# On the long-time behavior for a damped Navier-Stokes-Bardina model


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## Contents




**Abstract**

In this paper, we consider the damped Navier-Stokes-Bardina model posed on the whole three-dimensional. These equations arise from some oceanic model and, from the mathematical point of view, they write down as the well-know Navier-Stokes equations with an additional nonlocal operator in their nonlinear transport term, and moreover, with an additional damping term depending of a parameter. We study first the existence and *uniqueness* of global in time weak solutions in the *energy space*. Thereafter, we are interested in describing their *long time behavior*. For this, we use some tools in the theory of dynamical systems to prove the existence of a *global attractor*, which is *compact subset* in the energy space attracting all the weak solutions when the time goes to infinity. Moreover, we derive an upper bound for the *fractal dimension* of the global attractor associated to these equations. Finally, for a particular choice of the damping parameter, we are also able to give an acutely description of its internal structure.



**Keywords:** Narvier-Stokes equations; Bardina's model; Global attractor; Stationary solutions; Asymptotically and orbital stability.

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## 1 Introduction

The theory of partial differential equations (PDEs) is a broad research field, rapidly growing in close connections with other mathematical disciplines and applied sciences. The connections between the theories of dynamical systems and PDEs will be explored from several points of view. Infinite-dimensional dynamical systems generated by evolutionary PDEs provide the most immediate examples of interplay between the two theories. Extensions of well established results and techniques from finite-dimensional dynamical systems (invariant manifolds and bifurcations) have proved very useful in qualitative studies of PDEs. On the other hand, specific questions for PDEs brought about stimulating problems in the theory of dynamical systems, such as the existence of finite-dimensional attractors and their behavior under (regular or singular) perturbations. Entire (or eternal) solutions, which emerged as key objects in these problems, have long served as organizing centers for qualitative investigations of dissipative evolutionary PDEs and they continue to play an important role in other modern approaches to PDEs.

It is well-known that the dynamic of an incompressible fluid, which we will assume on the whole space $\mathbb{R}^3$, is successfully described by the classical, homogeneous and incompressible Navier-Stokes equations:

$$\partial_t\vec{v} + \text{div}(\vec{v}\otimes\vec{v}) - \nu\Delta\vec{v} + \vec{\nabla}q = 0, \quad \text{div}(\vec{v}) = 0, \tag{1}$$

where, $\vec{v} : [0, +\infty[\times\mathbb{R}^3 \to \mathbb{R}^3$ and $q : [0, +\infty[\times\mathbb{R}^3 \to \mathbb{R}$ are the velocity of the fluid and the pressure term respectively, the parameter $\nu > 0$ represents the kinetic viscosity parameter which we will keep fix. Moreover, the equation $\text{div}(\vec{v}) = 0$ describes the fluid's incompressibility.

Although the Navier-Stokes equations are a relevant physical model used in many applications, see for instance the book [26], the mathematical theory of these equations is not yet sufficient to prove the global well-posedness of the so-called Leray's solutions, and the uniqueness of these solutions remains a very challenging open question. In order to contour this problem, researchers who are investigating the use of the Navier-Stokes equations in practical applications [4] have applied some operators to these equations, to obtain regularized versions of the Navier-Stokes equations where the weak solutions are well-posed.

The idea is to introduce a regularized velocity field $\vec{u}(t, x)$ in terms of the original velocity field $\vec{v}(t, x)$ given by the equation (1) leading a variety of so-called $\alpha-$ models for the Navier-Stokes equations. See for instance the Chapter 17 of the book [21]. In this context, for a parameter $\alpha > 0$, J. Bardina, J. H. Ferziger, & W. C. Reynolds introduced in [5] the operator $(\cdot)_\alpha$, given by solving the Helmholtz equation:

$$-\alpha^2\Delta(\varphi)_\alpha + (\varphi)_\alpha = \varphi.$$

The operator $(\cdot)_\alpha$ is also called the *filtering/averaging* operator, due to the fact that this operator allows us to obtain an accurate model describing the large-scale motion of the fluid while filtering or averaging the fluid motion at scales smaller than $\alpha$.

On the other hand, from the mathematical point of view, we may observe in the whole space $\mathbb{R}^3$, and denoting by $I_d$ the identity operator, the expression $(\varphi)_\alpha$ is given by

$$(\varphi)_\alpha = (-\alpha^2\Delta + I_d)^{-1}\varphi, \tag{2}$$

where the action of the operator $(-\alpha^2\Delta + I_d)^{-1}$, also known as the *Bessel Potential*, could be easily defined in the Fourier variable as $\mathcal{F}\left((-\alpha^2\Delta + I_d)^{-1}\varphi\right)(\xi) = (\alpha^2|\xi|^2 + 1)^{-1}\widehat{\varphi}(\xi)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^3)$. For a more exhaustive study of *Bessel Potentials* see the Chapter 6 of the book [14].

Applying the operator $(\cdot)_\alpha$ to the Navier-Stokes equations (1), we get the following equations for the regularized velocity $(\vec{v})_\alpha$, and the regularized pressure $(q)_\alpha$:

$$\partial_t(\vec{v})_\alpha + \text{div}((\vec{v}\otimes\vec{v}))_\alpha - \nu\Delta(\vec{v})_\alpha + \vec{\nabla}(q)_\alpha = 0, \quad \text{div}((\vec{v})_\alpha) = 0.$$

However, there is still a problem to overcome. These regularized Navier-Stokes equations are not a closed system in the sense that, due to the regularized nonlinear term: $\text{div}((\vec{v}\otimes\vec{v}))_\alpha$, the system does not write down only in terms of the regularized velocity $(\vec{v})_\alpha$ and the regularized pressure $(q)_\alpha$. More precisely, we have the identity

$$\text{div}((\vec{v}\otimes\vec{v}))_\alpha = \text{div}\left((\vec{v})_\alpha \otimes (\vec{v})_a\right) + \text{div}(R(\vec{v},\vec{v})),$$

where the remainder term

$$R(\vec{v},\vec{v}) = (\vec{v}\otimes\vec{v})_\alpha - (\vec{v})_\alpha \otimes (\vec{v})_\alpha,$$

is known as the Reynolds stress tensor. See, *e.g.*, [7] for more details on this term.

In order to obtain a closed system, W. Layton & R. Lewandowski propose in [19] to approximate the Reynolds stress tensor as follows:

$$R(\vec{v},\vec{v}) \approx \left((\vec{v})_\alpha \otimes (\vec{v})_\alpha\right)_\alpha - (\vec{v})_\alpha \otimes (\vec{v})_\alpha.$$

Hence, replacing this approximation of the term $R(\vec{v},\vec{v})$ in the identity for the nonlinear term above, we obtain the following approximation for the nonlinear term:

$$\text{div}((\vec{v}\otimes\vec{v}))_\alpha \approx \text{div}\left(\left((\vec{v})_\alpha \otimes (\vec{v})_\alpha\right)_\alpha\right).$$

This approximation of the nonlinear term has been successfully used in many practical applications [7, 10]; and it finally leads us to the following closed system:

$$\partial_t(\vec{u})_\alpha + \text{div}\left(\left((\vec{v})_\alpha \otimes (\vec{v})_\alpha\right)_\alpha\right) - \nu\Delta(\vec{u})_\alpha + \vec{\nabla}(p)_\alpha = 0. \quad \text{div}((\vec{u})_\alpha) = 0.$$

To simplify the notation, we shall write the regularized functions $(\vec{v})_\alpha$ and $(q)_\alpha$ as $\vec{u}$ and $p$ respectively; and we thus obtain the so-called *Navier-Stokes-Bardina* model:

$$\partial_t\vec{u} + \text{div}((\vec{u}\otimes\vec{u})_\alpha) - \nu\Delta\vec{u} + \vec{\nabla}p = 0, \quad \text{div}(\vec{u}) = 0, \quad \vec{u}(0,\cdot) = \vec{u}_0, \quad \alpha > 0,\ \nu > 0. \tag{3}$$

where the initial $\vec{u}_0$ denotes the (regularized) velocity of the fluid at the time $t = 0$.

The first mathematical studies for these equations were given by W. Layton & R. Lewandowski in [18] in the space-periodic case. In this setting, for an initial datum $\vec{u}_0 \in H^1(\mathbb{T}^2)$, where for $L > 0$, $\mathbb{T}^3$ denotes the periodic box $[0, 2\pi L]^3$, and for all time $T > 0$, it is proven the existence and the *uniqueness of a weak solution* $(\vec{u}, p)$ which verifies $\vec{u} \in L^\infty([0,T[, H^1(\mathbb{T}^3)) \cap L^2([0,T[, \dot{H}^2(\mathbb{T}^3))$. More recently, using a variational formulation of the equation (3), and some *a priori* energy estimates, L. C. Berselli & R. Lewandowski extended in [6] the previous well-posedness results ( given for the periodic case) to the whole space $\mathbb{R}^3$. See also the Chapter 17.5 of the book [21] for an alternative proof of this result. Moreover, in the same work [6], the regularity of weak solutions is also studied and it is proven that, for all time $t > 0$, the unique weak solution $(\vec{u}(t,\cdot), p(t,\cdot))$ of the Navier-Stokes-Bardina equations (3) belongs to the Sobolev space $H^m(\mathbb{R}^3)$ for all entire $m \geq 0$.

On the other hand, another relevant issue for the Navier-Stokes Bardina's model is the *asymptotic properties* of weak solutions when times goes to infinity. From a physical perspective, this problem is also interesting when the Navier-Stokes Bardina's model is used to perform numerical simulations related to the turbulence description [4]. The main idea is to consider an external force term $\vec{f}$ in the equations Navier-Stokes-Bardina's model, which is assumed to be a stationary (time-independing) vector field. This stationarity assumption is a simplification of the physical model. Indeed, the idea behind of this physical model is that we will assume that a time independing external source acts on the fluid and put its in a perpetual turbulent state. In this scenario, we are thus interested in understanding the behavior of the velocity $\vec{u}(t,\cdot)$ when the time $t$ is large enough. It is also worth mention that if we consider a time-dependent force then we will need to find an appropriate time interval in which the fluid is turbulent and this is a highly non-trivial issue. See the Chapter 1 of [17] for a more detailed discussion. Thus, for a stationary external force $\vec{f}:\mathbb{R}^3\to\mathbb{R}^3$, we obtain the following forced equations:

$$\partial_t \vec{u} + \text{div}((\vec{u}\otimes\vec{u})_\alpha) - \nu\Delta\vec{u} + \vec{\nabla}p = \vec{f}, \quad \text{div}(\vec{u})=0, \quad \vec{u}(0,\cdot)=\vec{u}_0, \quad \alpha>0,\ \nu>0.$$

In the space-periodic setting, the long-time behavior of solutions was studied in [7]. In that work, using the *Poincaré* inequality the authors obtain good controls on the quantity $\|\vec{u}(t,\cdot)\|_{H^1(\mathbb{T}^3)}$ when the time $t$ goes to infinity. More precisely, for a constant $\eta>0$, and for all time $t\geq 0$, they get the following estimate:

$$\|\vec{u}(t,\cdot)\|^2_{L^2(\mathbb{T}^3)} + \alpha\,\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1(\mathbb{T}^3)} \lesssim e^{-\eta t}\left(\|\vec{u}_0\|^2_{L^2(\mathbb{T}^3)} + \alpha\|\vec{u}_0\|^2_{\dot{H}^1(\mathbb{T}^3)}\right) + \|\vec{f}\|^2_{L^2(\mathbb{T}^3)} + \alpha\|\vec{f}\|^2_{\dot{H}^1(\mathbb{T}^3)}.$$

Hence, for $t>0$ large enough, we may observe that the norm $\|\vec{u}(t,\cdot)\|_{H^1(\mathbb{T}^3)}$, expressed by the *equivalent* quantity $\|\vec{u}(t,\cdot)\|^2_{L^2(\mathbb{T}^3)} + \alpha\,\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1(\mathbb{T}^3)}$, is controlled *uniformly in time* by the quantity involving the external force: $\|\vec{f}\|^2_{L^2(\mathbb{T}^3)} + \alpha\|\vec{f}\|^2_{\dot{H}^1(\mathbb{T}^3)}$; and this control is one of the key tools to apply the classical theory of *dynamical systems* to study the long time behavior of solutions $\vec{u}(t,\cdot)$. Specifically, the authors prove the existence of a *global attractor* for the Navier-Stokes-Bardina equations with space-periodic conditions. In Section 2 below, we recall the definition of a *global attractor* and introduce more detail all the tools in the theory of dynamical systems used to perform this study.

Now, in the non-periodic setting of the whole space $\mathbb{R}^3$, and due to the lack of the *Poincaré* inequality, we can only obtain the following not so useful estimate in time (see the details in Appendix ())

$$\|\vec{u}(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} + \alpha\,\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)} \lesssim \|\vec{u}_0\|^2_{L^2(\mathbb{R}^3)} + \alpha\|\vec{u}_0\|^2_{\dot{H}^1(\mathbb{R}^3)} + t\left(\|\vec{f}\|^2_{L^2(\mathbb{R}^3)} + \alpha\|\vec{f}\|^2_{\dot{H}^1(\mathbb{R}^3)}\right),$$

and here we clearly loose any control in time when $t$ goes to infinity. To contour this problematic, some previous works related to the study of the long-time behavior for the Navier-Stokes equations and related models [13, 16, 17] suggest to compensate the lack of the Poincaré inequality by adding in the forced equation a supplementary *damping term* of the form $-\beta\vec{u}$, where $\beta>0$ is a damping parameter. It is worth mention that another (merely technical) damping terms can be considered to study the long-time behavior of the Navier-Stokes equations on the whole space [9, 20]. However, we will consider here the damping term $-\beta\vec{u}$, since this term has a physical meaning as a drag-friction term in some oceanic models [23], and then, it is also interesting from the physical point of view.

Thus, we shall consider here the following Cauchy problem of the *damped Navier-Stokes-Bardina's model* for incompressible fluids in the whole space $\mathbb{R}^3$:

$$\begin{cases} \partial_t \vec{u} + \text{div}((\vec{u}\otimes\vec{u})_\alpha) - \nu\Delta\vec{u} + \vec{\nabla}p = \vec{f} - \beta\vec{u}, \quad \alpha>0,\ \beta>0,\ \nu>0, \\ \text{div}(\vec{u})=0, \\ \vec{u}(0,\cdot)=\vec{u}_0, \quad \text{div}(\vec{u}_0)=0. \end{cases} \tag{4}$$

When $\beta=0$, the system (4) writes down as the classical Navier-Stokes-Bardina's model. However, all the results we obtained in this article deeply depend on the the parameter $\beta>0$ and, from now on, we will focus on this model, where, our main objective is to *describe the long-time behavior of its solutions.*

## 2 Statement of the results

Our first main result is devoted to the well-posedness of equations (4) in the energy space.

**Theorem 1** *Let $\vec{u}_0 \in H^1(\mathbb{R}^3)$ be the initial datum such that $div(\vec{u}_0) = 0$. Moreover, let $\vec{f} \in H^1(\mathbb{R}^3)$ be a stationary external force such that $div(\vec{f}) = 0$. Then, for all $\alpha > 0$ and $\beta > 0$ there exists a couple of functions $\vec{u} = \vec{u}_{\alpha,\beta} \in L^\infty([0,+\infty[, H^1(\mathbb{R}^3)) \cap L^2_{loc}([0,+\infty[, \dot{H}^2(\mathbb{R}^3))$, and $p = p_{\alpha,\beta} \in L^2_{loc}([0,+\infty[, H^3(\mathbb{R}^3))$, such that $(\vec{u}, p)$ is the* unique *weak solution of (4). Moreover, this solution verifies the following energy* equality *for all $t \geq 0$:*

$$\begin{aligned} &\|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} + 2\nu\int_0^t \|\vec{u}(s,\cdot)\|^2_{\dot{H}^1} ds + 2\alpha^2\int_0^t \|\vec{u}(s,\cdot)\|^2_{\dot{H}^2} ds = \|\vec{u}_0\|^2_{L^2} + \alpha^2\|\vec{u}_0\|^2_{\dot{H}^1} \\ &+ 2\int_0^t \left(\langle \vec{f}, \vec{u}(s,\cdot)\rangle_{L^2\times L^2} + \alpha^2\langle \vec{\nabla}\otimes\vec{f}, \vec{\nabla}\otimes\vec{u}(s,\cdot)\rangle_{L^2\times L^2}\right) ds - 2\beta\int_0^t \left(\|\vec{u}(s,)\|^2_{L^2} + \alpha^2\|\vec{u}(s,\cdot)\|^2_{\dot{H}^1}\right) ds. \end{aligned} \tag{5}$$

Comparing this result with the classical result on the existence of Leray's weak solutions for the Navier-Stokes equations (1), we may observe two main differences. First, we have here the *uniqueness* of the weak solution in the energy space, and moreover, this solution verifies an energy *equality*. These facts are due to filtering operator $(\cdot)_\alpha$ (defined in expression (2)) in the nonlinear transport term. As mentioned, the filtering operator regularizes the classical nonlinear term in the framework of the non homogeneous Sobolev spaces, providing the weak solutions of the equations (4) these good properties. In particular, the uniqueness of weak solutions is the key idea to study their long-time behavior and we will get back to this point later.

Another important contribution of the filtering operator $(\cdot)_\alpha$ is the fact that, following some of the ideas in [6], for $t > 0$ the regularity (in the spatial variable) of the weak solution $\vec{u}(t,\cdot)$ constructed in this theorem could be improved to Sobolev spaces of higher order (provided that the external force is regular enough). However, the natural regularity given by the energy space is enough to study the asymptotic time behavior of the solution $\vec{u}(t,\cdot)$.

Concerning the damping term $-\beta\vec{u}$, is it worth mention this term allows us to derive the following controls in time. These estimates will be very useful when we studying the large-time behavior of weak solutions.

