial_t(\frac{|\mathbf{u}_n|^2}{2}) = \Delta(\frac{|\mathbf{u}_n|^2}{2}) - |\nabla \mathbf{u}_n|^2 - \nabla \cdot \left(\frac{|\mathbf{u}_n|^2}{2}\mathbf{b}_n\right) - \nabla \cdot (p_n \mathbf{u}_n) + \mathbf{u}_n \cdot (\nabla \cdot \mathbb{F}_n) - \mu_n.$$

If $\mathbf{u}_{0,n}$ is strongly convergent to $\mathbf{u}_{0,\infty}$ in $L^2_{w_\gamma}$, if the sequence $\mathbb{F}_n$ is strongly convergent to $\mathbb{F}_\infty$ in $L^2((0,T), L^2_{w_\gamma})$, and if the sequence $\mathbf{b}_n$ is bounded in $L^3((0,T), L^3_{w_{3\gamma/2}})$, then there exists p_∞, $\mathbf{u}_\infty$, $\mathbf{b}_\infty$ and an increasing sequence $(n_k)_{k \in \mathbb{N}}$ with values in $\mathbb{N}$ such that

- *$\mathbf{u}_{n_k}$ converges *-weakly to $\mathbf{u}_\infty$ in $L^\infty((0,T), L^2_{w_\gamma})$, $\nabla \mathbf{u}_{n_k}$ converges weakly to $\nabla \mathbf{u}_\infty$ in $L^2((0,T), L^2_{w_\gamma})$*
- *$\mathbf{b}_{n_k}$ converges weakly to $\mathbf{b}_\infty$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$, p_{n_k} converges weakly to p_∞ in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T), L^2_{w_\gamma})$*

- $\mathbf{u}_{n_k}$ *converges strongly to* $\mathbf{u}_\infty$ *in* $L^2_{\rm loc}([0,T)\times\mathbb{R}^3)$ *: for every* $T_0\in(0,T)$ *and every* $R>0$*, we have*

$$\lim_{k\to+\infty}\int_0^{T_0}\int_{|y|<R}|\mathbf{u}_{n_k}(s,y)-\mathbf{u}_\infty(s,y)|^2\,ds\,dy=0.$$

Moreover, $\mathbf{u}_\infty$ *is a solution of the advection-diffusion problem*

$$(AD_\infty)\begin{cases}\partial_t\mathbf{u}_\infty=\Delta\mathbf{u}_\infty-(\mathbf{b}_\infty\cdot\nabla)\mathbf{u}_\infty-\nabla p_\infty+\nabla\cdot\mathbb{F}_\infty\\[2ex] \nabla\cdot\mathbf{u}_\infty=0, \qquad\qquad \mathbf{u}_\infty(0,.)=\mathbf{u}_{0,\infty}\end{cases}$$

and is such that :

- *the map* $t\in[0,T)\mapsto\mathbf{u}_\infty(t,.)$ *is weakly continuous from* $[0,T)$ *to* $L^2_{w_\gamma}$*, and is strongly continuous at* $t=0$ *:*

$$\lim_{t\to0}\|\mathbf{u}_\infty(t,.)-\mathbf{u}_{0,\infty}\|_{L^2_{w_\gamma}}=0.$$

- *there exists a non-negative locally finite measure* μ_∞ *on* $(0,T)\times\mathbb{R}^3$ *such that*

$$\partial_t(\frac{|\mathbf{u}_\infty|^2}{2})=\Delta(\frac{|\mathbf{u}_\infty|^2}{2})-|\nabla\mathbf{u}_\infty|^2-\nabla\cdot\left(\frac{|\mathbf{u}_\infty|^2}{2}\mathbf{b}_\infty\right)-\nabla\cdot(p_\infty\mathbf{u}_\infty)+\mathbf{u}_\infty\cdot(\nabla\cdot\mathbb{F}_\infty)-\mu_\infty.$$

Notations.

All along the text, C_γ is a positive constant whose value may change from line to line but which depends only on γ.

2 The weights w_δ.

We consider the weights $w_\delta=\frac{1}{(1+|x|)^\delta}$ where $0<\delta$ and $x\in\mathbb{R}^3$. A very important feature of those weights is the control of their gradients :

$$|\nabla w_\delta(x)|=\delta\frac{w_\delta(x)}{1+|x|} \tag{2}$$

Lemma 1 (Muckenhoupt weights) *If* $0<\delta<3$ *and* $1<p<+\infty$*, then* w_δ *belongs to the Muckenhoupt class* $\mathcal{A}_p$*.*

Proof : We recall that a weight w belongs to $\mathcal{A}_p(\mathbb{R}^3)$ for $1 < p < +\infty$ if and only if it satisfies the reverse Hölder inequality

$$\sup_{x\in\mathbb{R}^3, R>0}\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} w(y)\,dy\right)^{\frac{1}{p}}\left(\frac{1}{|B(x,R)|}\int_{B(x,R)}\frac{dy}{w(y)^{\frac{1}{p-1}}}\right)^{1-\frac{1}{p}} < +\infty. \tag{3}$$

For all $0 < R \leq 1$ the inequality $|x-y| < R$ implies $\frac{1}{2}(1+|x|) \leq 1+|y| \leq 2(1+|x|)$, thus we can control the left side in (3) for w_δ by $4^{\frac{\delta}{p}}$.

For all $R > 1$ and $|x| > 10R$, we have that the inequality $|x-y| < R$ implies $\frac{9}{10}(1+|x|) \leq 1+|y| \leq \frac{11}{10}(1+|x|)$, thus we can control the left side in (3) for w_δ by $(\frac{11}{9})^{\frac{\delta}{p}}$.

Finally, for $R > 1$ and $|x| \leq 10R$, we write

$$\begin{aligned}
&\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} w(y)\,dy\right)^{\frac{1}{p}}\left(\frac{1}{|B(x,R)|}\int_{B(0,R)}\frac{dy}{w(y)^{\frac{1}{p-1}}}\right)^{1-\frac{1}{p}}\\
\leq&\left(\frac{1}{|B(0,R)|}\int_{B(x,11\,R)} w(y)\,dy\right)^{\frac{1}{p}}\left(\frac{1}{|B(0,R)|}\int_{B(0,11\,R)}\frac{dy}{w(y)^{\frac{1}{p-1}}}\right)^{1-\frac{1}{p}}\\
=&\left(\frac{1}{R^3}\int_0^{11\,R} r^2\frac{dr}{(1+r)^\delta}\right)^{\frac{1}{p}}\left(\frac{1}{R^3}\int_0^{11\,R} r^2(1+r)^{\frac{\delta}{p-1}}\,dr\right)^{1-\frac{1}{p}}\\
\leq& c_{\delta,p}\left(\frac{1}{R^3}\int_0^{11\,R} r^2\frac{dr}{r^\delta}\right)^{\frac{1}{p}}\left(\left(\frac{1}{R^3}\int_0^{11\,R} r^2 dr\right)^{1-\frac{1}{p}}+\left(\frac{1}{R^3}\int_0^{11\,R} r^{2+\frac{\delta}{p-1}}\,dr\right)^{1-\frac{1}{p}}\right)\\
=& c_{\delta,p}\frac{11^3}{(3-\delta)^{\frac{1}{p}}}\left(\frac{(11R)^{-\frac{\delta}{p}}}{3^{1-\frac{1}{p}}}+\frac{1}{(3+\frac{\delta}{p-1})^{1-\frac{1}{p}}}\right).
\end{aligned}$$

The lemma is proved. ◇

Lemma 2 *If $0 < \delta < 3$ and $1 < p < +\infty$, then the Riesz transforms R_i and the Hardy–Littlewood maximal function operator are bounded on $L^p_{w_\delta} = L^p(w_\delta(x)\,dx)$:*

$$\|R_j f\|_{L^p_{w_\delta}} \leq C_{p,\delta}\|f\|_{L^p_{w_\delta}} \text{ and } \|\mathcal{M}_f\|_{L^p_{w_\delta}} \leq C_{p,\delta}\|f\|_{L^p_{w_\delta}}.$$

Proof : The boundedness of the Riesz transforms or of the Hardy–Littlewwod maximal function on $L^p(w_\gamma\,dx)$ are basic properties of the Muckenhoupt class $\mathcal{A}_p$ [5]. ◇

We will use strategically the next corollary, which is specially useful to obtain discretely self-similar solutions.

Corollary 1 (Non-increasing kernels) *Let $\theta \in L^1(\mathbb{R}^3)$ be a non-negative radial function which is radially non-increasing. Then, if $0 < \delta < 3$ and $1 < p < +\infty$, we have, for $f \in L^p_{w_\delta}$, the inequality*

$$\|\theta * f\|_{L^p_{w_\delta}} \leq C_{p,\delta} \|f\|_{L^p_{w_\delta}} \|\theta\|_1.$$

Proof : We have the well-known inequality for radial non-increasing kernels [4]

$$|\theta * f(x)| \leq \|\theta\|_1 \mathcal{M}_f(x)$$

so that we may conclude with Lemma 2. ◇

We illustrate the utility of Lemma 2 with the following corollaries:

Corollary 2 *Let $0 < \gamma < \frac{5}{2}$ and $0 < T < +\infty$. Let $\mathbb{F}$ be a tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1 \leq i,j \leq 3}$ such that $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$. Let $\mathbf{b}$ be a time-dependent divergence free vector-field ($\nabla \cdot \mathbf{b} = 0$) such that $\mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}})$.*

Let $\mathbf{u}$ be a solution of the following advection-diffusion problem

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{b} \cdot \nabla)\mathbf{u} - \nabla q + \nabla \cdot \mathbb{F} \\ \\ \qquad \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{4}$$

be such that : $\mathbf{u}$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla \mathbf{u}$ belongs to $L^2((0,T), L^2_{w_\gamma})$, and the pressure q belongs to $\mathcal{D}'((0,T) \times \mathbb{R}^3)$.

Then, the gradient of the pressure ∇q is necessarily related to $\mathbf{u}$, $\mathbf{b}$ and $\mathbb{F}$ through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula

$$\nabla q = \nabla \left(\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (b_i u_j - F_{i,j}) \right)$$

and $\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (b_i u_j)$ belongs to $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j F_{i,j}$ belongs to $L^2((0,T), L^2_{w_\gamma})$.

Proof : We define

$$p = \left(\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (b_i u_j - F_{i,j}) \right).$$

As $0 < \gamma < \frac{5}{2}$ we can use Lemma 2 and (2) to obtain $\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (b_i u_j)$ belongs to $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j F_{i,j}$ belongs to $L^2((0,T), L^2_{w_\gamma})$.

Taking the divergence in (4), we obtain $\Delta(q-p)=0$. We take a test function $\alpha \in \mathcal{D}(\mathbb{R})$ such that $\alpha(t)=0$ for all $|t|\geq \varepsilon$, and a test function $\beta \in \mathcal{D}(\mathbb{R}^3)$; then the distribution $\nabla q*(\alpha\otimes\beta)$ is well defined on $(\varepsilon, T-\varepsilon)\times\mathbb{R}^3$.

We fix $t\in(\varepsilon, T-\varepsilon)$ and define

$$A_{\alpha,\beta,t}=(\nabla q*(\alpha\otimes\beta)-\nabla p*(\alpha\otimes\beta))(t,.).$$

We have

$$\begin{aligned} A_{\alpha,\beta,t} =&(\mathbf{u}*(-\partial_t\alpha\otimes\beta+\alpha\otimes\Delta\beta)+(-\mathbf{u}\otimes\mathbf{b}+\mathbb{F})\cdot(\alpha\otimes\nabla\beta))(t,.) \\ &-(p*(\alpha\otimes\nabla\beta))(t,.). \end{aligned} \tag{5}$$

Convolution with a function in $\mathcal{D}(\mathbb{R}^3)$ is a bounded operator on $L^2_{w_\gamma}$ and on $L^{6/5}_{w_{6\gamma/5}}$ (as, for $\varphi\in\mathcal{D}(\mathbb{R}^3)$ we have $|f*\varphi|\leq C_\varphi\mathcal{M}_f$). Thus, we may conclude from (5) that $A_{\alpha,\beta,t}\in L^2_{w_\gamma}+L^{6/5}_{w_{6\gamma/5}}$. If $\max\{\gamma,\frac{\gamma+2}{2}\}<\delta<5/2$, we have $A_{\alpha,\beta,t}\in L^{6/5}_{w_{6\delta/5}}$.

In particular, $A_{\alpha,\beta,t}$ is a tempered distribution. As we have

$$\Delta A_{\alpha,\beta,t}=(\alpha\otimes\beta)*(\Delta(q-p))(t,.)=0,$$

we find that $A_{\alpha,\beta,t}$ is a polynomial. We remark that for all $1<r<+\infty$ and $0<\delta<3$, $L^r_{w_\delta}$ does not contain non-trivial polynomials. Thus, $A_{\alpha,\beta,t}=0$. We then use an approximation of identity $\frac{1}{\epsilon^4}\alpha(\frac{t}{\epsilon})\beta(\frac{x}{\epsilon})$ and conclude that $\nabla(q-p)=0$. ◇

Actually, we can answer a question posed by Bradshaw and Tsai in [2] about the nature of the pressure for self-similar solutions of the Navier–Stokes equations. In effect, we have the next corollary:

Corollary 3 *Let $1<\gamma<\frac{5}{2}$ and $0<T<+\infty$. Let $\mathbb{F}$ be a tensor $\mathbb{F}(t,x)=(F_{i,j}(t,x))_{1\leq i,j\leq 3}$ such that $\mathbb{F}\in L^2((0,T),L^2_{w_\gamma})$.*

Let $\mathbf{u}$ be a solution of the following problem

$$\begin{cases}\partial_t\mathbf{u}=\Delta\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{u}-\nabla p+\nabla\cdot\mathbb{F} \\ \qquad\quad \nabla\cdot\mathbf{u}=0,\end{cases}$$

be such that : $\mathbf{u}$ belongs to $L^\infty([0,+\infty),L^2)_{loc}$ and $\nabla\mathbf{u}$ belongs to $L^2([0,+\infty),L^2)_{loc}$, and the pressure q is in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$.