**Proposition 1** *Within the framework of Theorem 1, the solution $\vec{u}(t,\cdot)$ verifies the following estimates.*

1) *For all $t \geq 0$:*

$$\|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} \leq \left(\|\vec{u}_0\|^2_{L^2} + \alpha^2\|\vec{u}_0\|^2_{\dot{H}^1}\right) e^{-\beta t} + \frac{4}{\beta^2}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right). \tag{6}$$

2) *For all $t \geq 0$ and $T \geq 0$:*

$$\nu\int_t^{t+T} \|\vec{u}(s,\cdot)\|^2_{\dot{H}^1} ds + \alpha^2\int_t^{t+T} \|\vec{u}(s,\cdot)\|^2_{\dot{H}^2} ds \leq \frac{2T}{\beta}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right) + \|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}. \tag{7}$$

The uniqueness of global weak solutions constructed in Theorem 1 will be one of the key tools to study their long-time asymptotic behavior. Inspired by [16], our approach will be given through the language of the dynamical systems. Before to state our next result, for the sake of a complete exposition, we will set first some notations and we will recall some classical definitions in the theory of dynamical systems. We refer to [24, 25] and the references therein for more details.

From now on, we fix the filtering parameter $\alpha > 0$. Then, we define the space $\mathcal{H}^1_\alpha(\mathbb{R}^3)$ as the Banach space of *divergence free* vector fields $\vec{g} \in H^1(\mathbb{R}^3)$ with the norm $\|\vec{g}\|_{\mathcal{H}^1_\alpha} = \|\vec{g}\|_{L^2} + \alpha\|\vec{g}\|_{\dot{H}^1}$. This *equivalent* norm on the space $H^1(\mathbb{R}^3)$ naturally appears in the estimate (6), which will be useful in the proof of our next result below.

The key link between the equation (4) and the framework of dynamical systems is the fact that for a given (stationary and divergence free) external force $\vec{f} \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$, the equation (4) defines a *semigroup* acting on the Banach space $\mathcal{H}^1_\alpha(\mathbb{R}^3)$. More precisely, for a time $t \geq 0$ we define the semigroup $S(t) : \mathcal{H}^1_\alpha(\mathbb{R}^3) \to \mathcal{H}^1_\alpha(\mathbb{R}^3)$ as

$$S(t)\vec{u}_0 = \vec{u}(t,\cdot), \quad \text{for all} \quad \vec{u}_0 \in \mathcal{H}^1_\alpha(\mathbb{R}^3), \tag{8}$$

where $\vec{u}(t,\cdot)$ is the *unique* global weak solution of equation (4) arising from the initial datum $\vec{u}_0$ and constructed in Theorem 1. Due to the uniqueness of weak solutions, it is easy to verify that the family $(S(t))_{t\geq 0}$ given in (8) defines a (strongly continuous) semigroup on the Banach space $\mathcal{H}^1_\alpha(\mathbb{R}^3)$.

The study of the long-time asymptotics of weak solutions for equation (4) can be treated through the study of the semigroup $S(t)$ when $t \to +\infty$. More precisely, our aim is to prove that this semigroup has a *global attractor* whose definition we recall as follows.

**Definition 1** *Let $(S(t))_{t\geq 0}$ be the semigroup given in (8) acting on the Banach space $\mathcal{H}^1_\alpha(\mathbb{R}^3)$. A global attractor for the semigroup $(S(t))_{t\geq 0}$ is a set $\mathcal{A} \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$ which verifies:*

1) *The set $\mathcal{A}$ is compact in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$.*

2) *The set $\mathcal{A}$ is strictly invariant: for all time $t \geq 0$ we have $S(t)\mathcal{A} = \mathcal{A}$.*

3) *For every bound set $B \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$ and for every neighborhood $\mathcal{V} \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$ of the set $\mathcal{A}$, there exists a time $T = T(B,\mathcal{V}) > 0$, depending on the set $B$ and the neighborhood $\mathcal{V}$, such that for all time $t > T$ we have $S(t)B \subset \mathcal{V}$.*

In this definition we focus on point 3) to remark that, roughly speaking, a global attractor *attires* the image through the semigroup $S(t)$ of all bounded set $B \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$ when $t \to +\infty$. This property allows us to have a better comprehension of the long-time behavior of weak solutions for the equation (4). Indeed, we observe first that for all initial data $\vec{u}_0 \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$, setting the bounded set $B = \{\vec{u}_0\}$, and moreover, setting $\mathcal{V}$ any neighborhood of the attractor $\mathcal{A}$, then we find that the solution $\vec{u}(t,\cdot)$ of the equation (4) (arising from the initial datum $\vec{u}_0$) lies in the neighborhood $\mathcal{V}$ from a time $T = T(\vec{u}_0,\mathcal{V})$. Consequently, from *any initial datum* $\vec{u}_0$ the solution $\vec{u}(t,\cdot)$ is as close to the attractor $\mathcal{A}$ as we want when $t \to +\infty$.

In our second main result, we prove the existence of a global attractor for the semigroup $(S(t))_{t\geq 0}$ associated to equation (4).

**Theorem 2** *Let $\vec{f} \in H^1(\mathbb{R}^3)$ be a stationary and divergence free external force. Let $(S(t))_{t\geq 0}$ be the semigroup associated to (4) defined in (8). Then, this semigroup has a global attractor $\mathcal{A}_{\vec{f}}$, depending on the external force $\vec{f}$, given by Definition 1.*

Once we have proven the existence of a global attractor for the equation (4), our general aim is to give more precisely descriptions of the set $\mathcal{A}_{\vec{f}}$. Thus, to present all our results in an orderly fashion, we divided them in four sections below.

### Fractal dimension of the global attractor

As the global attractor is a compact set of $\mathcal{H}^1_\alpha(\mathbb{R}^3)$, it is natural to measure its *size* in some sense; and this is the aim of our next result. Specifically, we prove that the global attractor $\mathcal{A}_{\vec{f}}$ has a finite fractal dimension, and moreover, we derive an explicit upper bound for this dimension in terms of the parameter $\alpha, \beta, \nu$ and the norm of the external force $\|\vec{f}\|_{\mathcal{H}^1_\alpha}$. For this, we will quickly recall the definition of the fractal dimension through the so-called *box-counting* method. For more details we refer the reader to [2, 25] and the references therein.

Let $\mathcal{A}_{\vec{f}} \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$ be the global attractor associated to equation (4). Then, by the Hausdorff criterion, for every $\epsilon > 0$ it can be covered by the finite number of $\epsilon-$balls in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$. Let $N_\epsilon(\mathcal{A}_{\vec{f}})$ be the minimal number of such balls. We thus have the following definition.

**Definition 2** *The fractal (box-counting) dimension of the attarctor $\mathcal{A}_{\vec{f}}$ in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$ is defined via the following expression:*

$$dim\left(\mathcal{A}_{\vec{f}}\right) = \limsup_{\epsilon\to 0} \frac{\ln\; N_\epsilon(\mathcal{A}_{\vec{f}})}{\ln\left(\frac{1}{\epsilon}\right)}.$$

In our third result we derive an explicit upper bound of $\dim\left(\mathcal{A}_{\vec{f}}\right)$.

**Theorem 3** *Let the assumptions of Theorem (2) hold. Then the fractal dimension of the global attractor $\mathcal{A}$ associates to equation (4) satisfies the following estimate:*

$$dim\left(\mathcal{A}_{\vec{f}}\right) \leq c(\alpha,\beta,\nu)\max\left(\|\vec{f}\|^{14/5}_{\mathcal{H}^1_\alpha}, \|\vec{f}\|^2_{\mathcal{H}^1_\alpha}\right), \tag{9}$$

*where the constant $c(\alpha,\beta,\nu) > 0$, depending only of the parameters $\alpha, \beta$ and $\nu$, is explicitly given in the formula (57).*

We observe in this estimate that the fractal dimension of the global attractor $\mathcal{A}_{\vec{f}}$ is essentially controlled for above by the size of the external force $\vec{f}$ in the space $\mathcal{H}^1_\alpha(\mathbb{R}^3)$. This type of control was also pointed out in [16], for the case of the two-dimensional and damped Navier-Stokes equations, while, for the Navier-Stokes-Bardina's model in the space-periodic case, similar upper bounds on $\dim\left(\mathcal{A}_{\vec{f}}\right)$ were established in [7]. Finally, let us mention that this is a first estimation for an upper bound of $\dim\left(\mathcal{A}_{\vec{f}}\right)$ and the optimally of this upper bound, or moreover the derivation of some lower bounds, are matter of further investigations.

## Internal structure of the global attractor

We are also interested in characterizing the global attractor $\mathcal{A}_{\vec{f}}$. By [2],[3] and [24] it is well-known that the global attractor can be described through a particular kind of solution for the equations (4). Such solutions are called the *eternal solutions* which, as we will observe in the following definition, they do not arise from any initial data and they are actually defined for all time $t \in \mathbb{R}$.

**Definition 3** *Let $\vec{f} \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$ be a stationary and divergence-free external force. We say a couple $(\vec{v}, q)$ is an eternal solution for the damped Navier-Stokes-Bardina equations with force $\vec{f}$, if*

$$\vec{v} \in L^\infty_{loc}\left(\mathbb{R}, \mathcal{H}^1_\alpha(\mathbb{R}^3)\right) \cap L^2_{loc}(\mathbb{R}, \dot{H}^2(\mathbb{R}^3)), \quad q \in L^2_{loc}(\mathbb{R}, \dot{H}^3(\mathbb{R}^3)),$$

*and if $(\vec{v}, q)$ is a weak solution of the equations*

$$\partial_t \vec{v} + div((\vec{v}\otimes\vec{v})_\alpha) - \nu\Delta\vec{v} + \vec{\nabla} q = \vec{f} - \beta\vec{v}, \quad div(\vec{v}) = 0. \tag{10}$$

In Proposition 4.2 below, we prove the existence of eternal solutions for the damped Navier-Stokes-Bardina's model.

Denoting by $\mathcal{E}_{\vec{f}}$ the set of all the *bounded* eternal solutions $(\vec{v}, q)$ associated to the force $\vec{f}$, *i.e.*, the eternal solutions verifying $\vec{v} \in L^\infty(\mathbb{R}, \mathcal{H}^1_\alpha(\mathbb{R}^3))$, by Lemma 2.18 (page 16) in [24], we have that the global attractor $\mathcal{A}_{\vec{f}}$ given by the Theorem 2 has the following structure:

$$\mathcal{A}_{\vec{f}} = \mathcal{E}_{\vec{f}}\Big|_{t=0}. \tag{11}$$

In other words, the global attractor $\mathcal{A}_{\vec{f}}$ of the equations (4) consists of the set of functions $\vec{v}(0,\cdot)$, where $\vec{v}(t,\cdot)$ is an *bounded* eternal solution of the damped Navier-Stokes-Bardina equations given in Definition 3, and thus, its internal structure is explicitly described by the identity above.

On the other hand, a particular case of eternal solutions for the damped Navier-Stokes-Bardina equations are the *stationary solutions*. These solutions, which will be denoted by $(\vec{U}, P)$, only depend on the spatial variable and solve the following elliptic equation:

$$\begin{cases} -\nu\Delta\vec{U} + \text{div}((\vec{U}\otimes\vec{U})_\alpha) + \vec{\nabla}P = \vec{f} - \beta\vec{U}, \quad \alpha>0,\ \beta>0,\ \nu>0, \\ \text{div}(\vec{U}) = 0. \end{cases} \tag{12}$$

In our next result, we prove first the existence of stationary solutions, and moreover, we investigate their relation with the global attractor.

**Theorem 4** *Let $\vec{f} \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$ be the external force such that $div(\vec{f}) = 0$. Then, the following statements hold:*

1) *There exist $\vec{U} \in H^2(\mathbb{R}^3)$ and $P \in H^1(\mathbb{R}^3)$ such that the couple $(\vec{U}, P)$ is a solution of the equation (12).*

2) *All the stationary solutions verify the estimate: $\|\vec{U}\|^2_{\mathcal{H}^1_\alpha} \leq \nu\alpha^2\|\vec{U}\|^2_{\dot{H}^2} \leq \dfrac{2}{\beta^2}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha}$.*

3) *All the stationary solutions belong to the global attractor $\mathcal{A}_{\vec{f}}$.*

We have the following comments. The result given in point 1) establishes the existence of stationary solutions for the damped Navier-Stokes-Bardina equation with *any* external force, and thus, this is a general result for the elliptic equation (12) which is also of independent interest. In Section 6 we comment more in details our strategy to prove this point which is based in the Scheafer's fixed point argument. Moreover, it is worth mention the uniqueness issue for the stationary solutions, in the general case of any external force, seems to be more delicate and it is matter of further investigations.

On the other hand, in point 2) we show that *all* the stationary solutions belong to the space $H^1_\alpha(\mathbb{R}^3)$ and their norms are always controlled by the norm of the external force. Finally, maybe the most interesting feature on the stationary solutions is given in point 3), where we ensure that *all* the stationary solutions fallen inside the global attractor $\mathcal{A}_{\vec{f}}$.

### Additional properties of the global attractor driven by the damping parameter

In this section, we study the role of the damping parameter $\beta > 0$ in the description of the global attractor for the equation (4). We start by setting some notation. For the external force $\vec{f} \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$, and for a numerical constant $c > 0$, we introduce the following quantity depending on the damping parameter $\beta > 0$:

$$\eta(\beta) = \frac{c}{\alpha^{5/2}\beta}\|\vec{f}\|_{\mathcal{H}^1_\alpha} - \beta. \tag{13}$$

In our next result we prove that in the case when the parameter $\beta > 0$ is big enough, in the sense that this quantity verifies $\eta(\beta) \leq 0$, we are able to give sharp properties of the global attractor. More precisely, we will consider first the case when $\beta > 0$ is such that $\eta(\beta) = 0$. In this case, we study some kind of stability of the elements of the global attractor, also called the *orbital stability*, which its definition we recall below. For more references see the Section 1.1 of [8].

We say that $\vec{u}_0 \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$ is orbitally stable if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that for all $\vec{v}_0 \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$ verifying

$$\|\vec{u}_0 - \vec{v}_0\|_{\mathcal{H}^1_\alpha} \leq \delta,$$

then the solutions $\vec{u}(t,\cdot)$ and $\vec{v}(t,\cdot)$ to the equation (4) arising from $\vec{u}_0$ and $\vec{v}_0$ respectively, satisfy

$$\sup_{t\geq 0} \|\vec{u}(t,\cdot) - \vec{v}(t,\cdot)\|_{\mathcal{H}^1_\alpha} \leq \varepsilon.$$

On the other hand, when $\beta > 0$ is such that $\eta(\beta) < 0$, we go further in the description of the global attractor $\mathcal{A}_{\vec{f}}$. In this case, surprisingly, the global attractor contains a single element given by the *unique* solution of the stationary equation (12).

Summarizing, our result on the role of the damping parameter $\beta > 0$ in the description of the global attractor reads as follows.

**Theorem 5** *Let $\mathcal{A}_{\vec{f}} \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$ be global attractor of the equations (4) given by Theorem 2. Then, the following statements hold:*

1) *If $\beta > 0$ is such that $\eta(\beta) = 0$, then all the elements in the attractor $\mathcal{A}_{\vec{f}}$ are* orbitally stable.

2) *If $\beta > 0$ is such that $\eta(\beta) < 0$, then the stationary solution $(\vec{U}, P)$ of equation (4) given by Theorem 4 is unique. Moreover, the global attractor $\mathcal{A}_{\vec{f}}$ only contains the unique stationary solution of equation (4).*

The result given in point 2) ensures that as long as the damping parameter $\beta$ is such that $\eta(\beta) < 0$, all the weak solutions of the damped Navier-Stokes-Bardina equation (4) are attracted by the unique stationary solution of this equation when the time is large enough. In this case, we are also able to give a sharp *asymptotic profile* in time of the solutions to the equation (4).

**Proposition 2** *Within the framework of point* 2) *in Theorem 5, let $\vec{U} \in \mathcal{A}_{\vec{f}}$ be the unique solution of the stationary problem (12). Then, for all $x \in \mathbb{R}^3$ fixed, all the solutions of the equation (4) have the following* asymptotic profile *in time:*

$$\vec{u}(t,x) = \vec{U}(x) + \mathcal{R}_{\vec{u}}(t,x), \quad t > 0, \tag{14}$$

*where the remainder term $\mathcal{R}_{\vec{u}}(t,x)$ is a vector field depending on $\vec{u}$, which verify the following time decaying*

$$\|\mathcal{R}_{\vec{u}}(t,\cdot)\|_{L^\infty} \leq C\, t^{-3/4}, \quad t \gg 1, \tag{15}$$

*with a constant $C > 0$ depending on the the parameters $\alpha, \beta, \nu$, the initial value $\vec{u}(0,\cdot)$, and $\vec{f}$, $\vec{U}$.*

### The damped Navier-Stokes-Bardina's model without external force

Finally, in this last part we consider the damped Navier-Stokes-Bardina equations in the particular case of a zero external force.

$$\begin{cases} \partial_t \vec{u} + \text{div}((\vec{u}\otimes\vec{u})_\alpha) - \nu\Delta\vec{u} + \vec{\nabla}p = -\beta\vec{u}, \quad \alpha > 0,\ \beta > 0,\ \nu > 0, \\ \text{div}(\vec{u}) = 0, \\ \vec{u}(0,\cdot) = \vec{u}_0, \quad \text{div}(\vec{u}_0) = 0. \end{cases} \tag{16}$$

In this case, we give a sharp description of the global attractor associated with the zero force, more precisely, we show that the global attractor of this equation only contains the zero function. Moreover, we prove that all the weak solutions to the equation (16) have a fast (exponential) convergence rate to zero when the time goes to infinity.