We suppose that there exists $\lambda > 1$ such that $\lambda^2 \mathbb{F}(\lambda^2 t, \lambda x) = \mathbb{F}(t,x)$ and $\lambda \mathbf{u}(\lambda^2 t, \lambda x) = \mathbf{u}(t,x)$. Then, the gradient of the pressure ∇q is necessarily related to $\mathbf{u}$ and $\mathbb{F}$ through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula

$$\nabla q = \nabla \left(\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (u_i u_j - F_{i,j}) \right)$$

and $\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (u_i u_j)$ belongs to $L^4((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j F_{i,j}$ belongs to $L^2((0,T), L^2_{w_\gamma})$.

Proof : We shall use Corollary 2, and thus we need to show that $\mathbf{u}$ belongs to $L^\infty((0,T), L^2_{w_\gamma} \cap L^3((0,T), L^3_{3\gamma/2}))$ and $\nabla \mathbf{u}$ belongs to $L^2((0,T), L^2_{w_\gamma})$. In fact,

$$\|u\|_{L^\infty((0,T), L^2_{w_\gamma})} \leq \sup_{0 \leq t \leq T} \int_{|x|<1} |\mathbf{u}(t,x)|^2 \, dx + c \sup_{0 \leq t \leq T} \sum_{k \in \mathbb{N}} \int_{\lambda^{k-1} < |x| < \lambda^k} \frac{|\mathbf{u}(t,x)|^2}{\lambda^{\gamma k}} \, dx$$

and

$$\begin{aligned} \sup_{0 \leq t \leq T} \sum_{k \geq 1} \int_{\lambda^{k-1} < |x| < \lambda^k} \frac{|\mathbf{u}(t,x)|^2}{\lambda^{\gamma k}} \, dx &\leq \sup_{0 \leq t \leq T} \sum_{k \in \mathbb{N}} \lambda^{(1-\gamma)k} \int_{\lambda^{-1} < |x| < 1} |\mathbf{u}(\frac{t}{\lambda^{2k}}, x)|^2 \, dx \\ &\leq c \sup_{0 \leq t \leq T} \int_{\lambda^{-1} < |x| < 1} |\mathbf{u}(t,x)|^2 \, dx < +\infty. \end{aligned}$$

For $\nabla \mathbf{u}$, we compute for $k \in \mathbb{N}$,

$$\int_0^T \int_{\lambda^{k-1} < |x| < \lambda^k} |\nabla \mathbf{u}(t,x)|^2 \, dt \, dx = \lambda^k \int_0^{\frac{T}{\lambda^{2k}}} \int_{\frac{1}{\lambda} < |x| < 1} |\nabla \mathbf{u}(t,x)|^2 \, dx \, dt.$$

We may conclude that $\nabla \mathbf{u}$ belongs to $L^2((0,T), L^2_{w_\gamma})$, since for $\gamma > 1$ we have $\sum_{k \in \mathbb{N}} \lambda^{(1-\gamma)k} < +\infty$.

Now, we use the Sobolev embeddings described in next Lemma (Lemma 3) to get that $\mathbf{u}$ belongs to $L^2((0,T), L^6_{w_{3\gamma}})$, and thus (by interpolation with $L^\infty((0,T), L^2_{w_\gamma})$) to $L^4((0,T), L^3_{w_{3\gamma/2}})$.
In particular, $\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (u_i u_j)$ belongs to $L^4((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$, since we have

$$\|(\mathbf{u} \otimes \mathbf{u}) w_\gamma\|_{L^{6/5}} \leq \|\sqrt{w_\gamma} \mathbf{u}\|_{L^2} \|\sqrt{w_\gamma} \mathbf{u}\|_{L^3} \leq \|\sqrt{w_\gamma} \mathbf{u}\|_{L^2}^{\frac{3}{2}} \|\sqrt{w_\gamma} \mathbf{u}\|_{L^6}^{\frac{1}{2}}.$$

◇

Lemma 3 (Sobolev embeddings) *Let $\delta > 0$. If $f \in L^2_{w_\delta}$ and $\nabla f \in L^2_{w_\delta}$ then $f \in L^6_{w_{3\delta}}$ and*

$$\|f\|_{L^6_{w_{3\delta}}} \leq C_\delta(\|f\|_{L^2_{w_\delta}} + \|\nabla f\|_{L^2_{w_\delta}}).$$

Proof : Since both f and $w_{\delta/2}$ are locally in H^1, we write

$$\partial_i(f w_{\delta/2}) = w_{\delta/2}\partial_i f + f\partial_i(w_{\delta/2}) = w_{\delta/2}\partial_i f - \frac{\delta}{2}\frac{x_i}{|x|}w_{\delta/2}f$$

and thus

$$\|w_{\delta/2}f\|_2^2 + \|\nabla(w_{\delta/2}f)\|_2^2 \leq (1 + \frac{\delta^2}{2})\|w_{\delta/2}f\|_2^2 + 2\|w_{\delta/2}\nabla f\|_2^2.$$

Thus, $w_{\delta/2}f$ belongs to L^6 (since $H^1 \subset L^6$), or equivalently $f \in L^6_{w_{3\delta}}$. ⋄

3 A priori estimates for the advection-diffusion problem.

3.1 Proof of Theorem 2.

Let $0 < t_0 < t_1 < T$. We take a function $\alpha \in \mathcal{C}^\infty(\mathbb{R})$ which is non-decreasing, with $\alpha(t)$ equal to 0 for $t < 1/2$ and equal to 1 for $t > 1$. For $0 < \eta < \min(\frac{t_0}{2}, T - t_1)$, we define

$$\alpha_{\eta,t_0,t_1}(t) = \alpha(\frac{t - t_0}{\eta}) - \alpha(\frac{t - t_1}{\eta}).$$

We take as well a non-negative function $\phi \in \mathcal{D}(\mathbb{R}^3)$ which is equal to 1 for $|x| \leq 1$ and to 0 for $|x| \geq 2$. For $R > 0$, we define $\phi_R(x) = \phi(\frac{x}{R})$. Finally, we define, for $\epsilon > 0$, $w_{\gamma,\epsilon} = \frac{1}{(1+\sqrt{\epsilon^2+|x|^2})^\delta}$. We have $\alpha_{\eta,t_0,t_1}(t)\phi_R(x)w_{\gamma,\epsilon}(x) \in \mathcal{D}((0,T) \times \mathbb{R}^3)$ and $\alpha_{\eta,t_0,t_1}(t)\phi_R(x)w_{\gamma,\epsilon}(x) \geq 0$. Thus, using the local energy balance (1) and the fact that $\mu \geq 0$, we find

$$\begin{aligned}
&-\iint \frac{|\mathbf{u}|^2}{2} \partial_t \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\epsilon} \, dx \, ds \\
&\qquad \leq -\sum_{i=1}^3 \iint \partial_i \mathbf{u} \cdot \mathbf{u} \, \alpha_{\eta,t_0,t_1} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon}) \, dx \, ds \\
&\qquad\quad - \iint |\nabla \mathbf{u}|^2 \ \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\epsilon} dx \, ds \\
&\qquad\quad + \sum_{i=1}^3 \iint \frac{|\mathbf{u}|^2}{2} b_i \alpha_{\eta,t_0,t_1} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon}) \, dx \, ds \\
&\qquad\quad + \sum_{i=1}^3 \iint \alpha_{\eta,t_0,t_1} p u_i (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon}) \, dx \, ds \\
&\qquad\quad - \sum_{i=1}^3 \sum_{j=1}^3 \iint F_{i,j} u_j \alpha_{\eta,t_0,t_1} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon}) \, dx \, ds \\
&\qquad\quad - \sum_{i=1}^3 \sum_{j=1}^3 \iint F_{i,j} \partial_i u_j \ \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\epsilon} \, dx \, ds.
\end{aligned}$$

We remark that, independently from $R > 1$ and $\epsilon > 0$, we have (for $0 < \gamma \leq 2$)

$$|w_{\gamma,\epsilon} \partial_i \phi_R| + |\phi_R \partial_i w_{\gamma,\epsilon}| \leq C_\gamma \frac{w_\gamma(x)}{1 + |x|} \leq C_\gamma w_{3\gamma/2}(x).$$

Moreover, we know that $\mathbf{u}$ belongs to $L^\infty((0,T), L^2_{w_\gamma}) \cap L^2((0,T), L^6_{w_{3\gamma}})$ hence to $L^4((0,T), L^3_{w_{3\gamma/2}})$. Since $T < +\infty$, we have as well $\mathbf{u} \in L^3((0,T), L^3_{w_{3\gamma/2}})$. (This is the same type of integrability as required for $\mathbf{b}$). Moreover, we have $p u_i \in L^1_{w_{3\gamma/2}}$ since $w_\gamma p \in L^2((0,T), L^{6/5} + L^2)$ and $w_{\gamma/2} \mathbf{u} \in L^2((0,T), L^2 \cap L^6)$. All those remarks will allow us to use dominated convergence.

We first let η go to 0. We find that

$$
\begin{aligned}
-\lim_{\eta\to 0}\iint \frac{|\mathbf{u}|^2}{2}\partial_t\alpha_{\eta,t_0,t_1}\phi_R w_{\gamma,\epsilon}\,dx\,ds&\\
\leq -\sum_{i=1}^3\int_{t_0}^{t_1}\int \partial_i\mathbf{u}\cdot\mathbf{u}\,(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds&\\
-\int_{t_0}^{t_1}\int |\nabla\mathbf{u}|^2\ \phi_R w_{\gamma,\epsilon}dx\,ds&\\
+\sum_{i=1}^3\int_{t_0}^{t_1}\int \frac{|\mathbf{u}|^2}{2}b_i(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds&\\
+\sum_{i=1}^3\int_{t_0}^{t_1}\int pu_i(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds&\\
-\sum_{i=1}^3\sum_{j=1}^3\int_{t_0}^{t_1}\int F_{i,j}u_j(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds&\\
-\sum_{i=1}^3\sum_{j=1}^3\int_{t_0}^{t_1}\int F_{i,j}\partial_i u_j\ \phi_R w_{\gamma,\epsilon}\,dx\,ds.&
\end{aligned}
$$

Let us define

$$A_{R,\epsilon}(t)=\int |\mathbf{u}(t,x)|^2\phi_R(x)w_{\gamma,\epsilon}(x)\,dx.$$

As we have

$$-\iint \frac{|\mathbf{u}|^2}{2}\partial_t\alpha_{\eta,t_0,t_1}\phi_R w_{\gamma,\epsilon}\,dx\,ds=-\frac{1}{2}\int \partial_t\alpha_{\eta,t_0,t_1}A_{R,\epsilon}(s)\,ds$$

we find that, when t_0 and t_1 are Lebesgue points of the measurable function $A_{R,\epsilon}$

$$\lim_{\eta\to 0}-\iint \frac{|\mathbf{u}|^2}{2}\partial_t\alpha_{\eta,t_0,t_1}\phi_R w_{\gamma,\epsilon}\,dx\,ds=\frac{1}{2}(A_{R,\epsilon}(t_1)-A_{R,\epsilon}(t_0)).$$

Then, by continuity, we can let t_0 go to 0 and thus replace t_0 by 0 in the inequality. Moreover, if we let t_1 go to t, then by weak continuity, we find that $A_{R,\epsilon}(t)\leq \lim_{t_1\to t}A_{R,\epsilon}(t_1)$, so that we may as well replace t_1 by $t\in(0,T)$. Thus we find that for every $t\in(0,T)$, we have

$$\begin{aligned}\int \frac{|\mathbf{u}(t,x)|^2}{2}\phi_R w_{\gamma,\epsilon}\,dx& \\ \leq \int \frac{|\mathbf{u}_0(x)|^2}{2}&\phi_R w_{\gamma,\epsilon}\,dx \\ &- \sum_{i=1}^3 \int_0^t \int \partial_i \mathbf{u}\cdot\mathbf{u}\,(w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds \\ &- \int_0^t \int |\nabla \mathbf{u}|^2\ \phi_R w_{\gamma,\epsilon} dx\,ds \\ &+ \sum_{i=1}^3 \int_0^t \int \frac{|\mathbf{u}|^2}{2} b_i(w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds \\ &+ \sum_{i=1}^3 \int_0^t \int p u_i(w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds \\ &- \sum_{i=1}^3\sum_{j=1}^3 \int_0^t \int F_{i,j} u_j(w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds \\ &- \sum_{i=1}^3\sum_{j=1}^3 \int_0^t \int F_{i,j}\partial_i u_j\ \phi_R w_{\gamma,\epsilon}\,dx\,ds.\end{aligned} \tag{6}$$

Thus, letting R go to $+\infty$ and then ϵ go to 0, we find by dominated convergence that, for every $t \in (0,T)$, we have

$$\begin{aligned}\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}} &+ 2\int_0^t \|\nabla\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}}\,ds \\ \leq& \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} - \int_0^t \int \nabla|\mathbf{u}|^2\cdot\nabla w_\gamma\,dx\,ds + \int_0^t \int (|\mathbf{u}|^2\mathbf{b} + 2p\mathbf{u})\cdot\nabla(w_\gamma)\,dx\,ds \\ &- 2\sum_{i=1}^3\sum_{j=1}^3 \int_0^t \int F_{i,j}(\partial_i u_j)w_\gamma + F_{i,j}u_i\partial_j(w_\gamma)\,dx\,ds.\end{aligned}$$

Now we write

$$\begin{aligned}\left|\int_0^t \int \nabla|\mathbf{u}|^2\cdot\nabla w_\gamma\,ds\,ds\right| \leq& 2\gamma\int_0^t \int |\mathbf{u}||\nabla\mathbf{u}|\,w_\gamma\,dx\,ds \\ \leq& \frac{1}{4}\int_0^t \|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\,ds + 4\gamma^2\int_0^t \|\mathbf{u}\|^2_{L^2_{w_\gamma}}\,ds.\end{aligned}$$