**Proposition 3** *For the equation (16), the unique global attractor verifies $\mathcal{A}_0 = \{0\}$. Moreover, all the weak solutions $u(t,x)$ given by Theorem 1 verify:*

$$\|\vec{u}(t,\cdot)\|_{L^p} \leq C\, e^{-\frac{2\beta}{p}t}, \quad 2 \leq p < +\infty, \quad t \gg 1,$$

*where the constant $C > 0$ depends on the initial datum $\vec{u}(0,\cdot)$ and the parameter $p$.*

# 3 Global well-posedness in the energy space

## Proof of Theorem 1

The proof of this theorem is rather straightforward and it follows essentially the same lines of the classical framework. The first step is to solve the following integral problem:

$$\vec{u}(t,\cdot) = e^{\nu t\Delta}\vec{u}_0 + \int_0^t e^{\nu(t-s)\Delta}\vec{f}ds - \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha)(s,\cdot)ds - \beta\int_0^t e^{\nu(t-s)\Delta}\vec{u}(s,\cdot)ds, \tag{17}$$

in the energy space $E_T = L^\infty([0,T], H^1(\mathbb{R}^3)) \cap L^2([0,T], \dot{H}^2(\mathbb{R}^3))$ (with $0 < T < +\infty$) with the norm $\|\cdot\|_T = \|\cdot\|_{L^\infty_t H^1_x} + \|\cdot\|_{L^2_t \dot{H}^2_x}$. We write

$$\|\vec{u}(t,\cdot)\|_T \leq \underbrace{\left\| e^{\nu t\Delta}\vec{u}_0 + \int_0^t e^{\nu(t-s)\Delta}\vec{f}ds\right\|_T}_{(a)} + \underbrace{\left\|\int_0^t e^{\nu(t-s)\Delta}\mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha)(s,\cdot)ds\right\|_T}_{(b)} + \beta\underbrace{\left\|\int_0^t e^{\nu(t-s)\Delta}\vec{u}(s,\cdot)ds\right\|_T}_{(c)}.$$

The term $(a)$ is classical to treat. For the term $e^{\nu t\Delta}\vec{u}_0$ we have the following estimate (see the proof of Theorem 7.1, page 131 of the book [21]):

$$\|e^{\nu t\Delta}\vec{u}_0\|_{E_T} \leq c(1 + 1/\sqrt{2\nu})\|\vec{u}_0\|_{H^1}. \tag{18}$$

Moreover, to study the term $\displaystyle\int_0^t e^{\nu(t-s)\Delta}\vec{f}ds$, we shall use the following well-known estimates. See the Lemma 7.2, page 129 of the book [21].

**Lemma 3.1** *Let $g \in L^2([0,T], L^2(\mathbb{R}^3))$ and let $G(t,x) = \displaystyle\int_0^t e^{\nu(t-s)\Delta}g(s,x)ds$. Then, $G(t,x)$ belongs to the space $E_T$ and we have the following estimates:*

1) $\|G(t,\cdot)\|_{L^\infty_t L^2_x} \leq c\sqrt{T}\,\|g\|_{L^2_t L^2_x}$.

2) $\|G(t,\cdot)\|_{L^\infty_t \dot{H}^1_x} \leq \dfrac{c}{\sqrt{2\nu}}\,\|g\|_{L^2_t L^2_x}$.

3) $\|G(t,\cdot)\|_{L^2_t \dot{H}^2_x} \leq \dfrac{c}{\nu}\,\|g\|_{L^2_t L^2_x}$.

Thus, in this lemma we set $g = \vec{f}$, and moreover, since $\vec{f}$ is a time-independing function then we have

$$\begin{aligned}\left\|\int_0^t e^{\nu(t-s)\Delta}\vec{f}ds\right\|_{E_T} &\leq c(\sqrt{T} + 1/\sqrt{2\nu} + 1/\nu)\|\vec{f}\|_{L^2_t L^2_x} \leq c(\sqrt{T} + 1/\sqrt{2\nu} + 1/\nu)\sqrt{T}\|\vec{f}\|_{L^2} \\ &\leq c(\sqrt{T} + 1/\sqrt{2\nu} + 1/\nu)\sqrt{T}\|\vec{f}\|_{H^1}.\end{aligned} \tag{19}$$

At this point, we set the time $T \leq 1$ and then by the estimates (18) and (19) we can write

$$\left\| e^{\nu t\Delta}\vec{u}_0 + \int_0^t e^{\nu(t-s)\Delta}\vec{f}ds\right\|_T \leq c(1 + 1/\sqrt{2\nu} + 1/\nu)(\|\vec{u}_0\|_{H^1} + \|\vec{f}\|_{H^1}) \leq c_\nu(\|\vec{u}_0\|_{H^1} + \|\vec{f}\|_{H^1}). \tag{20}$$

We study now the term $(b)$. We recall first that by (2) we have $(\vec{u}\otimes\vec{u})_\alpha = (-\alpha^2\Delta + I_d)(\vec{u}\otimes\vec{u})$ and then, by well-known properties of the Bessel potential $(-\alpha^2\Delta + I_d)^{-1}$, we can write $(-\alpha^2\Delta + I_d)^{-1}(\vec{u}\otimes\vec{u}) = K_\alpha * (\vec{u}\otimes\vec{u})$, where the kernel $K_\alpha(x)$ has good decaying properties. In particular we have $\|K_\alpha\|_{L^1} \leq c_\alpha$, for a constant $0 < c_\alpha < +\infty$ depending on $\alpha > 0$ (see the Section 6.1.2 of the book [14] for all the details).

With this remark in mind, we start by estimating the quantity $\left\| \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha)(s,\cdot)ds \right\|_{L^\infty_t L^2_x}$. For $0 < t \leq T$ fix, by the Young inequalities, and moreover, by the continuity of the Leray projector in the Lebesgue spaces, we write

$$
\begin{aligned}
&\left\| \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha)(s,\cdot)ds \right\|_{L^2} \leq \int_0^t \|e^{\nu(t-s)\Delta}\mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha)(s,\cdot)\|_{L^2}ds\\
\leq & \int_0^t \|e^{\nu(t-s)\Delta}\mathbb{P}div(K_\alpha*(\vec{u}\otimes\vec{u}))(s,\cdot)\|_{L^2}ds \leq \int_0^t \left\| K_\alpha * \left(e^{\nu(t-s)\Delta}\mathbb{P}div(\vec{u}\otimes\vec{u})\right)(s,\cdot)\right\|_{L^2} ds\\
\leq & \; c_\alpha \int_0^t \left\| e^{\nu(t-s)\Delta}\mathbb{P}div(\vec{u}\otimes\vec{u})(s,\cdot)\right\|_{L^2} ds \leq c_{\nu,\alpha}\int_0^t \|div(\vec{u}\otimes\vec{u})(s,\cdot)\|_{L^2}\, ds\\
\leq & \; c_{\nu,\alpha}\int_0^t \|\vec{u}\otimes\vec{u}(s,\cdot)\|_{\dot{H}^1}ds \leq c_{\nu,\alpha}\, T^{1/2}\, \|\vec{u}\otimes\vec{u}\|_{L^2_t\dot{H}^1_x}.
\end{aligned}
$$

Here we shall need the following technical lemma.

**Lemma 3.2** *We have* $\|\vec{u}\otimes\vec{u}\|_{L^2_t\dot{H}^1_x} \leq c\,T^{1/4}\,\|\vec{u}\|^2_{E_T}$.

**Proof.** Using first the product laws in the homogeneous Sobolev spaces (see the Lemma 7.3, page 130 of the book [21]) and then, using the Hölder inequalities in the time variable we get $\|\vec{u}\otimes\vec{u}\|_{L^2_t\dot{H}^1_x} \leq c\,\|\vec{u}\|_{L^4_t\dot{H}^{3/2}_x}\|\vec{u}\|_{L^4_t\dot{H}^1_x}$. Thereafter, to estimate the quantity $\|\vec{u}\|_{L^4_t\dot{H}^{3/2}_x}$, we use the interpolation inequalities (first in the spatial variable and then in the temporal variable) and we have $\|\vec{u}\|_{L^4_t\dot{H}^{3/2}_x} \leq c\|\vec{u}\|^{1/2}_{L^\infty_t\dot{H}^1_x}\|\vec{u}\|^{1/2}_{L^2_t\dot{H}^2_x} \leq c\|\vec{u}\|_{E_T}$. Finally, the quantity $\|\vec{u}\|_{L^4_t\dot{H}^1_x}$ is directly estimated as follows: $\|\vec{u}\|_{L^4_t\dot{H}^1_x} \leq c\,T^{1/4}\|\vec{u}\|_{L^\infty_t\dot{H}^1_x} \leq c\,T^{1/4}\|\vec{u}\|_{E_T}$. ■

With this estimate at hand, and recalling that we have assumed $T \leq 1$, we can write

$$
\left\| \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha)(s,\cdot)ds \right\|_{L^\infty_t L^2_x} \leq c_{\nu,\alpha}\, T^{1/4}\|\vec{u}\|^2_{E_T}. \tag{21}
$$

We study now the quantity $\left\| \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha)(s,\cdot)ds \right\|_{L^\infty_t \dot{H}^1_x}$. By point 2) in Lemma 3.1, where we set now $g = \mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha)$, and moreover, by Lemma 3.2 we can write

$$
\begin{aligned}
\left\| \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha)(s,\cdot)ds \right\|_{L^\infty_t \dot{H}^1} &\leq c_\nu\,\|\mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha\|_{L^2_tL^2_x} \leq c_\nu\,\|K_\alpha*(\mathbb{P}div(\vec{u}\otimes\vec{u}))\,\|_{L^2_tL^2_x}\\
&\leq c_{\nu,\alpha}\,\|\vec{u}\otimes\vec{u}\|_{L^2_t\dot{H}^1_x} \leq c_{\nu,\alpha}\,T^{1/4}\,\|\vec{u}\|^2_{E_T}.
\end{aligned} \tag{22}
$$

Finally, we study the quantity $\left\| \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha)(s,\cdot)ds \right\|_{L^2_t \dot{H}^2_x}$. As the previous quantity, by point 3) in Lemma 3.1 and moreover by Lemma 3.2 we have

$$
\left\| \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha)(s,\cdot)ds \right\|_{L^2_t \dot{H}^2} \leq c_\nu\,\|\mathbb{P}div((\vec{u}\otimes\vec{u})_\alpha\|_{L^2_tL^2_x} \leq c_{\nu,\alpha}\,T^{1/4}\,\|\vec{u}\|^2_{E_T}. \tag{23}
$$

Thus, gathering the estimates (21), (22) and (23) we obtain

$$\left\| \int_0^t e^{\nu(t-s)\Delta} \mathbb{P} div((\vec{u}\otimes\vec{u})_\alpha)(s,\cdot)ds \right\|_{E_T} \leq c_{\nu,\alpha}\, T^{1/4}\, \|\vec{u}\|^2_{E_T}. \tag{24}$$

It remains to estimate the term $(c)$. By Lemma 3.1 (where we set now $g=\vec{u}$, and in point 1) we recall that $T\leq 1$) we write

$$\left\| \int_0^t e^{\nu(t-s)\Delta} \vec{u}(s,\cdot)ds \right\|_{E_T} \leq c_\nu\, \|\vec{u}\|_{L^2_t L^2_t} \leq c_\nu\, T^{1/2}\, \|\vec{u}\|_{E_T}. \tag{25}$$

Once we have inequalities (20), (24) and (25), for a time $T>0$ small enough by the Banach contraction principle we obtain a local solution $\vec{u}\in E_T$ of equations (17).

The second step is to prove that this solution is global in time. Remark first that the solution $\vec{u}$ obtained above also solves the problem

$$\partial_t \vec{u} + \mathbb{P}(\mathrm{div}((\vec{u}\otimes\vec{u})_\alpha)) - \nu\Delta\vec{u} = \vec{f} - \beta\vec{u},$$

in the distributional sense, where, as $\vec{u}\in E_T$ then each term in this equation belong to the space $L^2([0,T],L^2(\mathbb{R}^3))$. By the identity (2) we can write

$$\partial_t \vec{u} + \mathbb{P}(\mathrm{div}((-\alpha^2\Delta + I_d)^{-1}(\vec{u}\otimes\vec{u})) - \nu\Delta\vec{u} = \vec{f} - \beta\vec{u}, \tag{26}$$

and applying the operator $(-\alpha\Delta + I_d)$ in each term we get that the solution $\vec{u}$ also verifies the following equation

$$(-\alpha^2\Delta + I_d)\partial_t\vec{u} = -\mathbb{P}(\mathrm{div}(\vec{u}\otimes\vec{u})) + \nu(-\alpha^2\Delta+I_d)\Delta\vec{u} + (-\alpha^2\Delta+I_d)\vec{f} - \beta(-\alpha^2\Delta+I_d)\vec{u}.$$

Here each term belong to the space $L^2([0,T],H^{-2}(\mathbb{R}^3))$. Now, always by the fact $\vec{u}\in E_T$ we get $\vec{u}\in L^2([0,T],H^2(\mathbb{R}^3))$ and then we can write

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left( \|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right) =& \langle(-\alpha^2\Delta+I_d)\partial_t\vec{u}(t,\cdot),\vec{u}(t,\cdot)\rangle_{H^{-2}\times H^2} \\ =& -\nu\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2} + \langle\vec{f},\vec{u}(t,\cdot)\rangle_{L^2\times L^2} \\ &+ \alpha^2\langle\vec{\nabla}\otimes\vec{f},\vec{\nabla}\otimes\vec{u}(t,\cdot)\rangle_{L^2\times L^2} - \beta\left(\|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right). \end{aligned} \tag{27}$$

As the quantity $-\beta\left(\|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right)$ is negative, and moreover, applying the Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left( \|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right) \\ &\leq -\nu\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2} + \langle\vec{f},\vec{u}(t,\cdot)\rangle_{L^2\times L^2} + \alpha^2\langle\vec{\nabla}\otimes\vec{f},\vec{\nabla}\otimes\vec{u}(t,\cdot)\rangle_{L^2\times L^2} \\ &\leq -\nu\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2} + \|\vec{f}\|_{L^2}\|\vec{u}(t,\cdot)\|_{L^2} + \alpha^2\|\vec{f}\|_{\dot{H}^1}\|\vec{u}(t,\cdot)\|_{\dot{H}^1} \\ &\leq -\nu\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2} + \|\vec{f}\|^2_{L^2} + \|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} \\ &\leq \left(\|\vec{u}(t,\cdot)\|_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right) - \nu\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2} + \|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}. \end{aligned}$$

Then, applying the Grönwall inequalities, for all $t\in[0,T]$ we have

$$\begin{aligned} \|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} \leq& \left(\|\vec{u}_0\|^2_{L^2} + \alpha^2\|\vec{u}_0\|^2_{\dot{H}^1}\right) e^{2t} + \int_0^t e^{2(t-s)}\left(-2\nu\|\vec{u}(s,\cdot)\|^2_{\dot{H}^1} - 2\alpha^2\|\vec{u}(s,\cdot)\|^2_{\dot{H}^2}\right) ds \\ &+ \int_0^t e^{2(t-s)}\left(2\|\vec{f}\|^2_{L^2} + 2\alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right) ds, \end{aligned}$$

hence we get

$$\|\vec{u}(t,\cdot)\|_{L^2}^2 + alpha^2\|\vec{u}(t,\cdot)\|_{\dot{H}^1}^2 \leq \left(\|\vec{u}_0\|_{L^2}^2 + \alpha^2\|\vec{u}_0\|_{\dot{H}^1}^2\right) e^{2t} - \int_0^t \left(2\nu\|\vec{u}(s,\cdot)\|_{\dot{H}^1}^2 + 2\alpha^2\|\vec{u}(s,\cdot)\|_{\dot{H}^2}^2\right) ds$$
$$+ e^{2t}t\left(\|\vec{f}\|_{L^2}^2 + \alpha^2\|\vec{f}\|_{\dot{H}^1}^2\right),$$

and we thus obtain the following control in time

$$\begin{aligned}\|\vec{u}(t,\cdot)\|_{L^2}^2 + \alpha^2\|\vec{u}(t,\cdot)\|_{\dot{H}^1}^2 + \int_0^t \left(\nu\|\vec{u}(s,\cdot)\|_{\dot{H}^1}^2 + \alpha^2\|\vec{u}(s,\cdot)\|_{\dot{H}^2}^2\right) ds \\ \leq e^{2t}\left(\|\vec{u}_0\|_{L^2}^2 + \alpha^2\|\vec{u}_0\|_{\dot{H}^1}^2\right) + e^{2t}t\left(\|\vec{f}\|_{L^2}^2 + t\,\alpha^2\|\vec{f}\|_{\dot{H}^1}^2\right).\end{aligned} \tag{28}$$

which allows us to extend the local solution $\vec{u}$ to the interval $[0,+\infty[$.

The third step is to obtain the global energy equality (5). It directly follows by integrating the identity (27) on the interval of time $[0,t]$.

The fourth step is to recover the pressure $p$ which is always related to the velocity $\vec{u}$. Indeed, as $\vec{u}$ verifies the equation (26), and moreover, as we have $div(\vec{u}) = 0$ and $div(\vec{f}) = 0$, then we can write

$$\mathbb{P}\left(\partial_t\vec{u} + \text{div}((-\alpha^2\Delta + I_d)^{-1}(\vec{u}\otimes\vec{u})) - \nu\Delta\vec{u} - \vec{f} + \beta\vec{u}\right) = 0,$$

hence, by the well-known properties of the Leray projector $\mathbb{P}$ (see the Lemma 6.3, page 118 of the book [21]) there exists $p \in \mathcal{D}'([0,+\infty[\times\mathbb{R}^3)$ such that we have

$$\partial_t\vec{u} + \text{div}((-\alpha^2\Delta + I_d)^{-1}(\vec{u}\otimes\vec{u})) - \nu\Delta\vec{u} - \vec{f} + \beta\vec{u} = \vec{\nabla}p.$$

Applying the divergence operator in each term of this identity, and moreover, denoting by $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ the Riesz transforms, we get the following identity

$$p = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_i\left((-\alpha^2\Delta + I_d)^{-1}(u_iu_j)\right). \tag{29}$$

By this identity, the pressure term verifies $p \in L^2_{loc}([0,+\infty[, H^3(\mathbb{R}^3))$. Indeed, we shall quickly verify that $\vec{u}\otimes\vec{u} \in (L^2_t)_{loc}\, H^1_x$. By Lemma 3.2 we have $\vec{u}\otimes\vec{u} \in (L^2_t)_{loc}\dot{H}^1_x$. On the other hand, as $\vec{u} \in (L^\infty_t)_{loc}\, H^1_x$ then by the product laws in the non-homogeneous Sobolev spaces (see always the Lemma 7.3, page 130 of the book [21]) we get $\vec{u}\otimes\vec{u} \in (L^\infty_t)_{loc}\, H^{1/2}_x$, hence we have $\vec{u}\otimes\vec{u} \in (L^2_t)_{loc}L^2_x$ and thereafter, joint with the fact that $\vec{u}\otimes\vec{u} \in (L^2_t)_{loc}\, \dot{H}^1_x$ we finally obtain $\vec{u}\otimes\vec{u} \in (L^2_t)_{loc}H^1_x$.

The fifth and last step is to prove the uniqueness of solutions. So, let $(\vec{u}_1, p_1) \in (L^\infty_t)_{loc}H^1_x \cap (L^2_t)_{loc}\dot{H}^2_x \times (L^2_t)_{loc}H^3_x$ and $(\vec{u}_2, p_2) \in (L^\infty_t)_{loc}H^1_x \cap (L^2_t)_{loc}\dot{H}^2_x \times (L^2_t)_{loc}H^3_x$ be two solutions of equation (4) arising from the initial data $\vec{u}_{0,1}$ and $\vec{u}_{0,2}$ respectively. We define $\vec{w} = \vec{u}_1 - \vec{u}_2$ and $q = p_1 - p_2$, and then we get that the couple $(\vec{w}, q)$ verifies the equation

$$\partial_t\vec{w} + \left((\vec{w}\cdot\vec{\nabla})\vec{u}_1 + (\vec{u}_2\cdot\vec{\nabla})\vec{w}\right)_\alpha - \nu\Delta\vec{w} + \vec{\nabla}q = -\beta\vec{w}, \quad \vec{w}(0,\cdot) = \vec{u}_1(0,\cdot) - \vec{u}_2(0,\cdot).$$

Following the computations done in (27) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\vec{w}(t,\cdot)\|_{L^2}^2 + \alpha^2\|\vec{w}(t,\cdot)\|_{\dot{H}^1}^2\right) + \nu\|\vec{w}(t,\cdot)\|_{\dot{H}^1}^2 + \alpha^2\|\vec{w}(t,\cdot)\|_{\dot{H}^2}^2 = -\beta\left(\|\vec{w}(t,\cdot)\|_{L^2}^2 + \alpha^2\|\vec{w}(t,\cdot)\|_{\dot{H}^1}^2\right) \\ - \left\langle (\vec{w}\cdot\vec{\nabla})\vec{u}_1(t,\cdot), \vec{w}(t,\cdot)\right\rangle_{\dot{H}^{-1}\times\dot{H}^1},\end{aligned} \tag{30}$$

where we must estimate the term $\left\langle (\vec{w}\cdot\vec{\nabla})\vec{u}_1(t,\cdot),\vec{w}(t,\cdot)\right\rangle_{\dot{H}^{-1}\times\dot{H}^1}$. For this, we use the Hardy-Littlewood-Sobolev inequalities and the Hölder inequalities to write

$$\begin{aligned}\left|\left\langle (\vec{w}\cdot\vec{\nabla})\vec{u}_1(t,\cdot),\vec{w}(t,\cdot)\right\rangle_{\dot{H}^{-1}\times\dot{H}^1}\right| &\leq c\|(\vec{w}\cdot\vec{\nabla})\vec{u}_1(t,\cdot)\|_{\dot{H}^{-1}}\|\vec{w}(t,\cdot)\|_{\dot{H}^1}\leq c\|(\vec{w}\cdot\vec{\nabla})\vec{u}_1(t,\cdot)\|_{L^{6/5}}\|\vec{w}(t,\cdot)\|_{\dot{H}^1}\\ &\leq c\|\vec{w}(t,\cdot)\|_{L^2}\|\vec{\nabla}\otimes\vec{u}_1(t,\cdot)\|_{L^3}\,\|\vec{w}(t,\cdot)\|_{\dot{H}^1}=(a).\end{aligned}$$

To treat the term $\|\vec{\nabla}\otimes\vec{u}_1(t,\cdot)\|_{L^3}$ in expression (a), we apply first the interpolation inequalities, and thereafter, by the Hardy-Littlewood-Sobolev we obtain

$$\begin{aligned}(a)\leq& c\,\|\vec{w}(t,\cdot)\|_{L^2}\,\|\vec{\nabla}\otimes\vec{u}_1(t,\cdot)\|^{1/2}_{L^2}\|\vec{\nabla}\otimes\vec{u}_1(t,\cdot)\|^{1/2}_{L^6}\,\|\vec{w}(t,\cdot)\|_{\dot{H}^1}\\ \leq& c\,\|\vec{w}(t,\cdot)\|_{L^2}\,\|\vec{\nabla}\otimes\vec{u}_1(t,\cdot)\|^{1/2}_{L^2}\|\vec{\nabla}\otimes\vec{u}_1(t,\cdot)\|^{1/2}_{\dot{H}^1}\,\|\vec{w}(t,\cdot)\|_{\dot{H}^1}\\ \leq& c\,\|\vec{w}(t,\cdot)\|_{L^2}\,\|\vec{u}_1(t,\cdot)\|^{1/2}_{\dot{H}^1}\|\vec{u}_1(t,\cdot)\|^{1/2}_{\dot{H}^2}\,\|\vec{w}(t,\cdot)\|_{\dot{H}^1}\\ \leq& \frac{c}{\nu}\|\vec{w}(t,\cdot)\|^2_{L^2}\,\|\vec{u}_1(t,\cdot)\|_{\dot{H}^1}\|\vec{u}_1(t,\cdot)\|_{\dot{H}^2}+\frac{\nu}{2}\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}\\ \leq& \frac{c}{\nu}\left(\|\vec{w}(t,\cdot)\|^2_{L^2}+\alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}\right)\,\|\vec{u}_1(t,\cdot)\|_{\dot{H}^1}\|\vec{u}_1(t,\cdot)\|_{\dot{H}^2}+\frac{\nu}{2}\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}.\end{aligned}$$

With this estimate at hand, we get back to (30) where we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\vec{w}(t,\cdot)\|^2_{L^2}+\alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}\right)+\nu\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}+\alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^2}\leq -\beta\left(\|\vec{w}(t,\cdot)\|^2_{L^2}+\alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}\right)\\ +\frac{c}{\nu}\left(\|\vec{w}(t,\cdot)\|^2_{L^2}+\alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}\right)\,\|\vec{u}_1(t,\cdot)\|_{\dot{H}^1}\|\vec{u}_1(t,\cdot)\|_{\dot{H}^2}+\frac{\nu}{2}\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1},\end{aligned}$$

hence we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\vec{w}(t,\cdot)\|^2_{L^2}+\alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}\right)\leq\frac{c}{\nu}\left(\|\vec{w}(t,\cdot)\|^2_{L^2}+\alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}\right)\,\|\vec{u}_1(t,\cdot)\|_{\dot{H}^1}\|\vec{u}_1(t,\cdot)\|_{\dot{H}^2}.$$

Thus, applying the Grönwall inequalities we have

$$\|\vec{w}(t,\cdot)\|^2_{L^2}+\alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}\leq\left(\|\vec{w}(0,\cdot)\|^2_{L^2}+\alpha^2\|\vec{w}(0,\cdot)\|^2_{\dot{H}^1}\right)\,e^{\frac{c}{\nu}\int_0^t\|\vec{u}_1(s,\cdot)\|_{\dot{H}^1}\|\vec{u}_1(s,\cdot)\|_{\dot{H}^2}\,ds}.$$

Moreover, using (28) the term $\frac{c}{\nu}\int_0^t\|\vec{u}_1(s,\cdot)\|_{\dot{H}^1}\|\vec{u}_1(s,\cdot)\|_{\dot{H}^2}\,ds$ is estimated as follows:

$$\begin{aligned}&\frac{c}{\nu}\int_0^t\|\vec{u}_1(s,\cdot)\|_{\dot{H}^1}\|\vec{u}_1(s,\cdot)\|_{\dot{H}^2}\,ds\leq c_{\nu,\alpha}\left(\nu\int_0^t\|\vec{u}_1(s,\cdot)\|^2_{\dot{H}^1}ds+\alpha^2\int_0^t\|\vec{u}_1(s,\cdot)\|^2_{\dot{H}^2}ds\right)\\ \leq& c_{\nu,\alpha}\left(e^{2t}\left(\|\vec{u}_{0,1}\|^2+\alpha^2\|\vec{u}_{0,1}\|^2_{\dot{H}^1}\right)+e^{2t}t\left(\|\vec{f}\|^2_{L^2}+t\,\alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right)\right)=c_2(\alpha,\nu,\vec{f},\vec{u}_{0,1},t).\end{aligned}$$

With this estimate, we can write the following inequality:

$$\|\vec{w}(t,\cdot)\|^2_{L^2}+\alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}\leq\left(\|\vec{w}(0,\cdot)\|^2_{L^2}+\alpha^2\|\vec{w}(0,\cdot)\|^2_{\dot{H}^1}\right)\,e^{c_2(\alpha,\nu,\vec{f},\vec{u}_{0,1},t)}. \tag{31}$$

Uniqueness of solutions directly follows from this inequality. Theorem 1 is now proven. ■

**Proof of Proposition 1**

1) The control (6) directly follows from the identity (27). Indeed, by the Cauchy-Schwarz inequalities, and as $-\nu\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2}$ is negative quantity, we write

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\vec{u}(t,\cdot)\|^2 + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right)\\
\leq& -\nu\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2} + \|\vec{f}\|_{L^2}\|\vec{u}(t,\cdot)\|_{L^2} + \alpha^2\|\vec{f}\|_{\dot{H}^1}\|\vec{u}(t,\cdot)\|_{\dot{H}^1} - \beta\left(\|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right)\\
\leq& -\nu\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2} + \frac{2}{\beta}\|\vec{f}\|^2_{L^2} + \frac{\beta}{2}\|\vec{u}(t,\cdot)\|^2_{L^2} + \frac{2\alpha^2}{\beta}\|\vec{f}\|^2_{\dot{H}^1} + \frac{\beta\alpha^2}{2}\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\\
&-\beta\left(\|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right)\\
\leq& \frac{2}{\beta}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right) - \frac{\beta}{2}\left(\|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right).
\end{aligned}
$$

Then, applying the Grönwall inequalities we have

$$
\|\vec{u}(t,\cdot)\|^2 + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} \leq \left(\|\vec{u}_0\|^2 + \alpha^2\|\vec{u}_0\|^2_{\dot{H}^1}\right)e^{-\beta t} + \frac{4}{\beta}\int_0^t e^{-\beta(t-s)}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right)ds,
$$

hence, the desired control (6) follows.

2) Always by the identity (27) and the Cauchy-Schwarz inequalities, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\vec{u}(t,\cdot)\|^2 + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right)\\
\leq& -\nu\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2} + \|\vec{f}\|_{L^2}\|\vec{u}(t,\cdot)\|_{L^2} + \alpha^2\|\vec{f}\|_{\dot{H}^1}\|\vec{u}(t,\cdot)\|_{\dot{H}^1} - \beta\left(\|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right)\\
\leq& -\nu\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2} + \frac{2}{\beta}\|\vec{f}\|^2_{L^2} + \frac{\beta}{2}\|\vec{u}(t,\cdot)\|^2_{L^2} + \frac{2\alpha^2}{\beta}\|\vec{f}\|^2_{\dot{H}^1} + \frac{\beta\alpha^2}{2}\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\\
&-\beta\left(\|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right)\\
\leq& -\nu\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2} + \frac{2}{\beta}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right) - \frac{\beta}{2}\left(\|\vec{u}(t,\cdot)\|^2 + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right)\\
\leq& -\nu\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2} + \frac{2}{\beta}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right).
\end{aligned}
$$

Then, integrating in the interval $[t, t+T]$ we get

$$
\begin{aligned}
\|\vec{u}(t+T,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t+T,\cdot)\|^2_{\dot{H}^1} \leq& -\nu\int_t^{t+T}\|\vec{u}(s,\cdot)\|^2_{\dot{H}^1}ds - \alpha^2\int_t^{t+T}\|\vec{u}(s,\cdot)\|^2_{\dot{H}^2}ds\\
&+\frac{2T}{\beta}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right) + \|\vec{u}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1},
\end{aligned}
$$

hence we have the desired control (7) ■

# 4 The global attractor

To prove the existence of a global attractor associated to the equation (4), stated in Theorem 2, we will use some results arising from the theory of dynamical systems which, for completeness of the paper and the reader's convenience, we shall state below.

We will start by recalling that for the parameter $\alpha > 0$ fixed, we consider the Banach space $\mathcal{H}^1_\alpha(\mathbb{R}^3) = \left\{ \vec{g} \in H^1(\mathbb{R}^3) : \; div(\vec{g}) = 0, \quad \|\vec{g}\|^2_{\mathcal{H}^1_\alpha} = \|\vec{g}\|^2_{L^2} + \alpha^2 \|\vec{g}\|^2_{\dot{H}^1} < +\infty \right\}$. Thereafter, for $t \geq 0$ let $S(t) : \mathcal{H}^1_\alpha(\mathbb{R}^3) \to \mathcal{H}^1_\alpha(\mathbb{R}^3)$ be the semigruop associated with equation (4) and defined in (8).

Before to state a general result leading the existence of a global attractor for the semigroup $S(t)$, we introduce the following definitions we shall need later.

**Definition 4.1 (Absorbig set)** *A closed set $\mathcal{B} \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$ is an absorbing set for the semigroup $S(t)$ if for every bounded set $B \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$, there exists a time $T = T(B) > 0$ such that, for all $t > T$ we have $S(t)B \subset \mathcal{B}$.*

In this definition, it is worth mention we use the notation $S(t)B = \{S(t)\vec{u}_0 : \vec{u}_0 \in B\}$.

**Definition 4.2 (Semigroup asymptotically compact)** *The semigroup $S(t)$ is asymptotically compact if for any bounded sequence $(\vec{u}_{0,n})_{n\in\mathbb{N}}$ in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$, and moreover, for any sequence of times $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to \infty$, the sequence $(S(t_n)\vec{u}_{0,n})_{n\in\mathbb{N}}$ is precompact in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$.*

Once we have introduced these definitions, we are able to state the following result on the existence of a global attractor. For a proof of this fact see [25].

**Proposition 4.1** *Suppose that:*

1. *The semigroup $S(t)$ has a bounded and closed absorbing set $\mathcal{B} \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$ given in Definition 4.1.*
2. *The semigroup $S(t)$ is asymptotically compact in the sense of Definition 4.2.*
3. *For every fixed $t \geq 0$ the map $S(t) : \mathcal{B} \to \mathcal{H}^1_\alpha(\mathbb{R}^3)$ is continuous.*

*Then, the semigroup $S(t)$ has a global attractor $\mathcal{A}_{\vec{f}} \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$ given in Definition 1.*

## Proof of Theorem 2

We will prove that the semigroup $S(t)$ associated to equation (4) and defined in (8) verify the points 1., 2. and 3. in Proposition 4.1. We start by verifying the point 1 with the following result.

**Lemma 4.1** *Let $\mathcal{B} = \left\{ \vec{u}_0 \in \mathcal{H}^1_\alpha(\mathbb{R}^3) : \|\vec{u}_0\|^2_{\mathcal{H}^1_\alpha} \leq \frac{8}{\beta^2}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha} \right\}$. Then, $\mathcal{B}$ is a absorbing set in the sense of Definition 4.1.*

**Proof.** We observe that $\mathcal{B}$ is a bounded and closed set in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$ and we will prove that $\mathcal{B}$ is moreover an absorbing set. Indeed, let $B \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$ be a bounded set. Then, for $R > 0$ (large enough) we have $\|\vec{u}_0\|^2_{\mathcal{H}^1_\alpha} \leq R^2$ for all $\vec{u}_0 \in B$. On the other hand, by point 1) in Proposition 1, for all $\vec{u}_0 \in \mathcal{B}$ we have

$$\|S(t)\vec{u}_0\|^2_{\mathcal{H}^1_\alpha} \leq \|\vec{u}_0\|^2_{\mathcal{H}^1_\alpha} e^{-\beta t} + \frac{4}{\beta^2}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha} \leq R^2 e^{-\beta t} + \frac{4}{\beta^2}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha}.$$

Here, we set a time $T = T(B) > 0$ such that for all $t > T$ we have $R^2 e^{-\beta t} \leq \frac{4}{\beta^2}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha}$, and then, and for all $\vec{u}_0 \in \mathcal{B}$ we get $S(t)\vec{u}_0 \in \mathcal{B}$. ∎

We verify now the point 2. Let $(\vec{u}_{0,n})_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$, and moreover, let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive times such that $t_n \to +\infty$. We must show that the sequence $(S(t_n)\vec{u}_{0,n})_{n\in\mathbb{N}}$ is precompact in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$ and for this, we will use a energy method. For each $n\in\mathbb{N}$, we consider the following Cauchy problem for the equation (4):

$$\left\{\begin{array}{l} \partial_t \vec{u}_n + \text{div}\,((\vec{u}_n\otimes\vec{u}_n)_\alpha) - \nu\Delta\vec{u}_n + \vec{\nabla}p_n = \vec{f} - \beta\vec{u}_n, \qquad \text{div}(\vec{u}_n)=0, \\ \vec{u}_n(-t_n,\cdot) = \vec{u}_{0,n}. \end{array}\right. \tag{32}$$

Let $\vec{u}_n : [-t_n,+\infty[\times\mathbb{R}^3 \to \mathbb{R}^3$ and $p_n : [-t_n,+\infty[\times\mathbb{R}^3 \to \mathbb{R}$, be the unique solution of this equation given by Theorem 1, where it verifies $\vec{u}_n \in L^\infty([-t_n,+\infty[, H^1(\mathbb{R}^3)) \cap L^2_{loc}([-t_n,+\infty[, \dot{H}^2(\mathbb{R}^3))$ and $p_n \in L^2_{loc}([-t_n,+\infty[, H^3(\mathbb{R}^3))$.