Writing

$$p_1 = \sum_{i=1}^{3}\sum_{j=1}^{3} R_iR_j(b_iu_j) \text{ and } p_2 = -\sum_{i=1}^{3}\sum_{j=1}^{3} R_iR_j(F_{i,j})$$

and using the fact that $w_{6\gamma/5} \in \mathcal{A}_{6/5}$ and $w_\gamma \in \mathcal{A}_2$, we get

$$\begin{aligned}
\left|\int_0^t \int (|\mathbf{u}|^2\mathbf{b} + 2p_1\mathbf{u}) \cdot \nabla(w_\gamma)\,dx\,ds\right| \leq & \gamma \int_0^t \int (|\mathbf{u}|^2|\mathbf{b}| + 2|p_1|\,|\mathbf{u}|)\, w_\gamma^{3/2}\,dx\,ds \\
\leq & \gamma \int_0^t \|w_\gamma^{1/2}\mathbf{u}\|_6(\|w_\gamma|\mathbf{b}||\mathbf{u}|\|_{6/5} + \|w_\gamma p_1\|_{6/5})ds \\
\leq & C_\gamma \int_0^t \|w_\gamma^{1/2}\mathbf{u}\|_6\|w_\gamma|\mathbf{b}||\mathbf{u}|\|_{6/5}\,ds \\
\leq & C_\gamma \int_0^t \|w_\gamma^{1/2}\mathbf{u}\|_6\|w_\gamma^{1/2}\mathbf{b}\|_3\|w_\gamma^{1/2}\mathbf{u}\|_2\,ds \\
\leq & C'_\gamma \int_0^t (\|\nabla\mathbf{u}\|_{L^2_{w_\gamma}} + \|\mathbf{u}\|_{L^2_{w_\gamma}})\ \|\mathbf{b}\|_{L^3_{w_{3\gamma/2}}}\|\mathbf{u}\|_{L^2_{w_\gamma}}\,ds \\
\leq & \frac{1}{4}\int_0^t \|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\,ds + C''_\gamma \int_0^t \|\mathbf{u}\|^2_{L^2_{w_\gamma}}(\|\mathbf{b}\|_{L^3_{w_{3\gamma/2}}} + \|\mathbf{b}\|^2_{L^3_{w_{3\gamma/2}}})\,ds
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_0^t \int 2p_2\mathbf{u} \cdot \nabla(w_\gamma)\,dx\,ds\right| \leq & 2\gamma \int_0^t \int |p_2|\,|\mathbf{u}|\,w_\gamma\,dx\,ds \\
\leq & \gamma \int_0^t \|\mathbf{u}\|^2_{L^2_{w_\gamma}} + \|p_2\|^2_{L^2_{w_\gamma}}\,ds \\
\leq & C_\gamma \int_0^t \|\mathbf{u}\|^2_{L^2_{w_\gamma}} + \|\mathbb{F}\|^2_{L^2_{w_\gamma}}\,ds.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\left|2\sum_{i=1}^{3}\sum_{j=1}^{3}\int_0^t \int F_{i,j}(\partial_i u_j)w_\gamma + F_{i,j}u_i\partial_j(w_\gamma)\,dx\,ds\right| \leq & 2\int_0^t \int |F|\,(|\nabla\mathbf{u}| + \gamma|\mathbf{u}|)\,w_\gamma\,dx\,ds \\
\leq & \frac{1}{4}\int_0^t \|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\,ds + C_\gamma \int_0^t \|\mathbf{u}\|^2_{L^2_{w_\gamma}} + \|\mathbb{F}\|^2_{L^2_{w_\gamma}}\,ds.
\end{aligned}$$

We have obtained

$$\begin{aligned}
&\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}} + \int_0^t \|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\,ds \\
&\leq \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + C_\gamma \int_0^t \|\mathbb{F}(s,.)\|^2_{L^2_{w_\gamma}}\,ds + C_\gamma \int_0^t (1 + \|\mathbf{b}(s,.)\|^2_{L^3_{w_{3\gamma/2}}})\|\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}}\,ds
\end{aligned} \tag{7}$$

and Theorem 2 is proven. $\diamond$

3.2 Passive transportation.

From inequality (7), we have the following direct consequence :

Corollary 4 *Under the assumptions of Theorem 2, we have*

$$\sup_{0<t<T} \|\mathbf{u}\|_{L^2_{w_\gamma}} \leq (\|\mathbf{u}_0\|_{L^2_{w_\gamma}} + C_\gamma \|\mathbb{F}\|_{L^2((0,T),L^2_{w_\gamma})})\ e^{C_\gamma (T + T^{1/3} \|\mathbf{b}\|^2_{L^3((0,T),L^3_{w_{3\gamma/2}})})}$$

and

$$\|\nabla \mathbf{u}\|_{L^2((0,T),L^2_{w_\gamma})} \leq (\|\mathbf{u}_0\|_{L^2_{w_\gamma}} + C_\gamma \|\mathbb{F}\|_{L^2((0,T),L^2_{w_\gamma})})\ e^{C_\gamma (T + T^{1/3} \|\mathbf{b}\|^2_{L^3((0,T),L^3_{w_{3\gamma/2}})})}$$

where the constant C_γ depends only on γ.

Another direct consequence is the following uniqueness result for the advection-diffusion problem with a (locally in time), bounded $\mathbf{b}$:

Corollary 5 . *Let $0 < \gamma \leq 2$. Let $0 < T < +\infty$. Let $\mathbf{u}_0$ be a divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ and $\mathbb{F}$ be a tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1\leq i,j \leq 3}$ such that $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$. Let $\mathbf{b}$ be a time-dependent divergence free vector-field ($\nabla \cdot \mathbf{b} = 0$) such that $\mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}})$. Assume moreover that $\mathbf{b}$ belongs to $L^2_t L^\infty_x(K)$ for every compact subset K of $(0,T) \times \mathbb{R}^3$.*

Let $(\mathbf{u}_1, p_1)$ and $(\mathbf{u}_2, p_2)$ be two solutions of the following advection-diffusion problem

$$(AD) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{b} \cdot \nabla)\mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \\ \nabla \cdot \mathbf{u} = 0, \qquad\qquad \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

be such that, for $k = 1$ and $k = 2$, :

- *$\mathbf{u}_k$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla \mathbf{u}_k$ belongs to $L^2((0,T), L^2_{w_\gamma})$*
- *the pressure p_k is related to $\mathbf{u}_k$, $\mathbf{b}$ and $\mathbb{F}$ through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula*

$$p_k = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (b_i u_{k,j} - F_{i,j})$$

- *the map $t \in [0,T) \mapsto \mathbf{u}_k(t,.)$ is weakly continuous from $[0,T)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$:*

$$\lim_{t\to 0} \|\mathbf{u}_k(t,.) - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0.$$

Then $\mathbf{u}_1 = \mathbf{u}_2$.

Proof : Let $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ and $q = p_1 - p_2$. Then we have

$$\begin{cases} \partial_t \mathbf{v} = \Delta \mathbf{v} - (\mathbf{b} \cdot \nabla)\mathbf{v} - \nabla q \\ \\ \nabla \cdot \mathbf{v} = 0, \qquad\qquad \mathbf{v}(0,.) = 0 \end{cases}$$

Moreover on every compact subset K of $(0,T) \times \mathbb{R}^3$, $\mathbf{b} \otimes \mathbf{v}$ is in $L^2_t L^2_x$, while it belongs globally to $L^3_t L^{6/5}_{w_{6\gamma/5}}$. Writing, for $\varphi, \psi \in \mathcal{D}((0,T) \times \mathbb{R}^3)$ such that $\psi = 1$ on the neigborhood of the support of φ,

$$\varphi q = q_1 + q_2 = \varphi \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (\psi b_i v_j) + \varphi \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j ((1-\psi) b_i v_j)$$

we find that $\|q_1\|_{L^2 L^2} \leq C_{\varphi,\psi} \|\psi \mathbf{b} \otimes \mathbf{v}\|_{L^2 L^2}$ and

$$\|q_2\|_{L^3 L^\infty} \leq C_{\varphi,\psi} \|\mathbf{b} \otimes \mathbf{v}\|_{L^3 L^{6/5}_{w_{6\gamma/5}}}$$

with

$$C_{\varphi,\psi} \leq C \|\varphi\|_\infty \|1-\psi\|_\infty \sup_{x \in \mathrm{Supp}\, \varphi} \left(\int_{y \in \mathrm{Supp}\,(1-\psi)} \left(\frac{(1+|y|)^\gamma}{|x-y|^3} \right)^6 \right)^{1/6} < +\infty.$$

Thus, we may take the scalar product of $\partial_t \mathbf{v}$ with $\mathbf{v}$ and find that

$$\partial_t (\frac{|\mathbf{v}|^2}{2}) = \Delta (\frac{|\mathbf{v}|^2}{2}) - |\nabla \mathbf{v}|^2 - \nabla \cdot \left(\frac{|\mathbf{v}|^2}{2} \mathbf{b} \right) - \nabla \cdot (q \mathbf{v}).$$

Thus we are under the assumptions of Theorem 2 and we may use Corollary 4 to find that $\mathbf{v} = 0$. ◇

3.3 Active transportation.

We begin with the following lemma :

Lemma 4 *Let* α *be a non-negative bounded measurable function on* $[0,T)$ *such that, for two constants* $A, B \geq 0$, *we have*

$$\alpha(t) \leq A + B \int_0^t \alpha(s) + \alpha(s)^3 \, ds.$$

If $T_0 > 0$ *and* $T_1 = \min(T, T_0, \frac{1}{4B(A+BT_0)^2})$, *we have, for every* $t \in [0, T_1]$, $\alpha(t) \leq \sqrt{2}(A + BT_0)$.

Proof : We write $\alpha \leq 1+\alpha^3$. We define

$$\Phi(t) = A + BT_0 + B\int_0^t \alpha^3\,ds \text{ and } \Psi(t) = A + BT_0 + B\int_0^t \Phi^3(s)\,ds.$$

We have, for $t\in[0,T_1]$, $\alpha\leq\Phi\leq\Psi$. Since Ψ is $\mathcal{C}^1$, we may write

$$\Psi'(t) = B\Phi(t)^3 \leq B\Psi(t)^3$$

and thus

$$\frac{1}{\Psi(0)^2} - \frac{1}{\Psi(t)^2} \leq 2Bt.$$

We thus find

$$\Psi(t)^2 \leq \frac{\Psi(0)^2}{1-2B\Psi(0)^2 t} \leq 2\Psi(0)^2.$$

The lemma is proven. ◇

Corollary 6 *Assume that* $\mathbf{u}_0$*,* $\mathbf{u}$*,* p*,* $\mathbb{F}$ *and* $\mathbf{b}$ *satisfy assumptions of Theorem 2, Assume moreover that* $\mathbf{b}$ *is controlled by* $\mathbf{u}$ *: for every* $t\in(0,T)$*,*

$$\|\mathbf{b}(t,.)\|_{L^3_{w_{3\gamma/2}}} \leq C_0\|\mathbf{u}(t,.)\|_{L^3_{w_{3\gamma/2}}}.$$

Then there exists a constant $C_\gamma\geq 1$ *such that if* $T_0<T$ *is such that*

$$C_\gamma(1+C_0^4)\left(1+C_0^4+\|\mathbf{u}_0\|^2_{L^2_{w_\gamma}}+\int_0^{T_0}\|\mathbb{F}\|^2_{L^2_{w_\gamma}}\,ds\right)^2 T_0 \leq 1$$

then

$$\sup_{0\leq t\leq T_0}\|\,\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}} \leq C_\gamma(1+C_0^4+\|\mathbf{u}_0\|^2_{L^2_{w_\gamma}}+\int_0^{T_0}\|\mathbb{F}\|^2_{L^2_{w_\gamma}}\,ds)$$

and

$$\int_0^{T_0}\|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\,ds \leq C_\gamma(1+C_0^4+\|\mathbf{u}_0\|^2_{L^2_{w_\gamma}}+\int_0^{T_0}\|\mathbb{F}\|^2_{L^2_{w_\gamma}}\,ds).$$

Proof : We start from inequality (7) :

$$\|\mathbf{u}(t,.)\|^2_{L^2_{w_\gamma}}+\int_0^t\|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\,ds$$
$$\leq \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}}+C_\gamma\int_0^t\|\mathbb{F}(s,.)\|^2_{L^2_{w_\gamma}}\,ds+C_\gamma\int_0^t(1+\|\mathbf{b}(s,.)\|^2_{L^3_{w_{3\gamma/2}}})\|\mathbf{u}(s,.)\|^2_{L^2_{w_\gamma}}\,ds$$

We write

$$\|\mathbf{b}(s,.)\|_{L^3_{w_{3\gamma/2}}}^2 \leq C_0^2 \|\mathbf{u}(s,.)\|_{L^3_{w_{3\gamma/2}}}^2 \leq C_0^2 C_\gamma \|u\|_{L^2_{w_\gamma}} (\|u\|_{L^2_{w_\gamma}} + \|\nabla \mathbf{u}\|_{L^2_{w_\gamma}}).$$

This gives

$$\begin{aligned}
&\|\mathbf{u}(t,.)\|_{L^2_{w_\gamma}}^2 + \frac{1}{2}\int \|\nabla \mathbf{u}\|_{L^2_{w_\gamma}}^2 \, ds \\
&\quad \leq \|\mathbf{u}_0\|_{L^2_{w_\gamma}}^2 + C_\gamma \int_0^t \|\mathbb{F}(s,.)\|_{L^2_{w_\gamma}}^2 \, ds \\
&\qquad + C_\gamma \int_0^t \|\mathbf{u}(s,.)\|_{L^2_{w_\gamma}}^2 + C_0^2 \|\mathbf{u}(s,.)\|_{L^2_{w_\gamma}}^4 + C_0^4 \|\mathbf{u}(s,.)\|_{L^2_{w_\gamma}}^6 \, ds \\
&\quad \leq \|\mathbf{u}_0\|_{L^2_{w_\gamma}}^2 + C_\gamma \int_0^t \|\mathbb{F}(s,.)\|_{L^2_{w_\gamma}}^2 \, ds + 2C_\gamma \int_0^t \|\mathbf{u}(s,.)\|_{L^2_{w_\gamma}}^2 + C_0^4 \|\mathbf{u}(s,.)\|_{L^2_{w_\gamma}}^6 \, ds.
\end{aligned}$$

For $t \leq T_0$, we get

$$\begin{aligned}
&\|\mathbf{u}(t,.)\|_{L^2_{w_\gamma}}^2 + \frac{1}{2}\int \|\nabla \mathbf{u}\|_{L^2_{w_\gamma}}^2 \, ds \\
&\qquad \leq \|\mathbf{u}_0\|_{L^2_{w_\gamma}}^2 + C_\gamma \int_0^{T_0} \|\mathbb{F}\|_{L^2_{w_\gamma}}^2 \, ds + C_\gamma (1 + C_0^4) \int_0^t \|\mathbf{u}(t,.)\|_{L^2_{w_\gamma}}^2 + \|\mathbf{u}(t,.)\|_{L^2_{w_\gamma}}^6 \, ds
\end{aligned}$$

and we may conclude with Lemma 4. $\diamond$

4 Stability of solutions for the advection-diffusion problem.

4.1 The Rellich lemma.

We recall the Rellich lemma :

Lemma 5 (Rellich) *If $s > 0$ and (f_n) is a sequence of functions on $\mathbb{R}^d$ such that*

- *the family (f_n) is bounded in $H^s(\mathbb{R}^d)$*
- *there is a compact subset of $\mathbb{R}^d$ such that the support of each f_n is included in K*

then there exists a subsequence (f_{n_k}) such that f_{n_k} is strongly convergent in $L^2(\mathbb{R}^d)$.