Now, by uniqueness of solution $\vec{u}_n$, and moreover, by definition of the semigroup $S(t)$ associated with the equation (32), defined in (8), for all $n\in\mathbb{N}$ we have the identity $S(t_n)\vec{u}_{0,n} = \vec{u}_n(0,\cdot)$ and thus, it is enough to verify that the sequence $(\vec{u}_n(0,\cdot))_{n\in\mathbb{N}}$ is precompact in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$. For this, our general strategy is the following one. First, we prove the existence of a solution $(\vec{v},q)$ to the equation (10), called the eternal given in Definition 3. Then, we will show that the sequence $(\vec{u}_n(0,\cdot))_{n\in\mathbb{N}}$ strongly converges (via a sub-sequence) to $v(0,\cdot)$ in the space $H^1_\alpha(\mathbb{R}^3)$.

We start by the construction of an eternal solution $(\vec{v},q)$.

**Proposition 4.2** *There exists a couple of functions $(\vec{v},q)$, with $\vec{v}\in L^\infty_{loc}(\mathbb{R}, H^1(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}, \dot{H}^2(\mathbb{R}^3))$ and $q \in L^2_{loc}(\mathbb{R}, \dot{H}^3(\mathbb{R}^3))$, which is a weak solution of the equation (10).*

**Proof.** This solution will be obtained as the limit of the solutions $\vec{u}_n : [-t_n,+\infty[\times\mathbb{R}^3 \to \mathbb{R}^3$ and $p_n : [-t_n,+\infty[\times\mathbb{R}^3 \to \mathbb{R}$ of equations (32) when $n\to+\infty$.

We observe that by point 1) in Proposition 1, for all $n\in\mathbb{N}$ and for all $t\geq -t_n$, we have

$$\|\vec{u}_n(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1} \leq \left(\|\vec{u}_{0,n}\|^2_{L^2} + \alpha^2\|\vec{u}_{0,n}\|^2_{\dot{H}^1}\right) e^{-\beta\,(t+t_n)} + \frac{4}{\beta^2}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right). \tag{33}$$

Moreover, by point 2) in Proposition 1, for all $t\geq -t_n$ and $T\geq 0$, we have

$$\begin{aligned} \nu\int_t^{t+T} \|\vec{u}_n(s,\cdot)\|^2_{\dot{H}^1}ds + \alpha^2\int_t^{t+T}\|\vec{u}_n(s,\cdot)\|^2_{\dot{H}^2}ds \leq & \frac{2T}{\beta}\left(\|\vec{f}\|^2_{L^2}+\alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right) \\ & + \|\vec{u}_n(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1}. \end{aligned} \tag{34}$$

By estimate (33), and recalling that the sequence $(\vec{u}_{0,n})_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$, we can write

$$\begin{aligned} \sup_{n\in\mathbb{N}}\sup_{t\geq -t_n}\left(\|\vec{u}_n(t,\cdot)\|^2_{L^2}+\alpha^2\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1}\right) \leq & R^2 e^{-\beta\,(t+t_n)} + \frac{4}{\beta^2}\left(\|\vec{f}\|^2_{L^2}+\alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right) \\ \leq & R^2 + \frac{4}{\beta^2}\left(\|\vec{f}\|^2_{L^2}+\alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right). \end{aligned} \tag{35}$$

Moreover, by estimate (34) (with $T=1$) we get

$$\begin{aligned} & \sup_{n\in\mathbb{N}}\sup_{t\geq -t_n}\left(\nu\int_t^{t+1}\|\vec{u}_n(s,\cdot)\|^2_{\dot{H}^1}ds + \alpha^2\int_t^{t+1}\|\vec{u}_n(s,\cdot)\|^2_{\dot{H}^2}ds\right) \\ \leq & \frac{2}{\beta}\left(\|\vec{f}\|^2_{L^2}+\alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right) + \|\vec{u}_n(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1} \\ \leq & \frac{2}{\beta}\left(\|\vec{f}\|^2_{L^2}+\alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right) + R^2 + \frac{4}{\beta^2}\left(\|\vec{f}\|^2_{L^2}+\alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right). \end{aligned} \tag{36}$$

Then, by estimates (35) and (36) and the Banach-Alaoglu theorem, there exists $\vec{v} \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}, \dot{H}^2(\mathbb{R}^3))$ such that the sequence $(\vec{u}_n)_{n\in\mathbb{N}}$ converges (via s sub-sequence) to $\vec{v}$ in the weak-* topology of the spaces $L^\infty([-\tau,\tau], H^1(\mathbb{R}^3))$, $L^2([-\tau,\tau], \dot{H}^1(\mathbb{R}^3))$ and $L^2([-\tau,\tau], \dot{H}^2(\mathbb{R}^3))$, for all $\tau > 0$.

It remains to prove that the sequence of pressure terms $(p_n)_{n\in\mathbb{N}}$ converges to a limit $q : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$. Recall that by identity (29), for all $n \in \mathbb{N}$, we have: $p_n = \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_i\left((-\alpha^2\Delta + I_d)^{-1}(u_{n,i}u_{n,j})\right)$, hence, for all $t \geq -t_n$ and $T \geq 0$, we get

$$\int_t^{t+T} \|p_n(s,\cdot)\|^2_{H^3} ds \leq c_\alpha \int_t^{t+T} \|\vec{u}_n \otimes \vec{u}_n(s,\cdot)\|^2_{H^1} ds \leq c_\alpha \int_t^{t+T} \|\vec{u}_n \otimes \vec{u}_n(s,\cdot)\|^2_{L^2} ds + c_\alpha \int_t^{t+T} \|\vec{u}_n \otimes \vec{u}_n(s,\cdot)\|^2_{\dot{H}^1} ds,$$

where we must study each term in the right side. For the first term, by the product laws in the non-homogeneous Sobolev spaces (Lemma 7.3, page 130 of [21]), and moreover, by estimate (33) we write

$$\begin{aligned} c_\alpha \int_t^{t+T} \|\vec{u}_n \otimes \vec{u}_n(s,\cdot)\|^2_{L^2} ds \leq& c_\alpha \int_t^{t+T} \|\vec{u}_n \otimes \vec{u}_n(s,\cdot)\|^2_{H^{1/2}} ds \leq c_\alpha \int_t^{t+T} \|\vec{u}_n(s,\cdot)\|^4_{H^1} ds \\ \leq& c_\alpha \int_t^{t+T} \left(\|\vec{u}_n(s,\cdot)\|^2_{L^2} + \|\vec{u}_n(s,\cdot)\|^2_{\dot{H}^1}\right)^2 ds \\ \leq& c_\alpha \left(\max\left(1, 1/\alpha^2\right)\right)^2 \int_t^{t+T} \left(\|\vec{u}_n(s,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}_n(s,\cdot)\|^2_{\dot{H}^1}\right)^2 ds \\ \leq& C_\alpha\, T \left(R^2 + \frac{4}{\beta^2}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right)\right)^2. \end{aligned}$$

We estimate now the second term. Recall first that we have $\|\cdot\|_{E_T} = \|\cdot\|_{L^\infty_t H^1_x} + \|\cdot\|_{L^2_t \dot{H}^2_x}$, and then, by Lemma 3.2 we get

$$\begin{aligned} c_\alpha \int_t^{t+T} \|\vec{u}_n \otimes \vec{u}_n(s,\cdot)\|^2_{\dot{H}^1} ds \leq& c_\alpha T^{1/2}\|\vec{u}_n\|^2_{E_T} \leq c_\alpha\, T^{1/2}\left(\sup_{t\leq s\leq t+T} \|\vec{u}_n(s,\cdot)\|^2_{H^1} + \int_t^{t+T} \|\vec{u}_n(s,\cdot)\|^2_{\dot{H}^2} ds\right) \\ \leq& c_\alpha \left(\max\left(1, 1/\alpha^2\right)\right)^2 T^{1/2}\left(\sup_{t\leq s\leq t+T}\left(\|\vec{u}_n(s,\cdot)\|^2_{L^2} + \alpha^2\|\vec{u}_n(s,\cdot)\|^2_{\dot{H}^1}\right) + \alpha^2 \int_t^{t+T} \|\vec{u}_n(s,\cdot)\|^2_{\dot{H}^2} ds\right) = (a). \end{aligned}$$

Thereafter, by estimates (33) and (34) we can write

$$(a) \leq C_\alpha T^{1/2}\left(2R^2 + \frac{8}{\beta^2}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right) + \frac{2}{\beta}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right)\right).$$

Once we have studied each term, gathering the estimates obtained we finally get

$$\begin{aligned} \int_t^{t+T} \|p_n(s,\cdot)\|^2_{H^3} ds \leq& C_\alpha\, T \left(R^2 + \frac{4}{\beta^2}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right)\right)^2 \\ &+ C_\alpha\, T^{1/2}\left(2R^2 + \frac{8}{\beta^2}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right) + \frac{2}{\beta}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right)\right). \end{aligned}$$

Hence, taking $T = 1$ we can write

$$\sup_{n\in\mathbb{N}} \sup_{t\geq -t_n} \int_t^{t+T} \|p_n(s,\cdot)\|^2_{H^3} ds \leq C_\alpha\left(3R^2 + \left(\frac{12}{\beta^2} + \frac{2}{\beta}\right)\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right)\right),$$

and always by the Banach-Alaoglu theorem there exists $q \in L^2_{loc}(\mathbb{R}, H^3(\mathbb{R}^3))$, such that the sequence $(p_n)_{n\in\mathbb{N}}$ converges (through a a subsequence) to a limit $q$ in the weak-* topology of the spaces $L^2([-\tau,\tau], H^3(\mathbb{R}^3))$, for all $\tau > 0$.

We want to show that the limit $(\vec{v}, q)$ verifies the equation (10) in the distributional sense. From equation (32) we can write

$$\begin{array}{ll} \partial_t(-\alpha^2\Delta + Id)\vec{u}_n + \text{div}\,(\vec{u}_n \otimes \vec{u}_n) - \nu\Delta(-\alpha^2\Delta + Id)\vec{u}_n + \vec{\nabla}(-\alpha^2\Delta + Id)p_n & \\ =(-\alpha^2\Delta + Id)\,\vec{f} - \beta(-\alpha^2\Delta + Id)\,\vec{u}_n, & div(\vec{u}_n) = 0, \end{array}$$

and then, it is enough to prove that the non-linear terms $\text{div}\,(\vec{u}_n \otimes \vec{u}_n)$ converge to $\text{div}\,(\vec{v} \otimes \vec{v})$ in the sense of distributions. For this, we will use the following Rellich-Lions lemma. For a proof see the Theorem 12.1, page 349, of [21].

**Lemma 4.2** *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on $\mathbb{R}\times\mathbb{R}^3$, such that, for every $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}\times\mathbb{R}^3)$, we have:*

1. *for some positive $\gamma > 0$, $\displaystyle\sup_{n\in\mathbb{N}} \|\varphi g_n\|_{L^2_t H^\gamma_x} < +\infty$, and*

2. *for some negative $\sigma < 0$, $\displaystyle\sup_{n\in\mathbb{N}} \|\varphi \partial_t g_n\|_{L^2_t H^\sigma_x} < +\infty$.*

*Then, the sequence $(g_n)_{n\in\mathbb{N}}$ strongly converges (via a sub-sequence) to a limit $g$ in the space $L^2_{loc}(\mathbb{R}\times\mathbb{R}^3)$.*

We will prove now that the sequence $(\vec{u}_n)_{n\in\mathbb{N}}$ verifies the points 1 and 2 in this lemma, and for this, we will extend first these functions to whole real line by setting $\vec{u}_n(t,\cdot) = 0$ for all $t < -t_n$. Then, let $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}\times\mathbb{R}^3)$ be a test function, and let $\tau > 0$ be such that $\varphi(t,\cdot) = 0$ for all $|t| > \tau$. To verify the point 1, we set $\gamma = 1$ to get

$$\begin{aligned} \|\varphi\vec{u}_n\|^2_{L^2_t H^1_x} =& \int_{-\tau}^{\tau} \|\varphi(t,\cdot)\vec{u}_n(t,\cdot)\|^2_{H^1} dt = \int_{-\tau}^{\tau} \|\varphi(t,\cdot)\vec{u}_n(t,\cdot)\|^2_{L^2} dt + \int_{-\tau}^{\tau} \|\varphi(t,\cdot)\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1} dt \\ \leq& c_\alpha\, 2\tau(\|\varphi\|_{L^\infty} + \|\vec{\nabla}\varphi\|_{L^\infty}) \left( \sup_{-\tau\leq t\leq\tau} \|\vec{u}_n(t,\cdot)\|^2_{L^2} + \sup_{-\tau\leq t\leq\tau} \alpha^2 \|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1} \right) \\ \leq& C(\tau,\varphi,\alpha) \left( \sup_{-\tau\leq t\leq\tau} \|\vec{u}_n(t,\cdot)\|^2_{L^2} + \sup_{-\tau\leq t\leq\tau} \alpha^2 \|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1} \right). \end{aligned}$$

Moreover, by estimate (33) we can write

$$C(\tau,\varphi,\alpha) \left( \sup_{-\tau\leq t\leq\tau} \|\vec{u}_n(t,\cdot)\|^2_{L^2} + \sup_{-\tau\leq t\leq\tau} \alpha^2 \|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1} \right) \leq C(\tau,\varphi,\alpha) \left( R^2 + \frac{4}{\beta^2} \left( \|\vec{f}\|^2_{L^2} + \alpha^2 \|\vec{f}\|^2_{\dot{H}^1} \right) \right). \tag{37}$$

Thus, for all $n \in \mathbb{N}$ we have

$$\|\varphi\vec{u}_n\|^2_{L^2_t H^1_x} \leq C(\tau,\varphi,\alpha) \left( R^2 + \frac{4}{\beta^2} \left( \|\vec{f}\|^2_{L^2} + \alpha^2 \|\vec{f}\|^2_{\dot{H}^1} \right) \right).$$

We verify now the point 2. We observe first that for all $n \in \mathbb{N}$ we have

$$\varphi\,\partial_t\vec{u}_n = -\varphi\,\mathbb{P}\,\text{div}\,((\vec{u}_n \otimes \vec{u}_n)_\alpha) + \nu\varphi\,\Delta\vec{u}_n + \varphi\,\vec{f} - \beta\varphi\,\vec{u}_n,$$

where, for $\tau > 0$ given above, we will show that each term in the right side is uniformly bounded in the space $L^2([-\tau,\tau], L^2(\mathbb{R}^3))$. Indeed, for the term $\varphi\,\mathbb{P}\,\text{div}\,((\vec{u}_n \otimes \vec{u}_n)_\alpha)$, using the product laws in the Sobolev spaces, and moreover, by estimate (35) we have

$$\begin{aligned}\|\varphi\,\mathbb{P}\,\text{div}\,((\vec{u}_n\otimes\vec{u}_n)_\alpha)\|_{L^2_tL^2_x} &\leq \|\varphi\|_{L^\infty_{t,x}}\|\mathbb{P}\,\text{div}\,((\vec{u}_n\otimes\vec{u}_n)_\alpha)\|_{L^2_tL^2_x} \leq C(\alpha,\varphi)\|\vec{u}_n\otimes\vec{u}_n\|_{L^2_tH^{-1}_x}\\ &\leq C(\alpha,\varphi)\|\vec{u}_n\otimes\vec{u}_n\|_{L^2_tH^{1/2}_x} \leq C(\tau,\alpha,\varphi)\|\vec{u}_n\otimes\vec{u}_n\|_{L^\infty_tH^{1/2}_x}\\ &\leq C(\tau,\alpha,\varphi)\|\vec{u}_n\|^2_{L^\infty_tH^1_x}\\ &\leq C(\tau,\alpha,\varphi)\left(R^2+\frac{4}{\beta^2}\left(\|\vec{f}\|^2_{L^2}+\alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right)\right).\end{aligned}$$

For the term $\nu\varphi\,\Delta\vec{u}_n$, using the estimate (36) we can write

$$\begin{aligned}\nu\|\varphi\,\Delta\vec{u}_n\|_{L^2_tL^2_x} &\leq \frac{\nu}{\alpha}\|\varphi\|_{L^\infty_{t,x}}\alpha\|\Delta\vec{u}_n\|_{L^2_tL^2_x}\\ &\leq C(\tau,\alpha,\nu,\varphi)\left(\frac{2}{\beta}\left(\|\vec{f}\|^2_{L^2}+\alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right)+R^2+\frac{4}{\beta^2}\left(\|\vec{f}\|^2_{L^2}+\alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right)\right)^{1/2}.\end{aligned}$$

Finally, we may observe that the third and fourth terms: $\varphi\,\vec{f}$ and $\beta\varphi\,\vec{u}_n$, are uniformly bounded in the space $L^2([-\tau,\tau],L^2(\mathbb{R}^3))$, and then, the point 2 hols true for any $\sigma<0$.