We shall use a variant of this lemma (see [9]) :

Lemma 6 (space-time Rellich) *If $s > 0$, $\sigma \in \mathbb{R}$ and (f_n) is a sequence of functions on $(0,T) \times \mathbb{R}^d$ such that, for all $T_0 \in (0,T)$ and all $\varphi \in \mathcal{D}(\mathbb{R}^3)$*

- *φf_n is bounded in $L^2((0,T_0),H^s)$*
- *$\varphi \partial_t f_n$ is bounded in $L^2((0,T_0),H^\sigma)$*

then there exists a subsequence (f_{n_k}) such that f_{n_k} is strongly convergent in $L^2_{\rm loc}([0,T) \times \mathbb{R}^3)$: if f_∞ is the limit, we have for all $T_0 \in (0,T)$ and all $R_0 > 0$

$$\lim_{n_k \to +\infty} \int_0^{T_0} \int_{|x| \leq R} |f_{n_k} - f_\infty|^2 \, dx \, dt = 0.$$

Proof : With no loss of generality, we may assume that $\sigma < \min(1,s)$. Define g by $g_n(t,x) = \alpha(t)\varphi(x)f_n(t,x)$ if $t > 0$ and $g_n(t,x) = \alpha(t)\varphi(x)f_n(-t,x)$ if $t < 0$, where $\alpha \in \mathcal{C}^\infty$ on $(0,T)$, is equal to 1 on $[0,T_0]$ and equal to 0 for $t > \frac{T+T_0}{2}$, and $\varphi(x) = 1$ on $B(0,R_0)$. Then the support of g_n is contained in $[-\frac{T+T_0}{2}, \frac{T+T_0}{2}] \times \operatorname{Supp}\varphi$. Moreover, g_n is bounded in $L^2_t H^s$ and $\partial_t g_n$ is bounded in $L^2 H^\sigma$ so that g_n is bounded in $H^\rho(\mathbb{R} \times \mathbb{R}^3)$ with $\rho = \frac{s}{s+1-\sigma}$ (just write $(1+\tau^2+\xi^2)^{\frac{s}{s+1-\sigma}} \leq ((1+\tau^2)(1+\xi^2)^\sigma)^{\frac{s}{s+1-\sigma}} ((1+\xi^2)^s)^{\frac{1-\sigma}{s+1-\sigma}}$).. By the Rellich lemma, we know that there is a subsequence g_{n_k} which is strongly convergent in $L^2(\mathbb{R}\times\mathbb{R}^3)$, thus a subsequence f_{n_k} which is strongly convergent in $L^2((0,T_0) \times B(0,R_0))$.

We then iterate this argument for an increasing sequence of times $T_0 < T_1 < \dots < T_N \to T$ and an increasing sequence of radii $R_0 < R_1 < \dots < R_N \to +\infty$ and finish the proof. by the classical diagonal process of Cantor. $\diamond$

4.2 Proof of Theorem 3.

Assume that $\mathbf{u}_{0,n}$ is strongly convergent to $\mathbf{u}_{0,\infty}$ in $L^2_{w_\gamma}$ and that the sequence $\mathbb{F}_n$ is strongly convergent to $\mathbb{F}_\infty$ in $L^2((0,T),L^2_{w_\gamma})$, and assume that the sequence $\mathbf{b}_n$ is bounded in $L^3((0,T),L^3_{w_{3\gamma/2}})$. Then, by Theorem 2 and Corollary 4, we know that $\mathbf{u}_n$ is bounded in $L^\infty((0,T),L^2_{w_\gamma})$ and $\nabla \mathbf{u}_n$ is bounded in $L^2((0,T),L^2_{w_\gamma})$. In particular, writing $p_n = p_{n,1} + p_{n,2}$ with

$$p_{n,1} = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (b_{n,i} u_{n,j}) \text{ and } p_{n,2} = -\sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (F_{n,i,j})$$

we get that $p_{n,1}$ is bounded in $L^3((0,T),L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $p_{n,2}$ is bounded in $L^2((0,T),L^2_{w_\gamma})$.

If $\varphi\in\mathcal{D}(\mathbb{R}^3)$, we find that $\varphi\mathbf{u}_n$ is bounded in $L^2((0,T),H^1)$ and, writing

$$\partial_t\mathbf{u}_n=\Delta\mathbf{u}_n-\left(\sum_{i=1}^3\partial_i(b_{n,i}\mathbf{u}_n)+\nabla p_{n,1}\right)+(\nabla\cdot\mathbb{F}_n-\nabla p_{n,2}),$$

$\varphi\partial_t\mathbf{u}_n$ is bounded in $L^2L^2+L^2W^{-1,6/5}+L^2H^{-1}\subset L^2((0,T),H^{-2})$. Thus, by Lemma 6, there exists $\mathbf{u}_\infty$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that $\mathbf{u}_{n_k}$ converges strongly to $\mathbf{u}_\infty$ in $L^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^3)$: for every $T_0\in(0,T)$ and every $R>0$, we have

$$\lim_{k\to+\infty}\int_0^{T_0}\int_{|y|<R}|\mathbf{u}_{n_k}(s,y)-\mathbf{u}_\infty(s,y)|^2\,dy\,ds=0.$$

As $\mathbf{u}_n$ is bounded in $L^\infty((0,T),L^2_{w_\gamma})$ and $\nabla\mathbf{u}_n$ is bounded in $L^2((0,T),L^2_{w_\gamma})$, the convergence of $\mathbf{u}_{n_k}$ to $\mathbf{u}_\infty$ in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ implies that $\mathbf{u}_{n_k}$ converges *-weakly to $\mathbf{u}_\infty$ in $L^\infty((0,T),L^2_{w_\gamma})$ and $\nabla\mathbf{u}_{n_k}$ converges weakly to $\nabla\mathbf{u}_\infty$ in $L^2((0,T),L^2_{w_\gamma})$.

By Banach–Alaoglu's theorem, we may assume that there exists $\mathbf{b}_\infty$ such that $\mathbf{b}_{n_k}$ converges weakly to $\mathbf{b}_\infty$ in $L^3((0,T),L^3_{w_{3\gamma/2}})$. In particular $b_{n_k,i}u_{n_k,j}$ is weakly convergent in $(L^{6/5}L^{6/5})_{\mathrm{loc}}$ and thus in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$; as it is bounded in $L^3((0,T),L^{6/5}_{w_{\frac{6\gamma}{5}}})$, it is weakly convergent in $L^3((0,T),L^{6/5}_{w_{\frac{6\gamma}{5}}})$ to $b_{\infty,i}u_{\infty,j}$. Let

$$p_{\infty,1}=\sum_{i=1}^3\sum_{j=1}^3R_iR_j(b_{\infty,i}u_{\infty,j})\text{ and }p_{\infty,2}=-\sum_{i=1}^3\sum_{j=1}^3R_iR_j(F_{\infty,i,j}).$$

As the Riesz transforms are bounded on $L^{6/5}_{w_{\frac{6\gamma}{5}}}$ and on $L^2_{w_\gamma}$, we find that $p_{n_k,1}$ is weakly convergent in $L^3((0,T),L^{6/5}_{w_{\frac{6\gamma}{5}}})$ to $p_{\infty,1}$ and that $p_{n_k,2}$ is strongly convergent in $L^2((0,T),L^2_{w_\gamma})$ to $p_{\infty,2}$.

In particular, we find that in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$

$$\partial_t\mathbf{u}_\infty=\Delta\mathbf{u}_\infty-\sum_{i=1}^3\partial_i(b_{\infty,i}\mathbf{u}_\infty)-\nabla(p_{\infty,1}+p_{\infty,2})+\nabla\cdot\mathbb{F}_\infty.$$

In particular, $\partial_t\mathbf{u}_\infty$ is locally in L^2H^{-2}, and thus $\mathbf{u}_\infty$ has representative such that $t\mapsto\mathbf{u}_\infty(t,.)$ is continuous from $[0,T)$ to $\mathcal{D}'(\mathbb{R}^3)$ and coincides with $\mathbf{u}_\infty(0,.)+\int_0^t\partial_t\mathbf{u}_\infty\,ds$. In $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, we have that

$$\mathbf{u}_\infty(0,.)+\int_0^t\partial_t\mathbf{u}_\infty\,ds=\mathbf{u}_\infty=\lim_{n_k\to+\infty}\mathbf{u}_{n_k}=\lim_{n_k\to+\infty}\mathbf{u}_{0,n_k}+\int_0^t\partial_t\mathbf{u}_{n_k}\,ds=\mathbf{u}_{0,\infty}+\int_0^t\partial_t\mathbf{u}_\infty\,ds$$

Thus, $\mathbf{u}_\infty(0,.) = \mathbf{u}_{0,\infty}$, and $\mathbf{u}_\infty$ is a solution of (AD_∞).

Next, we define

$$A_n = -\partial_t(\frac{|\mathbf{u}_n|^2}{2}) + \Delta(\frac{|\mathbf{u}_n|^2}{2}) - \nabla\cdot\left(\frac{|\mathbf{u}_n|^2}{2}\mathbf{b}_n\right) - \nabla\cdot(p_n\mathbf{u}_n) + \mathbf{u}_n\cdot(\nabla\cdot\mathbb{F}_n) = |\nabla\mathbf{u}_n|^2 + \mu_n.$$

As $\mathbf{u}_n$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{u}_n$ is bounded in $L^2((0,T), L^2_{w_\gamma})$, it is bounded in $L^2((0,T), L^6_{w_{3\gamma/2}})$ and by interpolation with $L^\infty((0,T), L^2_{w_\gamma})$ it is bounded in $L^{10/3}((0,T), L^{10/3}_{w_{5\gamma/3}})$. Thus, u_{n_k} is locally bounded in $L^{10/3}L^{10/3}$ and locally strongly convergent in L^2L^2; it is then strongly convergent in L^3L^3. Thus, A_{n_k} is convergent in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ to

$$A_\infty = -\partial_t(\frac{|\mathbf{u}_\infty|^2}{2}) + \Delta(\frac{|\mathbf{u}_\infty|^2}{2}) - \nabla\cdot\left(\frac{|\mathbf{u}_\infty|^2}{2}\mathbf{b}_\infty\right) - \nabla\cdot(p_\infty\mathbf{u}_\infty) + \mathbf{u}_\infty\cdot(\nabla\cdot\mathbb{F}_\infty).$$

In particular, $A_\infty = \lim_{n_k\to+\infty}|\nabla\mathbf{u}_{n_k}|^2 + \mu_{n_k}$. If $\Phi\in\mathcal{D}((0,T)\times\mathbb{R}^3)$ is non-negative, we have

$$\iint A_\infty\Phi\,dx\,ds = \lim_{n_k\to+\infty}\iint A_{n_k}\Phi\,dx\,ds \geq \limsup_{n_k\to+\infty}\iint|\nabla\mathbf{u}_{n_k}|^2\Phi\,dx\,ds \geq \iint|\nabla\mathbf{u}_\infty|^2\Phi\,dx\,ds$$

(since $\sqrt{\Phi}\nabla\mathbf{u}_{n_k}$ is weakly convergent to $\sqrt{\Phi}\nabla\mathbf{u}_\infty$ in L^2L^2). Thus, there exists a non-negative locally finite measure μ_∞ on $(0,T)\times\mathbb{R}^3$ such that $A_\infty = |\nabla\mathbf{u}_\infty|^2 + \mu_\infty$, i.e. such that

$$\partial_t(\frac{|\mathbf{u}_\infty|^2}{2}) = \Delta(\frac{|\mathbf{u}_\infty|^2}{2}) - |\nabla\mathbf{u}_\infty|^2 - \nabla\cdot\left(\frac{|\mathbf{u}_\infty|^2}{2}\mathbf{b}_\infty\right) - \nabla\cdot(p_\infty\mathbf{u}_\infty) + \mathbf{u}\cdot(\nabla\cdot\mathbb{F}_\infty) - \mu_\infty.$$

Finally, we start from inequality (6) :

$$
\begin{aligned}
\int \frac{|\mathbf{u}_n(t,x)|^2}{2}\phi_R w_{\gamma,\epsilon}\,dx \leq & \int \frac{|\mathbf{u}_{0,n}(x)|^2}{2}\phi_R w_{\gamma,\epsilon}\,dx \\
& - \sum_{i=1}^3 \int_0^t \int \partial_i \mathbf{u}_n \cdot \mathbf{u}_n \,(w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\,dx\,ds \\
& - \int_0^t \int |\nabla \mathbf{u}_n|^2 \ \phi_R w_{\gamma,\epsilon} dx\,ds \\
& + \sum_{i=1}^3 \int_0^t \int \frac{|\mathbf{u}_n|^2}{2} b_{n,i}(w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\,dx\,ds \\
& + \sum_{i=1}^3 \int_0^t \int p_n u_{n,i}(w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\,dx\,ds \\
& - \sum_{i=1}^3 \sum_{j=1}^3 \int_0^t \int F_{n,i,j} u_{n,j}(w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\,dx\,ds \\
& - \sum_{i=1}^3 \sum_{j=1}^3 \int_0^t \int F_{n,i,j} \partial_i u_{n,} \ \phi_R w_{\gamma,\epsilon}\,dx\,ds.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\limsup_{n_k\to+\infty} \int \frac{|\mathbf{u}_{n_k}(t,x)|^2}{2}\phi_R w_{\gamma,\epsilon}\,dx + \int_0^t \int |\nabla \mathbf{u}_{n_k}|^2 \ \phi_R w_{\gamma,\epsilon} dx\,ds & \\
\leq \int \frac{|\mathbf{u}_{0,\infty}(x)|^2}{2}\phi_R w_{\gamma,\epsilon}\,dx & \\
- \sum_{i=1}^3 \int_0^t \int \partial_i \mathbf{u}_\infty \cdot \mathbf{u}_\infty \,(w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\,dx\,ds & \\
+ \sum_{i=1}^3 \int_0^t \int \frac{|\mathbf{u}_\infty|^2}{2} b_{\infty,i}(w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\,dx\,ds & \\
+ \sum_{i=1}^3 \int_0^t \int p_\infty u_{\infty,i}(w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\,dx\,ds & \\
- \sum_{i=1}^3 \sum_{j=1}^3 \int_0^t \int F_{\infty,i,j} u_{\infty,j}(w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\,dx\,ds & \\
- \sum_{i=1}^3 \sum_{j=1}^3 \int_0^t \int F_{\infty,i,j} \partial_i u_{\infty,j} \ \phi_R w_{\gamma,\epsilon}\,dx\,ds. &
\end{aligned}
$$

As we have

$$\mathbf{u}_{n_k} = \mathbf{u}_{0,n_k} + \int_0^t \partial_t \mathbf{u}_{n_k}\, ds$$

we see that $\mathbf{u}_{n_k}(t,.)$ is convergent to $\mathbf{u}_\infty(t,.)$ in $\mathcal{D}'(\mathbb{R}^3)$, hence is weakly convergent in $L^2_{\rm loc}$ (as it is bounded in $L^2_{w_\gamma}$), so that :

$$\int \frac{|\mathbf{u}_\infty(t,x)|^2}{2}\phi_R w_{\gamma,\epsilon}\, dx \le \limsup_{n_k\to+\infty} \int \frac{|\mathbf{u}_{n_k}(t,x)|^2}{2}\phi_R w_{\gamma,\epsilon}\, dx.$$

Similarly, as $\nabla \mathbf{u}_{n_k}$ is weakly convergent in $L^2 L^2_{w_\gamma}$, we have

$$\int_0^t \int \frac{|\nabla \mathbf{u}_\infty(s,x)|^2}{2}\phi_R w_{\gamma,\epsilon}\, dx\, ds \le \limsup_{n_k\to+\infty} \int_0^t \int \frac{|\nabla \mathbf{u}_{n_k}(s,x)|^2}{2}\phi_R w_{\gamma,\epsilon}\, dx\, ds.$$

Thus, letting R go to $+\infty$ and then ϵ go to 0, we find by dominated convergence that, for every $t \in (0,T)$, we have

$$\begin{aligned}\|\mathbf{u}_\infty(t,.)\|^2_{L^2_{w_\gamma}} &+ 2\int_0^t \|\nabla \mathbf{u}_\infty(s,.)\|^2_{L^2_{w_\gamma}}\, ds \\ &\le \|\mathbf{u}_{0,\infty}\|^2_{L^2_{w_\gamma}} - \int_0^t \int \nabla|\mathbf{u}_\infty|^2 \cdot \nabla w_\gamma\, dx\, ds + \int_0^t \int (|\mathbf{u}_\infty|^2 \mathbf{b}_\infty + 2p_\infty \mathbf{u}_\infty)\cdot \nabla(w_\gamma)\, dx\, ds \\ &\quad - 2\sum_{i=1}^3 \sum_{j=1}^3 \int_0^t \int F_{\infty,i,j}(\partial_i u_{\infty,j}) w_\gamma + F_{\infty,i,j} u_{\infty,i} \partial_j(w_\gamma)\, dx\, ds.\end{aligned}$$

Letting t go to 0, we find

$$\limsup_{t\to 0} \|\mathbf{u}_\infty(t,.)\|^2_{L^2_{w_\gamma}} \le \|\mathbf{u}_{0,\infty}\|^2_{L^2_{w_\gamma}}.$$

On the other hand, we know that $\mathbf{u}_\infty$ is weakly continuous in $L^2_{w_\gamma}$ and thus we have

$$\|\mathbf{u}_{0,\infty}\|^2_{L^2_{w_\gamma}} \le \liminf_{t\to 0} \|\mathbf{u}_\infty(t,.)\|^2_{L^2_{w_\gamma}}.$$

This gives $\|\mathbf{u}_{0,\infty}\|^2_{L^2_{w_\gamma}} = \lim_{t\to 0} \|\mathbf{u}_\infty(t,.)\|^2_{L^2_{w_\gamma}}$, which allows to turn the weak convergence into a strong convergence. Theorem 3 is proven. $\diamond$

5 Solutions of the Navier–Stokes problem with initial data in $L^2_{w_\gamma}$.

We now prove Theorem 1. The idea is to approximate the problem by a Navier–Stokes problem in L^2, then use the a priori estimates (Theorem 2) and the stability theorem (Theorem 3) to find a solution to the Navier–Stokes problem with data in $L^2_{w_\gamma}$).

5.1 Approximation by square integrable data.

Lemma 7 (Leray's projection operator) *Let $0 < \delta < 3$ and $1 < r < +\infty$. If $\mathbf{v}$ is a vector field on $\mathbb{R}^3$ such that $\mathbf{v} \in L^r_{w_\delta}$, then there exists a unique decompostion*

$$\mathbf{v} = \mathbf{v}_\sigma + \mathbf{v}_\nabla$$

such that

- $\mathbf{v}_\sigma \in L^r_{w_\delta}$ *and* $\nabla \cdot \mathbf{v}_\sigma = 0$.
- $\mathbf{v}_\nabla \in L^r_{w_\delta}$ *and* $\nabla \wedge \mathbf{v}_\nabla = 0$.

We shall write $\mathbf{v}_\sigma = \mathbb{P}\mathbf{v}$, where $\mathbb{P}$ is Leray's projection operator.

Similarly, if $\mathbf{v}$ is a distribution vector field of the type $\mathbf{v} = \nabla \cdot \mathbb{G}$ with $\mathbb{G} \in L^r_{w_\delta}$ then there exists a unique decompostion

$$\mathbf{v} = \mathbf{v}_\sigma + \mathbf{v}_\nabla$$

such that

- *there exists $\mathbb{H} \in L^r_{w_\delta}$ such that $\mathbf{v}_\sigma = \nabla \cdot \mathbb{H}$ and $\nabla \cdot \mathbf{v}_\sigma = 0$.*
- *there exists $q \in L^r_{w_\delta}$ such that $\mathbf{v}_\nabla = \nabla q$ (and thus $\nabla \wedge \mathbf{v}_\nabla = 0$).*

We shall still write $\mathbf{v}_\sigma = \mathbb{P}\mathbf{v}$. Moreover, the function q is given by

$$q = -\sum_{i=1}^{3}\sum_{j=1}^{3} R_i R_j (G_{i,j}).$$

Proof : As $w_\delta \in \mathcal{A}_r$ the Riesz transforms are bounded on $L^r_{w_\delta}$. Using the identity

$$\Delta \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \wedge (\nabla \wedge \mathbf{v})$$

we find (if the decomposition exists) that

$$\Delta \mathbf{v}_\sigma = -\nabla \wedge (\nabla \wedge \mathbf{v}_\sigma) = -\nabla \wedge (\nabla \wedge \mathbf{v}) \text{ and } \Delta \mathbf{v}_\nabla = \nabla(\nabla \cdot \mathbf{v}_\nabla) = \nabla(\nabla \cdot \mathbf{v}).$$

This proves the uniqueness. By linearity, we just have to prove that $\mathbf{v} = 0 \implies \mathbf{v}_\nabla = 0$. We have $\Delta \mathbf{v}_\nabla = 0$, and thus $\mathbf{v}_\nabla$ is harmonic; as it belongs to $\mathcal{S}'$, we find that it is a polynomial. But a polynomial which belongs to $L^r_{w_\delta}$ must be equal to 0. Similarly, if $\mathbf{v}_\nabla = \nabla q$, then $\Delta q = \nabla \cdot \mathbf{v}_\nabla = \nabla \cdot \mathbf{v} = 0$; thus q is harmonic and belongs to $L^r_{w_\delta}$, hence $q = 0$.

For the existence, it is enough to check that $v_{\nabla,i} = -\sum_{j=1}^3 R_i R_j v_j$ in the first case and $\mathbf{v}_\nabla = \nabla q$ with $q = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (G_{i,j})$ in the second case fulfill the conclusions of the lemma. ◇

Lemma 8 *Let $0 < \gamma \leq 2$. Let $\mathbf{u}_0$ be a divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ and $\mathbb{F}$ be a tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1\leq i,j\leq 3}$ such that $\mathbb{F} \in L^2((0,+\infty), L^2_{w_\gamma})$. Let $\phi \in \mathcal{D}(\mathbb{R}^3)$ be a non-negative function which is equal to 1 for $|x| \leq 1$ and to 0 for $|x| \geq 2$. For $R > 0$, we define $\phi_R(x) = \phi(\frac{x}{R})$, $\mathbf{u}_{0,R} = \mathbb{P}(\phi_R \mathbf{u}_0)$ and $\mathbb{F}_R = \phi_R \mathbb{F}$. Then $\mathbf{u}_{0,R}$ is a divergence-free square integrable vector field and $\lim_{R\to+\infty} \|\mathbf{u}_{0,R} - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0$. Similarly, $\mathbb{F}_R$ belongs to L^2L^2 and $\lim_{R\to+\infty} \|\mathbb{F}_R - \mathbb{F}\|_{L^2((0,+\infty),L^2_{w_\gamma})} = 0$.*

Proof : By dominated convergence, we have $\lim_{R\to+\infty} \|\phi_R \mathbf{u}_0 - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0$. We conclude by writing $\mathbf{u}_{0,R} - \mathbf{u}_0 = \mathbb{P}(\phi_R \mathbf{u}_0 - \mathbf{u}_0)$. ◇

5.2 Leray's mollification.

We want to solve the Navier–Stokes equations with initial value $\mathbf{u}_0$:

$$(NS)\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \nabla\cdot\mathbb{F} \\ \\ \nabla\cdot\mathbf{u} = 0, \qquad \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

We begin with Leray's method [11] for solving the problem in L^2 :

$$(NS_R)\begin{cases} \partial_t \mathbf{u}_R = \Delta \mathbf{u}_R - (\mathbf{u}_R\cdot\nabla)\mathbf{u}_R - \nabla p_R + \nabla\cdot\mathbb{F}_{\mathbb{R}} \\ \\ \nabla\cdot\mathbf{u}_R = 0, \qquad \mathbf{u}_R(0,.) = \mathbf{u}_{0,R} \end{cases}$$

The idea of Leray is to mollify the non-linearity by replacing $\mathbf{u}_R \cdot \nabla$ by $(\mathbf{u}_R * \theta_\epsilon)\cdot\nabla$, where $\theta(x) = \frac{1}{\epsilon^3}\theta(\frac{x}{\epsilon})$, $\theta \in \mathcal{D}(\mathbb{R}^3)$, θ is non-negative and radially decreasing and $\int \theta\, dx = 1$. We thus solve the problem

$$(NS_{R,\epsilon})\begin{cases} \partial_t \mathbf{u}_{R,\epsilon} = \Delta \mathbf{u}_{R,\epsilon} - ((\mathbf{u}_{R,\epsilon} * \theta_\epsilon)\cdot\nabla)\mathbf{u}_{R,\epsilon} - \nabla p_{R,\epsilon} + \nabla\cdot\mathbb{F}_R \\ \\ \nabla\cdot\mathbf{u}_{R,\epsilon} = 0, \qquad \mathbf{u}_{R,\epsilon}(0,.) = \mathbf{u}_{0,R} \end{cases}$$

The classical result of Leray states that the problem $(NS_{R,\epsilon})$ is well-posed :

Lemma 9 *Let $\mathbf{v}_0 \in L^2$ be a divergence-free vector field. Let $\mathbb{G} \in L^2((0,+\infty), L^2)$. Then the problem*

$$(NS_\epsilon)\begin{cases} \partial_t \mathbf{v}_\epsilon = \Delta \mathbf{v}_\epsilon - ((\mathbf{v}_\epsilon * \theta_\epsilon)\cdot\nabla)\mathbf{v}_\epsilon - \nabla q_\epsilon + \nabla\cdot\mathbb{G} \\ \\ \nabla\cdot\mathbf{v}_\epsilon = 0, \qquad \mathbf{v}_\epsilon(0,.) = \mathbf{v}_0 \end{cases}$$

has a unique solution $\mathbf{v}_\epsilon$ in $L^\infty((0,+\infty), L^2) \cap L^2((0,+\infty), \dot H^1)$. Moreover, this solution belongs to $\mathcal{C}([0,+\infty), L^2)$.