We can use now the Lemma 4.2 to conclude that the sequence $(\vec{u}_n)_{n\in\mathbb{N}}$ converges to the limit $\vec{v}$ in the strong topology of the space $L^2_{loc}(\mathbb{R}\times\mathbb{R}^3)$. Then, we have that $\vec{u}_n\otimes\vec{u}_n$ strongly converges to $\vec{v}\otimes\vec{v}$ in the space $L^1_{loc}(\mathbb{R}\times\mathbb{R}^3)$, hence we get that $div(\vec{u}_n\otimes\vec{u}_n)$ converges to $div(\vec{v}\otimes\vec{v})$ in the distributional sense.

At this point, we have proven that $(\vec{v},q)$ verifies (in the sense of distributions) the equation

$$\begin{aligned}&\partial_t(-\alpha^2\Delta+Id)\vec{v}+\text{div}\,(\vec{v}\otimes\vec{v})-\nu\Delta(-\alpha^2\Delta+Id)\vec{v}+\vec{\nabla}(-\alpha^2\Delta+Id)q\\ =&(-\alpha^2\Delta+Id)\,\vec{f}-\beta(-\alpha^2\Delta+Id)\,\vec{v},\end{aligned}\qquad div(\vec{v})=0,$$

but, recalling that $\vec{v}\in L^\infty_{loc}(\mathbb{R},H^1(\mathbb{R}^3))\cap L^2_{loc}(\mathbb{R},\dot{H}^2(\mathbb{R}^3))$ and $q\in L^2_{loc}(\mathbb{R},\dot{H}^3(\mathbb{R}^3))$, we can apply the filtering operator $(\cdot)_\alpha=(-\alpha\Delta+Id)^{-1}$ to each term in this equation to finally obtain that $(\vec{v},q)$ verifies the equation (10). Proposition 4.2 is proven. ■

Once we have constructed a solution $(\vec{v},q)$ to the equation (10), we will prove now that the sequence $(\vec{u}_n(0,\cdot))_{n\in\mathbb{N}}$ strongly converges (through a sub-sequence) to $\vec{v}(0,\cdot)$ in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$. Recall first that this space is equipped with the norm $\|\cdot\|^2_{\mathcal{H}^1_\alpha}=\|\cdot\|^2_{L^2}+\alpha^2\|\cdot\|^2_{\dot{H}^1}$. Then, recall moreover that, for all $n\in\mathbb{N}$ and for all $t\geq -t_n$, the solution $\vec{u}_n\in L^\infty([-t_n,+\infty[,H^1(\mathbb{R}^3))\cap L^2_{loc}([-t_n,+\infty[,\dot{H}^2(\mathbb{R}^3))$ of equation (32) verifies the identity (27):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\vec{u}_n(t,\cdot)\|^2_{\mathcal{H}^1_\alpha}=&-\nu\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1}-\alpha^2\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^2}+\langle\vec{f},\vec{u}_n(t,\cdot)\rangle_{L^2\times L^2}\\ &+\alpha^2\langle\vec{\nabla}\otimes\vec{f},\vec{\nabla}\otimes\vec{u}_n(t,\cdot)\rangle_{L^2\times L^2}-\beta\|\vec{u}_n(t,\cdot)\|^2_{\mathcal{H}^1_\alpha}.\end{aligned}$$

We multiply each term in this identity by $e^{2\beta t}$, and moreover, we integrate in the interval $[-t_n,0]$ to get:

$$\begin{aligned}\frac{1}{2}\|\vec{u}_n(0,\cdot)\|^2_{\mathcal{H}^1_\alpha}-\frac{1}{2}e^{-2\beta t_n}\|\vec{u}_{0,n}\|^2_{\mathcal{H}^1_\alpha}-\beta\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\mathcal{H}^1_\alpha}dt=-\nu\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1}dt\\ -\alpha^2\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^2}dt+\int_{-t_n}^0 e^{2\beta t}\langle\vec{f},\vec{u}_n(t,\cdot)\rangle_{L^2\times L^2}dt\\ +\alpha^2\int_{-t_n}^0 e^{2\beta t}\langle\vec{\nabla}\otimes\vec{f},\vec{\nabla}\otimes\vec{u}_n(t,\cdot)\rangle_{L^2\times L^2}-\beta\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\mathcal{H}^1_\alpha}dt,\end{aligned}$$

hence we obtain

$$\begin{aligned}\|\vec{u}_n(0,\cdot)\|^2_{\mathcal{H}^1_\alpha} =& e^{-2\beta t_n}\|\vec{u}_{0,n}\|^2_{\mathcal{H}^1_\alpha} - 2\nu\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1}dt - 2\alpha^2\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^2}dt\\ &+ 2\int_{-t_n}^0 e^{2\beta t}\langle \vec{f},\vec{u}_n(t,\cdot)\rangle_{L^2\times L^2}dt + 2\alpha^2\int_{-t_n}^0 e^{2\beta t}\langle \vec{\nabla}\otimes\vec{f},\vec{\nabla}\otimes\vec{u}_n(t,\cdot)\rangle_{L^2\times L^2}.\end{aligned}$$

In each term of this identity, we take now the lim sup when $n\to+\infty$ to write:

$$\begin{aligned}\limsup_{n\to+\infty}\|\vec{u}_n(0,\cdot)\|^2_{\mathcal{H}^1_\alpha} \leq& \limsup_{n\to+\infty} e^{-2\beta t_n}\|\vec{u}_{0,n}\|^2_{\mathcal{H}^1_\alpha} + \limsup_{n\to+\infty}\left(-2\nu\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1}dt\right)\\ &+\limsup_{n\to+\infty}\left(-2\alpha^2\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^2}dt\right)\\ &+\limsup_{n\to+\infty}\left(2\int_{-t_n}^0 e^{2\beta t}\langle\vec{f},\vec{u}_n(t,\cdot)\rangle_{L^2\times L^2}dt\right)\\ &+\limsup_{n\to+\infty}\left(2\alpha^2\int_{-t_n}^0 e^{2\beta t}\langle\vec{\nabla}\otimes\vec{f},\vec{\nabla}\otimes\vec{u}_n(t,\cdot)\rangle_{L^2\times L^2}dt\right),\end{aligned} \tag{38}$$

where we must study each term in the right side. For the first term, recalling that the sequence $(\vec{u}_{0,n})_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$, we have

$$\limsup_{n\to+\infty} e^{-2\beta t_n}\|\vec{u}_{0,n}\|^2_{\mathcal{H}^1_\alpha} = 0. \tag{39}$$

For the second term, since by estimate (34) we have that the sequence $(\vec{u}_n)_{n\in\mathbb{N}}$ converges to $\vec{v}$ in the weak-* topology of the space $L^2_{loc}(\mathbb{R},\dot{H}^1(\mathbb{R}^3))$, then we can write

$$\liminf_{n\to+\infty}\left(2\nu\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1}dt\right)\geq 2\nu\int_{-\infty}^0 e^{2\beta t}\|\vec{v}(t,\cdot)\|^2_{\dot{H}^1}dt,$$

hence we have

$$\begin{aligned}\limsup_{n\to+\infty}\left(-2\nu\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1}dt\right)\leq& -\liminf_{n\to+\infty}\left(2\nu\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^1}dt\right)\\ \leq& -2\nu\int_{-\infty}^0 e^{2\beta t}\|\vec{v}(t,\cdot)\|^2_{\dot{H}^1}dt.\end{aligned} \tag{40}$$

For the third term, always by estimate (34), we have that the sequence $(\vec{u}_n)_{n\in\mathbb{N}}$ converges to $\vec{v}$ in the weak-* topology of the space $L^2_{loc}(\mathbb{R},\dot{H}^2(\mathbb{R}^3))$, and we write

$$\liminf_{n\to+\infty}\left(2\alpha^2\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^2}dt\right)\geq 2\alpha^2\int_{-\infty}^0 e^{2\beta t}\|\vec{v}(t,\cdot)\|^2_{\dot{H}^1}dt.$$

Then, we have

$$\begin{aligned}\limsup_{n\to+\infty}\left(-2\alpha^2\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^2}dt\right)\leq& -\liminf_{n\to+\infty}\left(2\alpha^2\int_{-t_n}^0 e^{2\beta t}\|\vec{u}_n(t,\cdot)\|^2_{\dot{H}^2}dt\right)\\ \leq& -2\alpha^2\int_{-\infty}^0 e^{2\beta t}\|\vec{v}(t,\cdot)\|^2_{\dot{H}^2}dt.\end{aligned} \tag{41}$$

Similarly, by the estimate (33), the sequence $(\vec{u}_n)_{n\in\mathbb{N}}$ converges to $\vec{v}$ in the weak-* topology of the space $L^2_{loc}(\mathbb{R},L^2(\mathbb{R}^3))$, and moreover, as we also have the weak-* convergence in the space $L^2_{loc}(\mathbb{R},\dot{H}^1(\mathbb{R}^3))$. Then, for the fourth and fifth terms we have

$$\begin{aligned}\limsup_{n\to+\infty}\left(2\int_{-t_n}^0 e^{2\beta t}\langle\vec{f},\vec{u}_n(t,\cdot)\rangle_{L^2\times L^2}dt\right) =& 2\int_{-\infty}^0 e^{2\beta t}\langle\vec{f},\vec{v}(t,\cdot)\rangle_{L^2\times L^2}\,dt,\\ \limsup_{n\to+\infty}\left(2\alpha^2\int_{-t_n}^0 e^{2\beta t}\langle\vec{\nabla}\otimes\vec{f},\vec{\nabla}\otimes\vec{u}_n(t,\cdot)\rangle_{L^2\times L^2}\right) =& 2\alpha^2\int_{-\infty}^0 e^{2\beta t}\,\langle\vec{\nabla}\otimes\vec{f},\vec{\nabla}\otimes\vec{v}(t,\cdot)\rangle_{L^2\times L^2}\,dt.\end{aligned} \tag{42}$$

Thus, gathering the estimates (39), (40), (41) and (42), we get back to (38) to write:

$$\limsup_{n\to+\infty} \|\vec{u}_n(0,\cdot)\|^2_{\mathcal{H}^1_\alpha} \leq -2\nu \int_{-\infty}^{0} e^{2\beta t}\|\vec{v}(t,\cdot)\|^2_{\dot{H}^1} dt - 2\alpha^2 \int_{-\infty}^{0} e^{2\beta t}\|\vec{v}(t,\cdot)\|^2_{\dot{H}^2} dt$$
$$+ 2\int_{-\infty}^{0} e^{2\beta t}\langle \vec{f}, \vec{v}(t,\cdot)\rangle_{L^2\times L^2}\, dt + 2\alpha^2 \int_{-\infty}^{0} e^{2\beta t}\, \langle \vec{\nabla}\otimes \vec{f}, \vec{\nabla}\otimes \vec{v}(t,\cdot)\rangle_{L^2\times L^2}\, dt = (b).$$

We shall study now the term $(b)$ above. Since the solution $(\vec{v}, q)$ of the equation (10) verifies $\vec{v} \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}, \dot{H}^2(\mathbb{R}^3))$ and $q \in L^2_{loc}(\mathbb{R}, H^3(\mathbb{R}^3))$ then, following the same computations done in (27), we have the following energy equality:

$$\frac{1}{2}\frac{d}{dt}\|\vec{v}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} = -\nu\|\vec{v}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{v}(t,\cdot)\|^2_{\dot{H}^2} + \langle \vec{f}, \vec{v}(t,\cdot)\rangle_{L^2\times L^2} + \alpha^2\langle \vec{\nabla}\otimes \vec{f}, \vec{\nabla}\otimes\vec{v}(t,\cdot)\rangle_{L^2\times L^2} - \beta\|\vec{v}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha}.$$

We multiply each term by $e^{2\beta t}$, and integrating in the interval $]-\infty, 0]$ we get:

$$\frac{1}{2}\|\vec{v}(0,\cdot)\|^2_{\mathcal{H}^1_\alpha} - \beta\int_{-\infty}^{0} e^{2\beta t}\|\vec{v}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} dt = -\nu\int_{-\infty}^{0} e^{2\beta t}\|\vec{v}(t,\cdot)\|^2_{\dot{H}^1} dt - \alpha^2\int_{-\infty}^{0} e^{2\beta t}\|\vec{v}(t,\cdot)\|^2_{\dot{H}^2} dt$$
$$+ \int_{-\infty}^{0} e^{2\beta t}\langle \vec{f}, \vec{v}(t,\cdot)\rangle_{L^2\times L^2} dt + \alpha^2\int_{-\infty}^{0} e^{2\beta t}\langle \vec{\nabla}\otimes\vec{f}, \vec{\nabla}\otimes\vec{v}(t,\cdot)\rangle_{L^2\times L^2} - \beta\int_{-\infty}^{0} e^{2\beta t}\|\vec{v}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} dt.$$

From this identity we have $(b) = \|\vec{v}(0,\cdot)\|^2_{\mathcal{H}^1_\alpha}$, and then, getting back to the previous estimate, we obtain $\limsup\limits_{n\to+\infty} \|\vec{u}_n(0,\cdot)\|^2_{\mathcal{H}^1_\alpha} \leq \|\vec{v}(0,\cdot)\|^2_{\mathcal{H}^1_\alpha}$. On the other hand, since by the estimate (33) we know that the sequence $(\vec{u}_n)_{n\in\mathbb{N}}$ converges (via a sub-sequence) to $\vec{v}$ in the weak-* topology of the space $L^\infty(\mathbb{R}, H^1(\mathbb{R}^3))$, we also have the inequality $\|\vec{v}(0,\cdot)\|^2_{\mathcal{H}^1_\alpha} \leq \liminf\limits_{n\to+\infty}\|\vec{u}_n(0,\cdot)\|^2_{\mathcal{H}^1_\alpha}$. Then, we obtain the desired strong convergence: $\lim\limits_{n\to+\infty}\|\vec{u}_n(0,\cdot)\|^2_{\mathcal{H}^1_\alpha} = \|\vec{v}(0,\cdot)\|^2_{\mathcal{H}^1_\alpha}$, and the point 2 in Proposition 4.1 is now verified.

To verify now the point 3 in Proposition 4.1, we just observe that by estimate (31), where we have $\vec{w}(t,\cdot) = \vec{u}_1(t,\cdot) - \vec{u}_2(t,\cdot) = S(t)\vec{u}_{0,1} - S(t)\vec{u}_{0,2}$, and $\vec{w}(0,\cdot) = \vec{u}_{0,1} - \vec{u}_{0,2}$, then the continuity of the map $S(t): \mathcal{B} \to \mathcal{H}^1_\alpha(\mathbb{R}^3)$ follows directly.

Thus, by Proposition 4.1, the semigroup $S(t)$ has a global attractor $\mathcal{A}_{\vec{f}} \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$. Theorem 2 is now proven. ■

## 5 Fractal box counting dimension of the attractor

In this section, we prove that the global attractor $\mathcal{A}_{\vec{f}} \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$, constructed in Theorem 2, has finite fractal box counting dimension and we give an explicit upper bound. In order to estimate the fractal dimension of the attractor, we will use the following volume contraction method adapted from [16]. See also [12] and [25] for more details. We start by introducing some definitions that we shall use later.

The first definition concerns the following quasi-differential operator. Let $t \geq 0$ be a fixed time and let $\vec{u}_0 \in \mathcal{A}_{\vec{f}}$ be an initial datum. Moreover, let $u(t,\cdot)$ be the solution of the equation (4) arising from the initial datum $\vec{u}_0$ and given by Theorem 1. Thus, for $u(t,\cdot)$ fixed, let $\vec{v} \in L^\infty([0,+\infty[, \mathcal{H}^1_\alpha(\mathbb{R}^3)) \cap L^2_{loc}([0,+\infty[, \dot{H}^2(\mathbb{R}^3))$ be the solution of the following linearized version of the equation (4):

$$\begin{cases} \partial_t\vec{v} + \mathbb{P}\left(\left((\vec{v}\cdot\vec{\nabla})\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{v}\right)_\alpha\right) - \nu\Delta\vec{v} = -\beta\vec{v}, \quad \text{div}(\vec{v}) = 0, \\ \vec{v}(0,\cdot) = \vec{v}_0 \in \mathcal{H}^1_\alpha(\mathbb{R}^3), \end{cases} \tag{43}$$

where $\vec{v}_0$ denotes the initial datum. The existence and uniqueness of the solution $\vec{v}$ for this equation is straightforward (since we have a linear equation) and follows the main ideas and estimates in the proof of Theorem 1. So, we will omit the prove of this fact.

**Definition 5.1 (Quasi-differential operator)** *The quasi-differential operator $DS(t,\vec{u}_0)$, depending on the time $t \geq 0$ and the datum $\vec{u}_0 \in \mathcal{A}_{\vec{f}}$, is the linear and bounded operator $DS(t,\vec{u}_0) : \mathcal{H}^1_\alpha(\mathbb{R}^3) \to \mathcal{H}^1_\alpha(\mathbb{R}^3)$ defined as*

$$DS(t,\vec{u}_0)\vec{v}_0 = \vec{v}(t,\cdot),$$

*where $\vec{v}(t,\cdot)$ is the solution of the linearized equation (43).*

Once we have defined this operator, our second definition lies with the notion a semigroup uniformly quasi-differentiable.