5.3 Proof of Theorem 1 (local existence)

We use Lemma 9 and find a solution $\mathbf{u}_{R,\epsilon}$ to the problem $(NS_{R,\epsilon})$. Then we check that $\mathbf{u}_{R,\epsilon}$ fulfills the assumptions of Theorem 2 and of Corollary 6 :

- $\mathbf{u}_{R,\epsilon}$ belongs to $L^\infty((0,T),L^2_{w_\gamma})$ and $\nabla\mathbf{u}_{R,\epsilon}$ belongs to $L^2((0,T),L^2_{w_\gamma})$
- the map $t\in[0,+\infty)\mapsto\mathbf{u}_{R,\epsilon}(t,.)$ is weakly continuous from $[0,+\infty)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t=0$:
$$\lim_{t\to 0}\|\mathbf{u}_{R,\epsilon}(t,.)-\mathbf{u}_{0,R}\|_{L^2_{w_\gamma}}=0.$$
- on $(0,T)\times\mathbb{R}^3$, $\mathbf{u}_{R,\epsilon}$ fulfills the energy equality :
$$\partial_t(\frac{|\mathbf{u}_{R,\epsilon}|^2}{2})=\Delta(\frac{|\mathbf{u}_{R,\epsilon}|^2}{2})-|\nabla\mathbf{u}_{R,\epsilon}|^2-\nabla\cdot\left(\frac{|\mathbf{u}|^2}{2}\mathbf{b}_{R,\epsilon}\right)-\nabla\cdot(p_{R,\epsilon}\mathbf{u}_{R,\epsilon})+\mathbf{u}_{R,\epsilon}\cdot(\nabla\cdot\mathbb{F}_R).$$
with $\mathbf{b}_{R,\epsilon}=\mathbf{u}_{R,\epsilon}*\theta_\epsilon$.
- $\mathbf{b}_{R,\epsilon}$ is controlled by $\mathbf{u}_{R,\epsilon}$: for every $t\in(0,T)$,
$$\|\mathbf{b}_{R,\epsilon}(t,.)\|_{L^3_{w_{3\gamma/2}}}\le\|\mathcal{M}_{\mathbf{u}_{R,\epsilon}(t,.)}\|_{L^3_{w_{3\gamma/2}}}\le C_0\|\mathbf{u}_{R,\epsilon}(t,.)\|_{L^3_{w_{3\gamma/2}}}.$$

Thus, we know that, for every time T_0 such that
$$C_\gamma(1+C_0^4)\left(1+C_0^4+\|\mathbf{u}_{0,R}\|^2_{L^2_{w_\gamma}}+\int_0^{T_0}\|\mathbb{F}_R\|^2_{L^2_{w_\gamma}}\,ds\right)^2T_0\le 1$$
we have
$$\sup_{0\le t\le T_0}\|\,\mathbf{u}_{R,\epsilon}(t,.)\|^2_{L^2_{w_\gamma}}\le C_\gamma(1+C_0^4+\|\mathbf{u}_{0,R}\|^2_{L^2_{w_\gamma}}+\int_0^{T_0}\|\mathbb{F}_R\|^2_{L^2_{w_\gamma}}\,ds)$$
and
$$\int_0^{T_0}\|\nabla\mathbf{u}_{R,\epsilon}\|^2_{L^2_{w_\gamma}}\,ds\le C_\gamma(1+C_0^4+\|\mathbf{u}_{0,R}\|^2_{L^2_{w_\gamma}}+\int_0^{T_0}\|\mathbb{F}_R\|^2_{L^2_{w_\gamma}}\,ds).$$
Moreover, we have that
$$\|\mathbf{u}_{0,R}\|_{L^2_{w_\gamma}}\le C_\gamma\|\mathbf{u}_0\|_{L^2_{w_\gamma}}\text{ and }\|\mathbb{F}_R\|_{L^2_{w_\gamma}}\le\|\mathbb{F}\|_{L^2_{w_\gamma}}$$
so that
$$\begin{aligned}\|\mathbf{b}_{R,\epsilon}\|_{L^3((0,T_0),L^3_{w_{3\gamma/2}})}\le&C_\gamma\|\mathbf{u}_{R,\epsilon}\|_{L^3((0,T_0),L^3_{w_{3\gamma/2}})}\\ \le&C'_\gamma T_0^{\frac{1}{12}}\left((1+\sqrt{T_0})\|\mathbf{u}_{R,\epsilon}\|_{L^\infty((0,T_0),L^2_{w_\gamma})}+\|\nabla\mathbf{u}_{R,\epsilon}\|_{L^2((0,T_0),L^2_{w_\gamma})}\right)\\ \le&C''_\gamma\sqrt{1+C_0^4+\|\mathbf{u}_0\|^2_{L^2_{w_\gamma}}+\int_0^{T_0}\|\mathbb{F}\|^2_{L^2_{w_\gamma}}\,ds}.\end{aligned}$$

Let $R_n \to +\infty$ and $\epsilon_n \to 0$. Let $\mathbf{u}_{0,n} = \mathbf{u}_{0,R_n}$, $\mathbb{F}_n = \mathbb{F}_{R_n}$, $\mathbf{b}_n = \mathbf{b}_{R_n,\epsilon_n}$ and $\mathbf{u}_n = \mathbf{u}_{R_n,\epsilon_n}$. We may then apply Theorem 3, since $\mathbf{u}_{0,n}$ is strongly convergent to $\mathbf{u}_0$ in $L^2_{w_\gamma}$, $\mathbb{F}_n$ is strongly convergent to $\mathbb{F}$ in $L^2((0,T_0), L^2_{w_\gamma})$, and the sequence $\mathbf{b}_n$ is bounded in $L^3((0,T_0), L^3_{w_{3\gamma/2}})$. Thus there exists p, $\mathbf{u}$, $\mathbf{b}$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that

- $\mathbf{u}_{n_k}$ converges $*$-weakly to $\mathbf{u}$ in $L^\infty((0,T_0), L^2_{w_\gamma})$, $\nabla\mathbf{u}_{n_k}$ converges weakly to $\nabla\mathbf{u}$ in $L^2((0,T_0), L^2_{w_\gamma})$
- $\mathbf{b}_{n_k}$ converges weakly to $\mathbf{b}$ in $L^3((0,T_0), L^3_{w_{3\gamma/2}})$, p_{n_k} converges weakly to p in $L^3((0,T_0), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T_0), L^2_{w_\gamma})$
- $\mathbf{u}_{n_k}$ converges strongly to $\mathbf{u}$ in $L^2_{\mathrm{loc}}([0,T_0)\times\mathbb{R}^3)$.

Moreover, $\mathbf{u}$ is a solution of the advection-diffusion problem

$$\begin{cases} \partial_t \mathbf{u} = \Delta\mathbf{u} - (\mathbf{b}\cdot\nabla)\mathbf{u} - \nabla p + \nabla\cdot\mathbb{F} \\ \\ \nabla\cdot\mathbf{u} = 0, \qquad\qquad \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

and is such that :

- the map $t\in[0,T_0)\mapsto \mathbf{u}(t,.)$ is weakly continuous from $[0,T_0)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t=0$:
$$\lim_{t\to 0}\|\mathbf{u}(t,.)-\mathbf{u}_0\|_{L^2_{w_\gamma}} = 0.$$
- there exists a non-negative locally finite measure μ on $(0,T_0)\times\mathbb{R}^3$ such that
$$\partial_t(\frac{|\mathbf{u}|^2}{2}) = \Delta(\frac{|\mathbf{u}|^2}{2}) - |\nabla\mathbf{u}|^2 - \nabla\cdot\left(\frac{|\mathbf{u}|^2}{2}\mathbf{b}\right) - \nabla\cdot(p\mathbf{u}) + \mathbf{u}\cdot(\nabla\cdot\mathbb{F}) - \mu.$$

Finally, as $\mathbf{b}_n = \theta_{\epsilon_n} * (\mathbf{u}_n - \mathbf{u}) + \theta_{\epsilon_n} * \mathbf{u}$, we see that $\mathbf{b}_{n_k}$ is strongly convergent to $\mathbf{u}$ in $L^3_{\mathrm{loc}}([0,T_0)\times\mathbb{R}^3)$, so that $\mathbf{b}=\mathbf{u}$: thus, $\mathbf{u}$ is a solution of the Navier–Stokes problem on $(0,T_0)$. (It is easy to check that

$$p = \sum_{i=1}^3\sum_{j=1}^3 R_iR_j(u_iu_j - F_{i,j})$$

as $u_{i,n_k}u_{j,n_k}$ is weakly convergent to u_iu_j in $L^4((0,T_0), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $w_{\frac{6\gamma}{5}} \in \mathcal{A}_{6/5}$).

5.4 Proof of Theorem 1 (global existence)

In order to finish the proof, we shall use the scaling properties of the Navier–Stokes equations : if $\lambda > 0$, then $\mathbf{u}$ is a solution of the Cauchy initial value problem for the Navier–Stokes equations on $(0, T)$ with initial value $\mathbf{u}_0$ and forcing tensor $\mathbb{F}$ if and only if $\mathbf{u}_\lambda(t, x) = \lambda \mathbf{u}(\lambda^2 t, \lambda x)$ is a solution of the Navier–Stokes equations on $(0, T/\lambda^2)$ with initial value $\mathbf{u}_{0,\lambda}(x) = \lambda \mathbf{u}_0(\lambda x)$ and forcing tensor $\mathbb{F}_\lambda(t, x) = \lambda^2 \mathbb{F}(\lambda^2 t, \lambda x)$.

We take $\lambda > 1$ and for $n \in \mathbb{N}$ we consider the Navier–Stokes problem with initial value $\mathbf{v}_{0,n} = \lambda^n \mathbf{u}_0(\lambda^n \cdot)$ and forcing tensor $\mathbb{F}_n = \lambda^{2n} \mathbb{F}(\lambda^{2n} \cdot, \lambda^n \cdot)$. Then we have seen that we can find a solution $\mathbf{v}_n$ on $(0, T_n)$, with

$$C_\gamma \left(1 + \|\mathbf{v}_{0,n}\|^2_{L^2_{w_\gamma}} + \int_0^{+\infty} \|\mathbb{F}_n\|^2_{L^2_{w_\gamma}} \, ds\right)^2 T_n = 1.$$

Of course, we have $\mathbf{v}_n(t, x) = \lambda^n \mathbf{u}_n(\lambda^{2n} t, \lambda^n x)$ where $\mathbf{u}_n$ is a solution of the Navier–Stokes equations on $(0, \lambda^{2n} T_n)$ with initial value $\mathbf{u}_0$ and forcing tensor $\mathbb{F}$

Lemma 10

$$\lim_{n \to +\infty} \frac{\lambda^n}{1 + \|\mathbf{v}_{0,n}\|^2_{L^2_{w_\gamma}} + \int_0^{+\infty} \|\mathbb{F}_n\|^2_{L^2_{w_\gamma}} \, ds} = +\infty.$$

Proof : We have

$$\|\mathbf{v}_{0,n}\|^2_{L^2_{w_\gamma}} = \int |\mathbf{u}_0(x)|^2 \lambda^{n(\gamma-1)} \frac{(1 + |x|)^\gamma}{(\lambda^n + |x|)^\gamma} w_\gamma(x) \, dx.$$

We have

$$\lambda^{n(\gamma-1)} \le \lambda^n$$

as $\gamma \le 2$ and we have, by dominated convergence,

$$\lim_{n \to +\infty} \int |\mathbf{u}_0(x)|^2 \frac{(1 + |x|)^\gamma}{(\lambda^n + |x|)^\gamma} w_\gamma(x) \, dx = 0.$$

Similarly, we have

$$\int_0^{+\infty} \|\mathbb{F}_n\|^2_{L^2_{w_\gamma}} \, ds = \int_0^{+\infty} \int |\mathbb{F}(s, x)|^2 \lambda^{n(\gamma-1)} \frac{(1 + |x|)^\gamma}{(\lambda^n + |x|)^\gamma} w_\gamma(x) \, dx \, ds = o(\lambda^n).$$

Thus, $\lim_{n \to +\infty} \lambda^{2n} T_n = +\infty$. $\diamond$

Now, for a given $T > 0$, if $\lambda^{2n} T_n > T$ for $n \ge n_T$, then $\mathbf{u}_n$ is a solution of the Navier-Stokes problem on $(0, T)$. Let $\mathbf{w}_n(t, x) = \lambda^{n_T} \mathbf{u}_n(\lambda^{2n_T} t, \lambda^{n_T} x)$.

For $n \geq n_T$, $\mathbf{w}_n$ is a solution of the Navier-Stokes problem on $(0, \lambda^{-2n_T} T)$ with initial value $\mathbf{v}_{0,n_T}$ and forcing tensor $\mathbb{F}_{n_T}$. As $\lambda^{-2n_T} T \leq T_{n_T}$, we have

$$C_\gamma \left(1 + \|\mathbf{v}_{0,n_T}\|^2_{L^2_{w_\gamma}} + \int_0^{+\infty} \|\mathbb{F}_{n_T}\|^2_{L^2_{w_\gamma}} \, ds\right)^2 \lambda^{-2n_T} T \leq 1.$$

By corollary 6, we have

$$\sup_{0 \leq t \leq \lambda^{-2n_T} T} \| \mathbf{w}_n(t,.)\|^2_{L^2_{w_\gamma}} \leq C_\gamma (1 + \|\mathbf{v}_{0,n_T}\|^2_{L^2_{w_\gamma}} + \int_0^{\lambda^{-2n_T} T} \|\mathbb{F}_{n_T}\|^2_{L^2_{w_\gamma}} \, ds)$$

and

$$\int_0^{\lambda^{-2n_T} T} \|\nabla \mathbf{w}_n\|^2_{L^2_{w_\gamma}} \, ds \leq C_\gamma (1 + \|\mathbf{v}_{0,n_T}\|^2_{L^2_{w_\gamma}} + \int_0^{\lambda^{-2n_T} T} \|\mathbb{F}_{n_T}\|^2_{L^2_{w_\gamma}} \, ds).$$

We have

$$\|\mathbf{w}_n\|^2_{L^2_{w_\gamma}} = \int |\mathbf{u}_n(\lambda^{2n_T} t, x)|^2 \lambda^{n_T(\gamma-1)} \frac{(1+|x|)^\gamma}{(\lambda^{n_T} + |x|)^\gamma} w_\gamma(x) \, dx \geq \lambda^{n_T(\gamma-1)} \|\mathbf{u}_n(\lambda^{2n_T} t, .)\|^2_{L^2_{w_\gamma}}.$$

and

$$\begin{aligned}\int_0^{\lambda^{-2n_T} T} \|\nabla \mathbf{w}_n\|^2_{L^2_{w_\gamma}} \, ds = & \int_0^T \int |\nabla \mathbf{u}_n(s,x)|^2 \lambda^{n_T(\gamma-1)} \frac{(1+|x|)^\gamma}{(\lambda^{n_T} + |x|)^\gamma} w_\gamma(x) \, dx \, ds \\ \geq & \lambda^{n_T(\gamma-1)} \int_0^T \|\nabla \mathbf{u}_n\|^2_{L^2_{w_\gamma}} \, ds.\end{aligned}$$

Thus, we have a uniform control of $\mathbf{u}_n$ and of $\nabla \mathbf{u}_n$ on $(0,T)$ for $n \geq n_T$. We may then apply the Rellich lemma (Lemma 6) and Theorem 3 to find a subsequence $\mathbf{u}_{n_k}$ that converges to a global solution of the Navier–Stokes equations. Theorem 1 is proven. ⋄

6 Solutions of the advection-diffusion problem with initial data in $L^2_{w_\gamma}$.

The proof of Theorem 1 on the Navier–Stokes problem can be easily adapted to the case of the advection-diffusion problem :

Theorem 4 *Let $0 < \gamma \leq 2$. Let $0 < T < +\infty$. Let $\mathbf{u}_0$ be a divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ and $\mathbb{F}$ be a tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1 \leq i,j \leq 3}$*

such that $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$*. Let* $\mathbf{b}$ *be a time-dependent divergence free vector-field (*$\nabla \cdot \mathbf{b} = 0$*) such that* $\mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}})$*.*

Then the advection-diffusion problem

$$(AD) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{b} \cdot \nabla)\mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

has a solution $\mathbf{u}$ *such that :*

- $\mathbf{u}$ *belongs to* $L^\infty((0,T), L^2_{w_\gamma})$ *and* $\nabla \mathbf{u}$ *belongs to* $L^2((0,T), L^2_{w_\gamma})$
- *the pressure* p *is related to* $\mathbf{u}$*,* $\mathbf{b}$ *and* $\mathbb{F}$ *through the Riesz transforms* $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ *by the formula*

$$p = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (b_i u_j - F_{i,j})$$

- *the map* $t \in [0,T) \mapsto \mathbf{u}(t,.)$ *is weakly continuous from* $[0,T)$ *to* $L^2_{w_\gamma}$*, and is strongly continuous at* $t = 0$ *:*

$$\lim_{t \to 0} \|\mathbf{u}(t,.) - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0.$$

- *there exists a non-negative locally finite measure* μ *on* $(0,T) \times \mathbb{R}^3$ *such that*

$$\partial_t(\frac{|\mathbf{u}|^2}{2}) = \Delta(\frac{|\mathbf{u}|^2}{2}) - |\nabla \mathbf{u}|^2 - \nabla \cdot \left(\frac{|\mathbf{u}|^2}{2}\mathbf{b}\right) - \nabla \cdot (p\mathbf{u}) + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) - \mu.$$

Proof : Again, we define $\phi_R(x) = \phi(\frac{x}{R})$, $\mathbf{u}_{0,R} = \mathbb{P}(\phi_R \mathbf{u}_0)$ and $\mathbb{F}_R = \phi_R \mathbb{F}$. Moreover, we define $\mathbf{b}_R = \mathbb{P}(\phi_R \mathbf{b})$. We then solve the mollified problem

$$(AD_{R,\epsilon}) \begin{cases} \partial_t \mathbf{u}_{R,\epsilon} = \Delta \mathbf{u}_{R,\epsilon} - ((\mathbf{b}_R * \theta_\epsilon) \cdot \nabla)\mathbf{u}_{R,\epsilon} - \nabla p_{R,\epsilon} + \nabla \cdot \mathbb{F}_{R,\epsilon} \\ \nabla \cdot \mathbf{u}_{R,\epsilon} = 0, \qquad \mathbf{u}_{R,\epsilon}(0,.) = \mathbf{u}_{0,R} \end{cases}$$

for which we easily find a unique solution $\mathbf{u}_{R,\epsilon}$ in $L^\infty((0,T), L^2) \cap L^2((0,T), \dot{H}^1)$. Moreover, this solution belongs to $\mathcal{C}([0,T), L^2)$.

Again, $\mathbf{u}_{R,\epsilon}$ fulfills the assumptions of Theorem 2 :

- $\mathbf{u}_{R,\epsilon}$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla \mathbf{u}_{R,\epsilon}$ belongs to $L^2((0,T), L^2_{w_\gamma})$

- the map $t \in [0,T) \mapsto \mathbf{u}_{R,\epsilon}(t,.)$ is weakly continuous from $[0,T)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t=0$:

$$\lim_{t\to 0} \|\mathbf{u}_{R,\epsilon}(t,.) - \mathbf{u}_{0,R}\|_{L^2_{w_\gamma}} = 0.$$

- on $(0,T)\times\mathbb{R}^3$, $\mathbf{u}_{R,\epsilon}$ fulfills the energy equality :

$$\partial_t(\frac{|\mathbf{u}_{R,\epsilon}|^2}{2}) = \Delta(\frac{|\mathbf{u}_{R,\epsilon}|^2}{2}) - |\nabla\mathbf{u}_{R,\epsilon}|^2 - \nabla\cdot\left(\frac{|\mathbf{u}|^2}{2}\mathbf{b}_{R,\epsilon}\right) - \nabla\cdot(p_{R,\epsilon}\mathbf{u}_{R,\epsilon}) + \mathbf{u}_{R,\epsilon}\cdot(\nabla\cdot\mathbb{F}_R).$$

  with $\mathbf{b}_{R,\epsilon} = \mathbf{b}_R * \theta_\epsilon$.

Thus, by Corollary 4 we know that,

$$\sup_{0<t<T} \|\mathbf{u}_{R,\epsilon}\|_{L^2_{w_\gamma}} \leq (\|\mathbf{u}_{0,R}\|_{L^2_{w_\gamma}} + C_\gamma\|\mathbb{F}_R\|_{L^2((0,T),L^2_{w_\gamma})})\, e^{C_\gamma(T+T^{1/3}\|\mathbf{b}_{R,\epsilon}\|^2_{L^3((0,T),L^3_{w_{3\gamma/2}})})}$$

and

$$\|\nabla\mathbf{u}_{R,\epsilon}\|_{L^2((0,T),L^2_{w_\gamma})} \leq (\|\mathbf{u}_{0,R}\|_{L^2_{w_\gamma}} + C_\gamma\|\mathbb{F}_R\|_{L^2((0,T),L^2_{w_\gamma})})\, e^{C_\gamma(T+T^{1/3}\|\mathbf{b}_{R,\epsilon}\|^2_{L^3((0,T),L^3_{w_{3\gamma/2}})})}$$

where the constant C_γ depends only on γ.

Moreover, we have that

$$\|\mathbf{u}_{0,R}\|_{L^2_{w_\gamma}} \leq C_\gamma\|\mathbf{u}_0\|_{L^2_{w_\gamma}},\ \|\mathbb{F}_R\|_{L^2_{w_\gamma}} \leq \|\mathbb{F}\|_{L^2_{w_\gamma}}$$

and

$$\|\mathbf{b}_{R,\epsilon}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})} \leq \|\mathcal{M}_{\mathbf{b}_R}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})} \leq C'_\gamma\|\mathbf{b}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})}$$

Let $R_n \to +\infty$ and $\epsilon_n \to 0$. Let $\mathbf{u}_{0,n} = \mathbf{u}_{0,T_n}$, $\mathbb{F}_n = \mathbb{F}_{R_n}$, $\mathbf{b}_n = \mathbf{b}_{R_n,\epsilon_n}$ and $\mathbf{u}_n = \mathbf{u}_{R_n,\epsilon_n}$. We may then apply Theorem 3, since $\mathbf{u}_{0,n}$ is strongly convergent to $\mathbf{u}_0$ in $L^2_{w_\gamma}$, $\mathbb{F}_n$ is strongly convergent to $\mathbb{F}$ in $L^2((0,T),L^2_{w_\gamma})$, and the sequence $\mathbf{b}_n$ is strongly convergent to $\mathbf{b}$ in $L^3((0,T),L^3_{w_{3\gamma/2}})$. Thus there exists p, $\mathbf{u}$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that

- $\mathbf{u}_{n_k}$ converges *-weakly to $\mathbf{u}$ in $L^\infty((0,T),L^2_{w_\gamma})$, $\nabla\mathbf{u}_{n_k}$ converges weakly to $\nabla\mathbf{u}$ in $L^2((0,T),L^2_{w_\gamma})$
- p_{n_k} converges weakly to p in $L^3((0,T),L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T),L^2_{w_\gamma})$
- $\mathbf{u}_{n_k}$ converges strongly to $\mathbf{u}$ in $L^2_{\rm loc}([0,T)\times\mathbb{R}^3)$.

We then easily finish the proof. $\diamond$

7 Application to the study of λ-discretely self-similar solutions

We may now apply our results to the study of λ-discretely self-similar solutions for the Navier–Stokes equations.

Definition 1 *Let* $\mathbf{u}_0 \in L^2_{\rm loc}(\mathbb{R}^3)$*. We say that* $\mathbf{u}_0$ *is a* λ*-discretely self-similar function (*λ*-DSS) if there exists* $\lambda > 1$ *such that* $\lambda \mathbf{u}_0(\lambda x) = \mathbf{u}_0$*.*

A vector field $\mathbf{u} \in L^2_{\rm loc}([0,+\infty) \times \mathbb{R}^3)$ *is* λ*-DSS if there exists* $\lambda > 1$ *such that* $\lambda \mathbf{u}(\lambda^2 t, \lambda x) = \mathbf{u}(t,x)$*.*

A forcing tensor $\mathbb{F} \in L^2_{\rm loc}([0,+\infty) \times \mathbb{R}^3)$ *is* λ*-DSS if there exists* $\lambda > 1$ *such that* $\lambda^2 \mathbb{F}(\lambda^2 t, \lambda x) = \mathbb{F}(t,x)$*.*

We shall speak of self-similarity if $\mathbf{u}_0$, $\mathbf{u}$ or $\mathbb{F}$ are λ-DSS for every $\lambda > 1$.

Examples :

- Let $\gamma > 1$ and $\lambda > 1$. Then, for two positive constants $A_{\gamma,\lambda}$ and $B_{\gamma,\lambda}$, we have : if $\mathbf{u}_0 \in L^2_{\rm loc}(\mathbb{R}^3)$ is λ-DSS, then $\mathbf{u}_0 \in L^2_{w_\gamma}$ and

$$A_{\gamma,\lambda} \int_{1<|x|\leq\lambda} |\mathbf{u}_0(x)|^2 \, dx \leq \int |\mathbf{u}_0(x)|^2 w_\gamma(x) \, dx \leq B_{\gamma,\lambda} \int_{1<|x|\leq\lambda} |\mathbf{u}_0(x)|^2 \, dx$$

- $\mathbf{u}_0 \in L^2_{\rm loc}$ is self-similar if and only if it is of the form $\mathbf{u}_0 = \frac{\mathbf{w}_0(\frac{x}{|x|})}{|x|}$ with $\mathbf{w}_0 \in L^2(S^2)$.

- $\mathbb{F}$ belongs to $L^2((0,+\infty), L^2_{w_\gamma})$ with $\gamma > 1$ and is self-similar if and only if it is of the form $\mathbb{F}(t,x) = \frac{1}{t}\mathbb{F}_0(\frac{x}{\sqrt{t}})$ with $\int |\mathbb{F}_0(x)|^2 \frac{1}{|x|} \, dx < +\infty$.

Proof :

- If $\mathbf{u}_0$ is λ-DSS and if $k \in \mathbb{Z}$ we have

$$\int_{\lambda^k<|x|<\lambda^{k+1}} |\mathbf{u}_0(x)|^2 w_\gamma(x) \, dx \leq \frac{\lambda^k}{(1+\lambda^k)^\gamma} \int_{1<|x|<\lambda} |\mathbf{u}_0(x)|^2 \, dx$$

  with $\sum_{k\in\mathbb{Z}} \frac{\lambda^k}{(1+\lambda^k)^\gamma} < +\infty$ for $\gamma > 1$.

- If $\mathbf{u}_0$ is self-similar, we have $\mathbf{u}_0(x) = \frac{1}{|x|}\mathbf{u}_0(\frac{x}{|x|})$. From this equality, we find that, for $\lambda > 1$

$$\int_{1<|x|<\lambda} |\mathbf{u}_0(x)|^2 \, dx = (\lambda - 1) \int_{S^2} |\mathbf{u}_0(\sigma)|^2 \, d\sigma$$

- If $\mathbb{F}$ is self-similar, then it is of the form $\mathbb{F}(t,x) = \frac{1}{t}\mathbb{F}_0(\frac{x}{\sqrt{t}})$. Moreover, we have

$$\int_0^{+\infty}\int |\mathbb{F}(t,x)|^2\, w_\gamma(x)\, dx\, ds = \int_0^{+\infty}\int |\mathbb{F}_0(x)|^2 w_\gamma(\sqrt{t}\, x)\, dx\, \frac{dt}{\sqrt{t}} = C_\gamma \int |\mathbb{F}_0(x)|^2\, \frac{dx}{|x|}$$

with $C_\gamma = \int_0^{+\infty} \frac{1}{(1+\sqrt{\theta})^\gamma} \frac{d\theta}{\sqrt{\theta}} < +\infty$. ◇

In this section, we are going to give a new proof of the results of Chae and Wolf [3] and Bradshaw and Tsai [2] on the existence of λ-DSS solutions of the Navier–Stokes problem (and of Jia and Šverák [6] for self-similar solutions) :

Theorem 5 *Let $4/3 < \gamma \leq 2$ and $\lambda > 1$. If $\mathbf{u}_0$ is a λ-DSS divergence-free vector field (such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$) and if $\mathbb{F}$ is a λ-DSS tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1\leq i,j\leq 3}$ such that $\mathbb{F} \in L^2_{\rm loc}([0,+\infty)\times\mathbb{R}^3)$, then the Navier–Stokes equations with initial value $\mathbf{u}_0$*

$$(NS)\begin{cases} \partial_t \mathbf{u} = \Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \nabla\cdot\mathbb{F} \\ \\ \nabla\cdot\mathbf{u} = 0, \qquad\qquad \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

has a global weak solution $\mathbf{u}$ such that :

- *$\mathbf{u}$ is a λ-DSS vector field*
- *for every $0 < T < +\infty$, $\mathbf{u}$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{u}$ belongs to $L^2((0,T), L^2_{w_\gamma})$*
- *the map $t \in [0,+\infty) \mapsto \mathbf{u}(t,.)$ is weakly continuous from $[0,+\infty)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$:*

$$\lim_{t\to 0} \|\mathbf{u}(t,.) - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0.$$

- *the solution $\mathbf{u}$ is suitable : there exists a non-negative locally finite measure μ on $(0,+\infty)\times\mathbb{R}^3$ such that*

$$\partial_t(\frac{|\mathbf{u}|^2}{2}) = \Delta(\frac{|\mathbf{u}|^2}{2}) - |\nabla\mathbf{u}|^2 - \nabla\cdot\left((\frac{|\mathbf{u}|^2}{2} + p)\mathbf{u}\right) + \mathbf{u}\cdot(\nabla\cdot\mathbb{F}) - \mu.$$

7.1 The linear problem.

Following Chae and Wolf, we consider an approximation of the problem that is consistent with the scaling properties of the equations : let θ be a non-negative and radially decreasing function in $\mathcal{D}(\mathbb{R}^3)$ with $\int \theta\, dx = 1$; We define $\theta_{\epsilon,t}(x) = \frac{1}{(\epsilon\sqrt{t})^3}\ \theta(\frac{x}{\epsilon\sqrt{t}})$. We then will study the "mollified" problem

$$(NS_\epsilon)\begin{cases} \partial_t \mathbf{u}_\epsilon = \Delta \mathbf{u}_\epsilon - ((\mathbf{u}_\epsilon * \theta_{\epsilon,t})\cdot\nabla)\mathbf{u}_\epsilon - \nabla p_\epsilon + \nabla\cdot\mathbb{F} \\ \\ \qquad \nabla\cdot\mathbf{u} = 0, \qquad\qquad \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

and begin with the linearized problem

$$(LNS_\epsilon)\begin{cases} \partial_t \mathbf{v} = \Delta \mathbf{v} - ((\mathbf{b} * \theta_{\epsilon,t})\cdot\nabla)\mathbf{v} - \nabla q + \nabla\cdot\mathbb{F} \\ \\ \qquad \nabla\cdot\mathbf{v} = 0, \qquad\qquad \mathbf{v}(0,.) = \mathbf{u}_0 \end{cases}$$