**Definition 5.2 (Semigroup uniformly quasi-differentiable)** *For $t \geq 0$ fixed, let $S(t)$ be the semi-group associated to equation (4) and defined in (8). We say that this semigroup is uniformly quasi-differentiable on the global attractor $\mathcal{A}_{\vec{f}} \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$, if for all $\vec{u}_{0,1}, \vec{u}_{0,2} \in \mathcal{A}_{\vec{f}}$ we have*

$$\|S(t)\vec{u}_{0,2} - S(t)\vec{u}_{0,1} - DS(t,\vec{u}_{0,1})(\vec{u}_{0,2} - \vec{u}_{0,1})\|_{\mathcal{H}^1_\alpha} \leq \mathfrak{o}\left(\|\vec{u}_{0,2} - \vec{u}_{0,1}\|_{\mathcal{H}^1_\alpha}\right),$$

*where the quasi-differential operator $DS(t,\vec{u}_{0,1})$ is given in Definition 5.1, and moreover, the quantity $\mathfrak{o}(\cdot)$ verifies: $\lim\limits_{h\to 0^+} \mathfrak{o}(h)/h = 0$.*

Finally, in our last definition, we need to introduce the notion the $m-$ global Lyapunov exponents for a $m \in \mathbb{N}$ given. For this, we shall need to precise first some notation. On one hand, we denote by $\mathcal{O}_m$, the set of all the orthonormal families $(\vec{w}_i)_{1\leq i\leq m}$ in the space $\mathcal{H}^1_\alpha(\mathbb{R}^3)$ dotted with its natural scalar product:

$$[\vec{w}_i, \vec{w}_j]_\alpha = (\vec{w}_i, \vec{w}_j)_{L^2\times L^2} + \alpha^2 \left(\vec{\nabla} \otimes \vec{w}_i, \vec{\nabla} \otimes \vec{w}_j\right)_{L^2\times L^2}. \tag{44}$$

On the other hand, getting back to the linearized equation (43), we can write

$$\partial_t \vec{v} = -\mathbb{P}\left(\left((\vec{v}\cdot\vec{\nabla})\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{v}\right)_\alpha\right) + \nu\Delta\vec{v} - \beta\vec{v}.$$

and then, from the right side of this identity, and for all $\vec{w} \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$, we define now the linear operator

$$\mathcal{L}(t,\vec{u}_0)\vec{w} = -\mathbb{P}\left(\left((\vec{w}\cdot\vec{\nabla})\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{w}\right)_\alpha\right) + \nu\Delta\vec{w} - \beta\vec{w}. \tag{45}$$

Once we have introduced the set $\mathcal{O}_m$ and the linear operator $\mathcal{L}(t,\vec{u}_0)(\cdot)$ above, we have the following definition.

**Definition 5.3 ($m-$ global Lyapunov exponents)** *Let $m \in \mathbb{N}$ fixed. We define the $m-$ global Lyapunov exponent $\ell(m)$ as the quantity:*

$$\ell(m) = \limsup_{T\to+\infty}\left(\sup_{\vec{u}_0\in\mathcal{A}_{\vec{f}}}\ \sup_{(\vec{w}_i)_{1\leq i\leq m}\in\mathcal{O}_m}\left(\frac{1}{T}\int_0^T \sum_{i=1}^m [\mathcal{L}(t,\vec{u}_0)\vec{w}_i, \vec{w}_i]_\alpha\, dt\right)\right).$$

We have now all the tools to state the following technical result that we shall use to derive an upper bound of the fractal dimension of the attractor $\mathcal{A}_{\vec{f}}$. For a proof of this result see [11].

**Theorem 5.1 (Upper bound of the fractal dimension)** *Let $S(t)$ be the semigruop associated to equation (4) and defined in (8). Moreover, let $\mathcal{A}_{\vec{f}} \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$ be the global attractor of the semigroup $S(t)$ given by Theorem 2. Finally, let $\dim\left(\mathcal{A}_{\vec{f}}\right)$ be the fractal box counting dimension of the attractor $\mathcal{A}_{\vec{f}}$ given in Definition 2.*

*If the following statements hold:*

1. *The semigroup $S(t)$ is uniformly quasi-differentiable on the attractor $\mathcal{A}_{\vec{f}}$ in the sense of Definition 5.2.*

2. *The quasi-differential operator $DS(t, \vec{u}_0)(\cdot)$ given in Definition 5.1, depends continuously on the initial datum $\vec{u}_0 \in \mathcal{A}_{\vec{f}}$.*

3. *There exists $\gamma \geq 1$, and moreover, there exist two constants $c_1, c_2 > 0$ such that, for all $m \in \mathbb{N}$, the $m-$ global Lyapunov exponent $\ell(m)$ given in Definition 5.3 verifies:*

$$\ell(m) \leq -c_1 \, m^\gamma + c_2. \tag{46}$$

*Then, we have the following upper bound: dim* $\left(\mathcal{A}_{\vec{f}}\right) \leq \left(\frac{c_2}{c_1}\right)^{1/\gamma}$.

## Proof of Theorem 3

We must verify that the points 1, 2 and 3 in Theorem 5.1 hold. However, the points 1 and 2 are given in [2] where it is proven that the semigroup $S(t)$ is even differentiable for all $\vec{u}_0 \in \mathcal{A}_{\vec{f}}$ and the differential operator $DS(t, \vec{u}_0)$ depends continuously on $\vec{u}_0 \in \mathcal{A}_{\vec{f}}$. So, we will focus on the point 3.

To estimate the $m-$ global Lyapunov exponent $\ell(m)$ according to the desired estimate (46), we shall prove the following technical estimates. In the expression of the quantity $\ell(m)$ given in Definition 5.3, we derive first an upper bound for the term $\displaystyle\sum_{i=1}^{m} [\mathcal{L}(t, \vec{u}_0)\vec{w}_i, \vec{w}_i]_\alpha$ as follows:

**Proposition 5.1** *Let $m \in \mathbb{N}$ fixed and let $(\vec{w}_i)_{1 \leq i \leq m} \in \mathcal{O}_m$. Moreover, let $\mathcal{L}(t, \vec{u}_0)(\cdot)$ be the linear operator given in (45), and let $[\cdot, \cdot]_\alpha$ be the scalar product defined in (44). Then, we have:*

$$\sum_{i=1}^{m} [\mathcal{L}(t, \vec{u}_0)\vec{w}_i, \vec{w}_i]_\alpha \leq -\beta m + 2\frac{C^4_{LT}}{\nu^{12/5}\alpha^{6/5}} \|\vec{u}(t,\cdot)\|^{14/5}_{\dot{H}^1} + \frac{3}{8}\alpha^2 \|\vec{u}(t,\cdot)\|^2_{\dot{H}^2}, \tag{47}$$

*where $C_{LT} > 0$ is a numerical constant given in (49).*

**Proof.** By definition of the operator $\mathcal{L}(t, \vec{u}_0)(\cdot)$, we write

$$\sum_{i=1}^{m} [\mathcal{L}(t, \vec{u}_0)\vec{w}_i, \vec{w}_i]_\alpha = \sum_{i=1}^{m} \left[-\mathbb{P}\left(\left((\vec{w}_i \cdot \vec{\nabla})\vec{u} + (\vec{u} \cdot \vec{\nabla})\vec{w}_i\right)_\alpha\right), \vec{w}_i\right]_\alpha + \sum_{i=1}^{m} [\nu\Delta\vec{w}_i - \beta\vec{w}_i, \vec{w}_i]_\alpha = I_1 + I_2, \tag{48}$$

where we shall study each term $I_1$ and $I_2$ separately.

For the term $I_1$, as we have $div(\vec{w}_i) = 0$ (for $1 \leq i \leq m$), and moreover, by the well-known properties of the Leray's projector $\mathbb{P}$, we can write

$$I_1 = \sum_{i=1}^{m} \left[-\left((\vec{w}_i \cdot \vec{\nabla})\vec{u} + (\vec{u} \cdot \vec{\nabla})\vec{w}_i\right)_\alpha, \vec{w}_i\right]_\alpha$$

Then, we will use the following identity.

**Lemma 5.1** *By definition of the filtering operator $(\cdot)_\alpha = (-\alpha^2\Delta + I_d)^{-1}$, and moreover, by definition of the scalar product $[\cdot, \cdot]_\alpha$ given in (44), for $\vec{g}_1, \vec{g}_2 \in H^1(\mathbb{R}^3)$ we have: $[(\vec{g}_1)_\alpha, \vec{g}_2]_\alpha = (\vec{g}_1, \vec{g}_2)_{L^2 \times L^2}$.*

**Proof.** We write $[(\vec{g}_1)_\alpha, \vec{g}_2]_\alpha = [\vec{g}_1, (\vec{g}_2)_\alpha]_\alpha = (\vec{g}_1, (\vec{g}_2)_\alpha)_{L^2\times L^2} + \alpha^2 \left(\vec{\nabla}\otimes \vec{g}_1, \vec{\nabla}\otimes (\vec{g}_2)_\alpha\right)_{L^2\times L^2}$, and integrating by parts the last term we have

$$\begin{aligned}[(\vec{g}_1)_\alpha, \vec{g}_2]_\alpha &= (\vec{g}_1, (\vec{g}_2)_\alpha)_{L^2\times L^2} - \alpha^2 (\vec{g}_1, \Delta(\vec{g}_2)_\alpha)_{L^2\times L^2} = \left(\vec{g}_1, (\vec{g}_2)_\alpha - \alpha^2\Delta((\vec{g}_2)_\alpha)\right)_{L^2\times L^2} \\ &= \left(\vec{g}_1, (-\alpha^2\Delta + I_d)(\vec{g}_2)_\alpha\right)_{L^2\times L^2} = \left(\vec{g}_1, (-\alpha^2\Delta + I_d)(-\alpha^2\Delta + I_d)^{-1}\vec{g}_2\right)_{L^2\times L^2} = (\vec{g}_1, \vec{g}_2)_{L^2\times L^2}.\end{aligned}$$

■

Thus, applying by this identity, and morerover, as we have $div(\vec{w}_i) = 0$, we can write

$$\begin{aligned} I_1 &= -\sum_{i=1}^m \left((\vec{w}_i\cdot\vec{\nabla})\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{w}_i, \vec{w}_i\right)_{L^2\times L^2} = -\sum_{i=1}^m \left((\vec{w}_i\cdot\vec{\nabla})\vec{u}, \vec{w}_i\right)_{L^2\times L^2} - \sum_{i=1}^m \left((\vec{u}\cdot\vec{\nabla})\vec{w}_i, \vec{w}_i\right)_{L^2\times L^2} \\ &= \sum_{i=1}^m \left((\vec{w}_i\cdot\vec{\nabla})\vec{u}, \vec{w}_i\right)_{L^2\times L^2} = \sum_{i=1}^m \int_{\mathbb{R}^3} \left(\sum_{j,k=1}^3 w_{i,k}(\partial_k u_j)\, w_{i,j}\right) dx \leq \sum_{i=1}^m \int_{\mathbb{R}^3} \left|\sum_{j,k=1}^3 w_{i,k}(\partial_k u_j)\, w_{i,j}\right| dx. \end{aligned}$$

Here, we need to estimate the term $\left|\sum_{j,k=1}^3 w_{i,k}(\partial_k u_j)\, w_{i,j}\right|$, and following the same computations done in the estimate (3.5), page 16 in [16], we have $\left|\sum_{j,k=1}^3 w_{i,k}(\partial_k u_j)\, w_{i,j}\right| \leq |\vec{\nabla}\otimes\vec{u}|\, |\vec{w}_i|^2$.

With this pointwise inequality at hand, we get back to the estimate of the term $I_1$ above, where we can write

$$I_1 \leq \sum_{i=1}^m \int_{\mathbb{R}^3} |\vec{\nabla}\otimes\vec{u}||\vec{w}_i|^2 dx \leq \int_{\mathbb{R}^3} |\vec{\nabla}\otimes\vec{u}| \left(\sum_{i=1}^m |\vec{w}_i|^2\right) dx.$$

Then, applying the Hölder inequalities (with $2/5 + 3/5 = 1$) we have

$$\int_{\mathbb{R}^3} |\vec{\nabla}\otimes\vec{u}| \left(\sum_{i=1}^m |\vec{w}_i|^2\right) dx \leq \|\vec{\nabla}\otimes\vec{u}\|_{L^{5/2}} \left\|\sum_{i=1}^m |\vec{w}_i|^2\right\|_{L^{5/3}}.$$

In order to estimate the last term in the right we will use the following Lieb-Thirring inequality, for a proof see the equation (6), page 2 in [22]:

$$\left\|\sum_{i=1}^m |\vec{w}_i|^2\right\|_{L^{5/3}} \leq C_{LT} \left(\sum_{i=1}^m \|\vec{w}_i\|^2_{\dot{H}^1}\right)^{3/5},$$

where, for the function Gamma $\Gamma(\cdot)$, the constant $C_{LT} > 0$ writes down as:

$$C_{LT} = \frac{3}{5^{5/3}} \left(16\pi^{3/2}\frac{\Gamma(7/2)}{\Gamma(5)}\right)^{2/3}. \tag{49}$$

Thus, by the Lieb-Thirring inequality above, and moreover, by the Young inequalities (always with $2/5 + 3/5 = 1$), we have

$$\begin{aligned} \|\vec{\nabla}\otimes\vec{u}\|_{L^{5/2}} \left\|\sum_{i=1}^m |\vec{w}_i|^2\right\|_{L^{5/3}} &\leq C_{LT}\|\vec{\nabla}\otimes\vec{u}\|_{L^{5/2}} \left(\sum_{i=1}^m \|\vec{w}_i\|^2_{\dot{H}^1}\right)^{3/5} \leq \frac{C_{LT}}{\nu^{3/5}}\|\vec{\nabla}\otimes\vec{u}\|_{L^{5/2}} \left(\nu\sum_{i=1}^m \|\vec{w}_i\|^2_{\dot{H}^1}\right)^{3/5} \\ &\leq \frac{2}{5}\frac{C_{LT}^{5/2}}{\nu^{3/2}}\|\vec{\nabla}\otimes\vec{u}\|^{5/2}_{L^{5/2}} + \frac{3\nu}{5}\sum_{i=1}^m \|\vec{w}_i\|^2_{\dot{H}^1}. \end{aligned}$$

Now, in the first term to the right hand side, by the interpolation inequalities (with $2/5 = \theta/2 + (1-\theta)/6$, and $\theta = 7/10$), and moreover, by the Sobolev embedding, we write

$$\begin{aligned}
\frac{2}{5}\frac{C_{LT}^{5/2}}{\nu^{3/2}}\|\vec{\nabla}\otimes\vec{u}\|_{L^{5/2}}^{5/2} + \frac{3\nu}{5}\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 &\leq \frac{2}{5}\frac{C_{LT}^{5/2}}{\nu^{3/2}}\|\vec{\nabla}\otimes\vec{u}\|_{L^2}^{7/4}\,\|\vec{\nabla}\otimes\vec{u}\|_{L^6}^{3/4} + \frac{3\nu}{5}\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 \\
&\leq \frac{2}{5}\frac{C_{LT}^{5/2}}{\nu^{3/2}}\|\vec{u}\|_{\dot{H}^1}^{7/4}\,\left(4\|\vec{u}\|_{\dot{H}^2}\right)^{3/4} + \frac{3\nu}{5}\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 \\
&\leq \frac{2^{5/2}}{5}\frac{C_{LT}^{5/2}}{\nu^{3/2}}\|\vec{u}\|_{\dot{H}^1}^{7/4}\,\|\vec{u}\|_{\dot{H}^2}^{3/4} + \frac{3\nu}{5}\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 \\
&\leq 2\frac{C_{LT}^{5/2}}{\nu^{3/2}\alpha^{3/4}}\|\vec{u}\|_{\dot{H}^1}^{7/4}\,(\alpha\|\vec{u}\|_{\dot{H}^2})^{3/4} + \frac{3\nu}{5}\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2
\end{aligned}$$

At this point, always in the first term to the right hand side, we apply the Young inequalities (with $1 = 5/8 + 3/8$) to get

$$\begin{aligned}
2\frac{C_{LT}^{5/2}}{\nu^{3/2}\alpha^{3/4}}\|\vec{u}\|_{\dot{H}^1}^{7/4}\,(\alpha\|\vec{u}\|_{\dot{H}^2})^{3/4} + \frac{3\nu}{5}\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 &\leq \frac{5}{8}\left(2\frac{C_{LT}^{5/2}}{\nu^{3/2}\alpha^{3/4}}\|\vec{u}\|_{\dot{H}^1}^{7/4}\right)^{8/5} + \frac{3}{8}\alpha^2\|\vec{u}\|_{\dot{H}^2}^2 + \frac{3\nu}{5}\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 \\
&\leq 2\frac{C_{LT}^4}{\nu^{12/5}\alpha^{6/5}}\|\vec{u}\|_{\dot{H}^1}^{14/5} + \frac{3}{8}\alpha^2\|\vec{u}\|_{\dot{H}^2}^2 + \frac{3\nu}{5}\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2.
\end{aligned}$$

Finally, for the term $I_1$ in (48) we have the estimate

$$I_1 \leq 2\frac{C_{LT}^4}{\nu^{12/5}\alpha^{6/5}}\|\vec{u}(t,\cdot)\|_{\dot{H}^1}^{14/5} + \frac{3}{8}\alpha^2\|\vec{u}(t,\cdot)\|_{\dot{H}^2}^2 + \frac{3\nu}{5}\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2. \tag{50}$$

We estimate now the term $I_2$ in (48). By definition of the scalar product $[\cdot,\cdot]_\alpha$ given in (44), and integrating by parts, we write

$$\begin{aligned}
I_2 =& \sum_{i=1}^{m}\int_{\mathbb{R}^3}(\nu\Delta\vec{w}_i - \beta\vec{w}_i)\cdot\vec{w}_i\,dx + \alpha^2\sum_{i=1}^{m}\int_{\mathbb{R}^3}\vec{\nabla}\otimes(\nu\Delta\vec{w}_i - \beta\vec{w}_i)\cdot\vec{\nabla}\otimes\vec{w}_i\,dx \\
=& \nu\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 - \beta\sum_{i=1}^{m}\|\vec{w}_i\|_{L^2}^2 - \alpha^2\nu\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^2}^2 - \alpha^2\beta\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 \\
=& -\beta\sum_{i=1}^{m}\left(\|\vec{w}_i\|_{L^2}^2 + \alpha^2\|\vec{w}_i\|_{\dot{H}^1}^2\right) - \nu\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 - \alpha^2\nu\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^2}^2 \\
=& -\beta\sum_{i=1}^{m}\|\vec{w}_i\|_{\mathcal{H}_\alpha^1}^2 - \nu\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 - \alpha^2\nu\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^2}^2,
\end{aligned}$$

and recalling that $(\vec{w}_i)_{1\leq i\leq m}$ is an orthonormal family in $\mathcal{H}_\alpha^1(\mathbb{R}^3)$, we finally get

$$I_2 = -\beta m - \nu\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 - \alpha^2\nu\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^2}^2 \leq -\beta m - \nu\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2. \tag{51}$$