Lemma 11 *Let $1 < \gamma \le 2$. Let $\lambda > 1$ Let $\mathbf{u}_0$ be a λ-DSS divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ and $\mathbb{F}$ be a λ-DSS tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1\le i,j\le 3}$ such that, for every $T > 0$, $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$. Let $\mathbf{b}$ be a λ-DSS time-dependent divergence free vector-field ($\nabla\cdot\mathbf{b} = 0$) such that, for every $T > 0$, $\mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}})$.*

Then the advection-diffusion problem

$$(LNS_\epsilon)\begin{cases} \partial_t \mathbf{v} = \Delta \mathbf{v} - ((\mathbf{b} * \theta_{\epsilon,t})\cdot\nabla)\mathbf{v} - \nabla q + \nabla\cdot\mathbb{F} \\ \\ \qquad \nabla\cdot\mathbf{v} = 0, \qquad\qquad \mathbf{v}(0,.) = \mathbf{u}_0 \end{cases}$$

has a unique solution $\mathbf{v}$ such that :

- *for every positive T, $\mathbf{v}$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{v}$ belongs to $L^2((0,T), L^2_{w_\gamma})$*
- *the pressure p is related to $\mathbf{v}$, $\mathbf{b}$ and $\mathbb{F}$ through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula*

$$p = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j((b_i * \theta_{\epsilon,t})v_j - F_{i,j})$$

- *the map $t \in [0,+\infty) \mapsto \mathbf{v}(t,.)$ is weakly continuous from $[0,+\infty)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$:*

$$\lim_{t\to 0} \|\mathbf{v}(t,.) - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0.$$

This solution $\mathbf{v}$ *is a* λ*-DSS vector field.*

Proof : As we have $|\mathbf{b}(t,.)*\theta_{\epsilon,t}|\leq \mathcal{M}_{\mathbf{b}(t,.)}$ and thus

$$\|\mathbf{b}(t,.)*\theta_{\epsilon,t}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})}\leq C_\gamma\|\mathbf{b}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})}$$

we see that we can use Theorem 4 to get a solution $\mathbf{v}$ on $(0,T)$.

As clearly $\mathbf{b}*\theta_{\epsilon,t}$ belongs to $L^2_tL^\infty_x(K)$ for every compact subset K of $(0,T)\times\mathbb{R}^3$, we can use Corollary 5 to see that $\mathbf{v}$ is unique.

Let $\mathbf{w}(t,x)=\frac{1}{\lambda}\mathbf{v}(\frac{t}{\lambda^2},\frac{x}{\lambda})$. As $b*\theta_{\epsilon,t}$ is still λ-DSS, we see that $\mathbf{w}$ is solution of (LNS_ϵ) on $(0,T)$, so that $\mathbf{w}=\mathbf{v}$. This means that $\mathbf{v}$ is λ-DSS. $\diamond$

7.2 The mollified Navier–Stokes equations.

The solution $\mathbf{v}$ provided by Lemma 11 belongs to $L^3((0,T),L^3_{w_{3\gamma/2}})$ (as $\mathbf{v}$ belongs to $L^\infty((0,T),L^2_{w_\gamma})$ and $\nabla\mathbf{v}$ belongs to $L^2((0,T),L^2_{w_\gamma})$). Thus we have a mapping $L_\epsilon:\mathbf{b}\mapsto\mathbf{v}$ which is defined from

$$X_{T,\gamma}=\{\mathbf{b}\in L^3((0,T),L^3_{w_{3\gamma/2}})\ /\ \mathbf{b}\text{ is }\lambda-\mathrm{DSS}\}$$

to $X_{T,\gamma}$ by $L_\epsilon(\mathbf{b})=\mathbf{v}$.

Lemma 12 *For* $4/3<\gamma$*,* $X_{T,\gamma}$ *is a Banach space for the equivalent norms* $\|\mathbf{b}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})}$ *and* $\|\mathbf{b}\|_{L^3((0,T/\lambda^2),\times B(0,\frac{1}{\lambda}))}$.

Proof : We have

$$\int_0^T\int_{B(0,1)}|\mathbf{b}(t,x)|^3\,dx\,dt=\lambda^2\int_0^{\frac{T}{\lambda^2}}\int_{B(0,\frac{1}{\lambda})}|\mathbf{b}(t,x)|^3\,dx\,dt$$

and , for $k\in\mathbb{N}$,

$$\int_0^T\int_{\lambda^{k-1}<|x|<\lambda^k}|\mathbf{b}(t,x)|^3\,dx\,dt=\lambda^{2k}\int_0^{\frac{T}{\lambda^{2k}}}\int_{\frac{1}{\lambda}<|x|<1}|\mathbf{b}(t,x)|^3\,dx\,dt.$$

We may conclude, since for $\gamma>4/3$ we have $\sum_{k\in\mathbb{N}}\lambda^{k(2-\frac{3\gamma}{2})}<+\infty$.

Lemma 13 *For* $4/3<\gamma\leq 2$*, the mapping* L_ϵ *is continuous and compact on* $X_{T,\gamma}$.

Proof : Let $\mathbf{b}_n$ be a bounded sequence in $X_{T,\gamma}$ and let $\mathbf{v}_n = L_\epsilon(\mathbf{b}_n)$. We remark that the sequence $\mathbf{b}_n(t,.)*\theta_{\epsilon,t}$ is bounded in $X_{T,\gamma}$. Thus, by Theorem 2 and Corollary 4, the sequence $\mathbf{v}_n$ is bounded in $L^\infty((0,T),L^2_{w_\gamma})$ and $\nabla\mathbf{v}_n$ is bounded in $L^2((0,T),L^2_{w_\gamma})$.

We now use Theorem 3 and get that then there exists q_∞, $\mathbf{v}_\infty$, $\mathbf{B}_\infty$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that

- $\mathbf{v}_{n_k}$ converges *-weakly to $\mathbf{v}_\infty$ in $L^\infty((0,T),L^2_{w_\gamma})$, $\nabla\mathbf{v}_{n_k}$ converges weakly to $\nabla\mathbf{v}_\infty$ in $L^2((0,T),L^2_{w_\gamma})$
- $\mathbf{b}_{n_k}*\theta_{\epsilon,t}$ converges weakly to $\mathbf{B}_\infty$ in $L^3((0,T),L^3_{w_{3\gamma/2}})$, ,
- the associated pressures q_{n_k} converge weakly to q_∞ in $L^3((0,T),L^{6/5}_{w_{\frac{6\gamma}{5}}})+L^2((0,T),L^2_{w_\gamma})$
- $\mathbf{v}_{n_k}$ converges strongly to $\mathbf{v}_\infty$ in $L^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^3)$: for every $T_0\in(0,T)$ and every $R>0$, we have

$$\lim_{k\to+\infty}\int_0^{T_0}\int_{|y|<R}|\mathbf{v}_{n_k}(s,y)-\mathbf{v}_\infty(s,y)|^2\,ds\,dy=0.$$

As $\sqrt{w_\gamma}\mathbf{v}_n$ is bounded in $L^\infty((0,T),L^2)$ and in $L^2((0,T),L^6)$, it is bounded in $L^{10/3}((0,T)\times\mathbb{R}^3)$. The strong convergence of $\mathbf{v}_{n_k}$ in $L^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^3)$ then implies the strong convergence of $\mathbf{v}_{n_k}$ in $L^3_{\mathrm{loc}}((0,T)\times\mathbb{R}^3)$.

Moreover, $\mathbf{v}_\infty$ is still λ-DSS (a property that is stable under weak limits).We find that $\mathbf{v}_\infty\in X_{T,\gamma}$ and that

$$\lim_{n_k\to+\infty}\int_0^{\frac{T}{\lambda^2}}\int_{B(0,\frac{1}{\lambda})}|\mathbf{v}_{n_k}(s,y)-\mathbf{v}_\infty(s,y)|^3\,ds\,dy=0.$$

This proves that L_ϵ is compact.

If we assume moreover that $\mathbf{b}_n$ is convergent to $\mathbf{b}_\infty$ in $X_{T,\gamma}$, then necessarily we have $\mathbf{B}_\infty=\mathbf{b}_\infty*\theta_{\epsilon,t}$, and $\mathbf{v}_\infty=L_\epsilon(\mathbf{b}_\infty)$. Thus, the relatively compact sequence $\mathbf{v}_n$ can have only one limit point; thus it must be convergent. This proves that L_ϵ is continuous. ◇

Lemma 14 *Let $4/3<\gamma\le 2$. If, for some $\mu\in[0,1]$, $\mathbf{v}$ is a solution of $\mathbf{v}=\mu L_\epsilon(\mathbf{v})$ then*

$$\|\mathbf{v}\|_{X_{T,\gamma}}\le C_{\mathbf{u}_0,\mathbb{F},\gamma,T}$$

where the constant $C_{\mathbf{u}_0,\mathbb{F},\gamma,T}$ depends only on $\mathbf{u}_0$, $\mathbb{F}$, γ and T (but not on μ nor on ϵ).

Proof : We have $\mathbf{v} = \mu \mathbf{w}$; with

$$\begin{cases} \partial_t \mathbf{w} = \Delta \mathbf{w} - ((\mathbf{v} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{w} - \nabla q + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{w} = 0, \qquad \mathbf{w}(0,.) = \mathbf{u}_0 \end{cases}$$

Multiplying by μ, we find that

$$\begin{cases} \partial_t \mathbf{v} = \Delta \mathbf{v} - ((\mathbf{v} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{v} - \nabla(\mu q) + \nabla \cdot \mu\mathbb{F} \\ \nabla \cdot \mathbf{v} = 0, \qquad \mathbf{v}(0,.) = \mu\mathbf{u}_0 \end{cases}$$

We then use Corollary 6. We choose $T_0 \in (0,T)$ such that

$$C_\gamma \left(1 + \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}}\, ds\right)^2 T_0 \leq 1.$$

Then, as

$$C_\gamma \left(1 + \|\mu\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mu\mathbb{F}\|^2_{L^2_{w_\gamma}}\, ds\right)^2 T_0 \leq 1$$

we know that

$$\sup_{0 \leq t \leq T_0} \| \mathbf{v}(t,.)\|^2_{L^2_{w_\gamma}} \leq C_\gamma(1 + \mu^2 \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \mu^2 \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}}\, ds)$$

and

$$\int_0^{T_0} \|\nabla \mathbf{v}\|^2_{L^2_{w_\gamma}}\, ds \leq C_\gamma(1 + \mu^2 \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \mu^2 \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}}\, ds).$$

In particular, we have

$$\int_0^{T_0} \|\mathbf{v}\|^3_{L^3_{w_{3\gamma/2}}}\, ds \leq C_\gamma T_0^{1/4}(1 + \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}}\, ds)^{\frac{3}{2}}.$$

As $\mathbf{v}$ is λ-DSS, we can go back from T_0 to T. ◇

Lemma 15 *Let $4/3 < \gamma \leq 2$. There is at least one solution $\mathbf{u}_\epsilon$ of the equation $\mathbf{u}_\epsilon = L_\epsilon(\mathbf{u}_\epsilon)$.*

Proof : Obvious due to the Leray–Schauder principle (and the Schaefer theorem), since L_ϵ is continuous and compact and since we have uniform a priori estimates for the fixed points of μL_ϵ for $0 \leq \mu \leq 1$. ◇

7.3 Proof of Theorem 5.

We may now finish the proof of Theorem 5. We consider the solutions $\mathbf{u}_\epsilon$ of $\mathbf{u}_\epsilon = L_\epsilon(\mathbf{u}_\epsilon)$.

By Lemma 14, $\mathbf{u}_\epsilon$ is bounded in $L^3((0,T), L^3_{w_{3\gamma/2}})$, and so is $\mathbf{u}_\epsilon * \theta_{\epsilon,t}$. We then know, by Theorem 2 and Corollary 4, that the familly $\mathbf{u}_\epsilon$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{u}_\epsilon$ is bounded in $L^2((0,T), L^2_{w_\gamma})$.

We now use Theorem 3 and get that then there exists p, $\mathbf{u}$, $\mathbf{B}$ and a decreasing sequence $(\epsilon_k)_{k\in\mathbb{N}}$ (converging to 0) with values in $(0,+\infty)$ such that

- $\mathbf{u}_{\epsilon_k}$ converges *-weakly to $\mathbf{u}$ in $L^\infty((0,T), L^2_{w_\gamma})$, $\nabla\mathbf{u}_{\epsilon_k}$ converges weakly to $\nabla\mathbf{u}$ in $L^2((0,T), L^2_{w_\gamma})$
- $\mathbf{u}_{\epsilon_k} * \theta_{\epsilon_k,t}$ converges weakly to $\mathbf{B}$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$
- the associated pressures p_{ϵ_k} converge weakly to p in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T), L^2_{w_\gamma})$
- $\mathbf{u}_{\epsilon_k}$ converges strongly to $\mathbf{u}$ in $L^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^3)$.

Moreover we easily see that $\mathbf{B} = \mathbf{u}$. Indeed, we have that $\mathbf{u} * \theta_{\epsilon,t}$ converges strongly in $L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^3)$ as ϵ goes to 0 (since it is bounded by $\mathcal{M}_{\mathbf{u}}$ and converges, for each fixed t, strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$); moreover, we have $|(\mathbf{u}-\mathbf{u}_\epsilon) * \theta_{\epsilon,t}| \le \mathcal{M}_{\mathbf{u}-\mathbf{u}_\epsilon}$, so that the strong convergence of $\mathbf{u}_{\epsilon_k}$ to $\mathbf{u}$ is kept by convolution with $\theta_{\epsilon,t}$ as far as we work on compact subsets of $(0,T)\times\mathbb{R}^3$ (and thus don't allow t to go to 0).

Thus, Theorem 5 is proven. $\diamond$

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