Once we have the estimates (50) and (51), we get back to (48) to write

$$\begin{aligned}
\sum_{i=1}^{m}[\mathcal{L}(t,\vec{u}_0)\vec{w}_i,\vec{w}_i]_\alpha &\leq 2\frac{C_{LT}^4}{\nu^{12/5}\alpha^{6/5}}\|\vec{u}(t,\cdot)\|_{\dot{H}^1}^{14/5} + \frac{3}{8}\alpha^2\|\vec{u}(t,\cdot)\|_{\dot{H}^2}^2 + \frac{3\nu}{5}\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 - \beta m - \nu\sum_{i=1}^{m}\|\vec{w}_i\|_{\dot{H}^1}^2 \\
&\leq -\beta m + 2\frac{C_{LT}^4}{\nu^{12/5}\alpha^{6/5}}\|\vec{u}(t,\cdot)\|_{\dot{H}^1}^{14/5} + \frac{3}{8}\alpha^2\|\vec{u}(t,\cdot)\|_{\dot{H}^2}^2,
\end{aligned}$$

and then we obtain the desired estimate. Proposition 5.1 is proven. ■

Once we have the estimate on the quantity $\displaystyle\sum_{i=1}^{m}[\mathcal{L}(t,\vec{u}_0)\vec{w}_i,\vec{w}_i]_\alpha$ given in Proposition 5.1, we shall continue estimating the quantity $\ell(m)$ given in (5.3). So, for $T>0$, we take the time-average $\frac{1}{T}\int_0^T(\cdot)dt$ in each term of (47) to get

$$\frac{1}{T}\int_0^T\sum_{i=1}^{m}[\mathcal{L}(t,\vec{u}_0)\vec{w}_i,\vec{w}_i]_\alpha \leq -\beta m + 2\frac{C^4_{LT}}{\nu^{12/5}\alpha^{6/5}}\underbrace{\frac{1}{T}\int_0^T\|\vec{u}(t,\cdot)\|^{14/5}_{\dot{H}^1}dt}_{(a)} + \frac{3}{8}\underbrace{\alpha^2\frac{1}{T}\int_0^T\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2}dt}_{(b)}, \tag{52}$$

where we must study now the terms $(a)$ and $(b)$. For the term $(a)$, recalling that $\|\cdot\|^2_{\mathcal{H}^1_\alpha}=\|\cdot\|^2_{L^2}+\alpha^2\|\cdot\|^2_{\dot{H}^1}$, by estimate (6) we have

$$\frac{1}{T}\int_0^T\|\vec{u}(t,\cdot)\|^{14/5}_{\dot{H}^1}dt \leq \frac{1}{\alpha^{14/5}}\frac{1}{T}\int_0^T\left(\alpha^2\|\vec{u}(t,\cdot)\|^2_{\dot{H}^1}\right)^{7/5}dt \leq \frac{1}{\alpha^{14/5}}\frac{1}{T}\int_0^T\left(\|\vec{u}_0\|^2_{\mathcal{H}^1_\alpha}e^{-\beta t}+\frac{4}{\beta^2}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha}\right)^{7/5}dt$$
$$\leq \frac{2^{2/5}}{\alpha^{14/5}}\left(\|\vec{u}_0\|^{14/5}_{\mathcal{H}^1_\alpha}\frac{1}{T}\int_0^T e^{-\frac{7\beta}{5}t}dt+\frac{2^{14/5}}{\beta^{14/5}}\|\vec{f}\|^{14/5}_{\mathcal{H}^1_\alpha}\right) \leq \frac{2^{2/5}}{\alpha^{14/5}}\|\vec{u}_0\|^{14/5}_{\mathcal{H}^1_\alpha}\frac{5}{7\beta T}(1-e^{-\frac{7\beta}{5}T})+\frac{2^{16/5}}{\alpha^{14/5}\beta^{14/5}}\|\vec{f}\|^{14/5}_{\mathcal{H}^1_\alpha}.$$

Recall also that by definition of the quantity $\ell(m)$ (see always the expression (5.3)) we have $\vec{u}_0\in\mathcal{A}_{\vec{f}}$, and moreover, as the global attractor $\mathcal{A}_{\vec{f}}$ is a compact set in $\mathcal{H}^1_\alpha(\mathbb{R}^3)$ (see Definition 1) then there exists $M>0$ such that $\|\vec{u}_0\|_{\mathcal{H}^1_\alpha}\leq M$ for all $\vec{u}_0\in\mathcal{H}^1_\alpha(\mathbb{R}^3)$. Then, by the previous estimate we can write

$$\frac{1}{T}\int_0^T\|\vec{u}(t,\cdot)\|^{14/5}_{\dot{H}^1}dt \leq \frac{2^{2/5}}{\alpha^{14/5}}\frac{5M^{14/5}}{7\beta T}(1-e^{-\frac{7\beta}{5}T})+\frac{2^{16/5}}{\alpha^{14/5}\beta^{14/5}}\|\vec{f}\|^{14/5}_{\mathcal{H}^1_\alpha}. \tag{53}$$

In order to estimate the term $(b)$, we will use now the inequality (7) (with $t=0$ and $T>0$) to write

$$\alpha^2\frac{1}{T}\int_0^T\|\vec{u}(t,\cdot)\|^2_{\dot{H}^2}dt \leq \frac{2}{\beta}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha}+\frac{1}{T}\|\vec{u}_0\|^2_{\mathcal{H}^1_\alpha} \leq \frac{2}{\beta}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha}+\frac{1}{T}M^2. \tag{54}$$

Gathering the estimates (53) and (54), in (52) we have

$$\begin{aligned}\frac{1}{T}\int_0^T\sum_{i=1}^{m}[\mathcal{L}(t,\vec{u}_0)\vec{w}_i,\vec{w}_i]_\alpha\,dt \leq& -\beta m + 2\frac{C^4_{LT}}{\nu^{12/5}\alpha^{6/5}}\left(\frac{2^{2/5}}{\alpha^{14/5}}\frac{5M^{14/5}}{7\beta T}(1-e^{-\frac{7\beta}{5}T})+\frac{2^{16/5}}{\alpha^{14/5}\beta^{14/5}}\|\vec{f}\|^{14/5}_{\mathcal{H}^1_\alpha}\right)\\ &+\frac{3}{8}\left(\frac{2}{\beta}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha}+\frac{1}{T}M^2\right)\\ \leq& -\beta m+\frac{1}{T}\underbrace{\left(2\frac{C^4_{LT}}{\nu^{12/5}\alpha^{6/5}}\frac{2^{2/5}}{\alpha^{14/5}}\frac{5M^{14/5}}{7\beta}(1-e^{-\frac{7\beta}{5}T})+\frac{3}{8}M^2\right)}_{(c)}\\ &+2\frac{C^4_{LT}}{\nu^{12/5}\alpha^{6/5}}\frac{2^{16/5}}{\alpha^{14/5}\beta^{14/5}}\|\vec{f}\|^{14/5}_{\mathcal{H}^1_\alpha}+\frac{3}{4\beta}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha},\end{aligned}$$

and moreover, we set now the constant

$$C(\alpha,\beta,\nu)=2\frac{C^4_{LT}}{\nu^{12/5}\alpha^{6/5}}\frac{2^{16/5}}{\alpha^{14/5}}+\frac{3}{4\beta}, \tag{55}$$

to write

$$\frac{1}{T}\int_0^T \sum_{i=1}^m [\mathcal{L}(t,\vec{u}_0)\vec{w}_i,\vec{w}_i]_\alpha \, dt \leq -\beta m + \frac{1}{T}(c) + C(\alpha,\beta,\nu)\max\left(\|\vec{f}\|^{14/5}_{\mathcal{H}^1_\alpha}, \|\vec{f}\|^2_{\mathcal{H}^1_\alpha}\right).$$

Finally, by (5.3) we obtain the estimate

$$\ell(m) \leq -\beta m + c(\alpha,\beta,\nu)\max\left(\|\vec{f}\|^{14/5}_{\mathcal{H}^1_\alpha}, \|\vec{f}\|^2_{\mathcal{H}^1_\alpha}\right), \tag{56}$$

which is the desired estimate (46) with $c_1 = \beta$, $\gamma = 1$ and $c_2 = C(\alpha,\beta,\nu)\max\left(\|\vec{f}\|^{14/5}_{\mathcal{H}^1_\alpha}, \|\vec{f}\|^2_{\mathcal{H}^1_\alpha}\right)$. Thus, appliying the Theorem 5.1 we get the upper bound for the fractal dimension of the attractor $\mathcal{A}_{\vec{f}}$ given in (9), where the constant $c(\alpha,\beta,\nu)$ is defined through the constant $C(\alpha,\beta,\nu)$, given in (55) as

$$c(\alpha,\beta,\nu) = \frac{1}{\beta}C(\alpha,\beta,\nu). \tag{57}$$

Theorem 3 is now proven. ■

# 6 Internal estructure of the global attractor $\mathcal{A}_{\vec{f}}$

## Proof of Theorem 4

In order to prove this theorem, we will verify each point stated in this result separately.

### 1) The existence of stationary solutions

Given an external force $\vec{f} \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$, we will construct a solution $(\vec{U}, P)$ for the stationary problem (12). We shall start by explaining the strategy of our proof. Since we work on the whole space $\mathbb{R}^3$, and moreover, since we have $div(\vec{f}) = 0$ and $div(\vec{U}) = 0$, then we can apply the Leray's projector $\mathbb{P}$ to the equation (12) to obtain the following equation:

$$-\nu\Delta\vec{U} + \mathbb{P}\text{div}((\vec{U}\otimes\vec{U})_\alpha) = \vec{f} - \beta\vec{U}. \tag{58}$$

From this equation, as $div(\vec{U}) = 0$ we may observe that the velocity $\vec{U}$ formally solves the following *equivalent* fixed point problem:

$$\begin{aligned}\vec{U} &= -\frac{1}{-\nu\Delta+\beta I_d}\left[\mathbb{P}\text{div}((\vec{U}\otimes\vec{U})_\alpha)\right] + \frac{1}{-\nu\Delta+\beta I_d}\left[\vec{f}\right] \\ &= -\frac{1}{-\nu\Delta+\beta I_d}\left[\mathbb{P}\left((\vec{U}\cdot\vec{\nabla})\vec{U}\right)_\alpha\right] + \frac{1}{-\nu\Delta+\beta I_d}\left[\vec{f}\right] = T(\vec{U}),\end{aligned} \tag{59}$$

where the non-local operator $\dfrac{1}{-\nu\Delta+\beta I_d}$ is easily defined in the Fourier variable by its symbol $\dfrac{1}{\nu|\xi|^2+\beta}$. Thus, the main idea to construct a solution $(\vec{U}, P)$ to the equation (12) is to solve first this fixed point problem, and then, the pressure term $P$ is calculated through the velocity $\vec{U}$ by the formula (29).

To solve the fixed point problem (59), we could use the (classical) Picard's iterative scheme in the space $H^2(\mathbb{R}^3)$, however, this scheme needs a control on the quantity $\left\|\dfrac{1}{-\nu\Delta+\beta I_d}\left[\vec{f}\right]\right\|_{H^2}$. Moreover, as we have the estimate $\left\|\dfrac{1}{-\nu\Delta+\beta I_d}\left[\vec{f}\right]\right\|_{H^2} \lesssim \|\vec{f}\|_{H^1}$, we observe that the Picard's iterative scheme ultimately needs a control on the quantity $\|\vec{f}\|_{H^1}$. Consequently, this scheme works for the case of external forces small enough: $\|\vec{f}\|_{H^1} \lesssim 1$.

Since we want to construct stationary solutions $(\vec{U}, P)$ for any external force $\vec{f} \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$, we shall use here a different approach. Instead of the Picard's fixed point theorem, the key tool to construct a solution $(\vec{U}, P)$ will be the Sheafer's fixed point theorem. For a proof see the Theorem 16.1, page 529 of [21].

**Theorem 6.1 (Sheafer's fixed point)** *Let $E$ be a Banach space and let $T : E \to E$ such that:*

*1) $T$ is a continuous operator.*

*2) $T$ is a compact operator.*

*3) There exists a constant $M > 0$, such that for all $\lambda \in [0, 1]$, if $e \in E$ verifies $e = \lambda T(e)$ then we have $\|e\|_E \leq M$.*

*Then, there exists $\overline{e} \in E$ such that $\overline{e} = T(\overline{e})$.*

It is worth mention this result asserts the existence of at least a fixed point of the operator $T$, but there is not any supplementary information on its uniqueness.

We would like to apply this result to the Banach space $E = \left\{ \vec{U} \in H^2 : \; div(\vec{U}) = 0 \right\}$, and the operator $T$ given in (59), however, there is a technical problem to overcome: to the best of our knowledge, we do not how to verify that the compactness of the operator $T$ in the space $E$ above. To solve this problem, we will consider a family of *modified* operators $(T_r)_{r>0}$ which verify all the points of the Sheafer's fixed theorem above. Then, for all $r > 0$ fixed, by this theorem we will get a solution $\vec{U}_r \in E$ of the equation $\vec{U}_r = T_r(\vec{U}_r)$. Finally, using a Rellich-Lions lemma we will show that the sequence $(\vec{U}_r)_{r>0}$ converges to a solution of the stationary problem (12).

For $r > 0$ fixed, we define now the modified operator $T_r : E \to E$ as follows. Let $\theta \in \mathcal{C}^\infty_0(\mathbb{R}^3)$ be a test function such that $0 \leq \theta(x) \leq 1$, $\theta(x) = 1$ for $|x| < 1$, and $\theta(x) = 0$ for $|x| > 2$. Then, we define the cut-off function $\theta_r(x) = \theta(x/r)$, and moreover, we define $T_r$ by

$$T_r(\vec{U}) = -\frac{1}{-\nu\Delta + \beta I_d}\left[\mathbb{P}\left((\theta_r\vec{U}\cdot\vec{\nabla})(\theta_r\vec{U})\right)_\alpha\right] + \frac{1}{-\nu\Delta + \beta I_d}\left[\vec{f}\right]. \tag{60}$$

We will prove that this operator verifies the points stated in the Theorem 6.1. For the point 1), for $\vec{U}, \vec{V} \in E$ we write

$$\begin{aligned} \|T_r(\vec{U}) - T_r(\vec{V})\|_{H^2} &= \left\| -\frac{1}{-\nu\Delta + \beta I_d}\left[\mathbb{P}\left((\theta_r\vec{U}\cdot\vec{\nabla})(\theta_r\vec{U}) - (\theta_r\vec{V}\cdot\vec{\nabla})(\theta_r\vec{V})\right)_\alpha\right]\right\|_{H^2} \\ &= \left\| -\frac{1}{-\nu\Delta + \beta I_d}\left[\mathbb{P}\left(((\theta_r(\vec{U}-\vec{V}))\vec{\nabla})(\theta_r\vec{U}) + ((\theta_r\vec{V})\cdot\vec{\nabla})(\theta_r(\vec{U}-\vec{V}))\right)_\alpha\right]\right\|_{H^2}. \end{aligned} \tag{61}$$

Using the fact that the symbol of the operator $\dfrac{1}{-\nu\Delta + \beta I_d}$ is a bounded function in the frequency variable, and moreover, by the well-known properties of the Leray's projector $\mathbb{P}$, we can write

$$\begin{aligned} &\left\| -\frac{1}{-\nu\Delta + \beta I_d}\left[\mathbb{P}\left(((\theta_r(\vec{U}-\vec{V}))\vec{\nabla})(\theta_r\vec{U}) + ((\theta_r\vec{V})\cdot\vec{\nabla})(\theta_r(\vec{U}-\vec{V}))\right)_\alpha\right]\right\|_{H^2} \\ \leq & c(\beta,\nu)\left(\left\|\left(((\theta_r(\vec{U}-\vec{V}))\vec{\nabla})(\theta_r\vec{U}) + ((\theta_r\vec{V})\cdot\vec{\nabla})(\theta_r(\vec{U}-\vec{V}))\right)_\alpha\right\|_{H^2}\right), \end{aligned}$$

and then, by definition of the filterig operator $(\cdot)_\alpha$ given in formula (2) we can write

$$\begin{aligned}\|T_r(\vec{U})-T_r(\vec{V})\|_{H^2} \leq & c(\beta,\nu)\left(\left\|\left(((\theta_r(\vec{U}-\vec{V}))\vec{\nabla})(\theta_r\vec{U})+((\theta_r\vec{V})\cdot\vec{\nabla})(\theta_r(\vec{U}-\vec{V}))\right)_\alpha\right\|_{H^2}\right)\\ \leq & c(\alpha,\beta,\nu)\left\|((\theta_r(\vec{U}-\vec{V}))\vec{\nabla})(\theta_r\vec{U})+((\theta_r\vec{V})\cdot\vec{\nabla})(\theta_r(\vec{U}-\vec{V}))\right\|_{L^2}.\end{aligned}$$

Moreover, applying the Hölder inequalities and recalling that the space $H^2(\mathbb{R}^3)$ embeds in the spaces $L^\infty(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$, we obtain

$$\begin{aligned}& c(\alpha,\beta,\nu)\left\|((\theta_r(\vec{U}-\vec{V}))\vec{\nabla})(\theta_r\vec{U})+((\theta_r\vec{V})\cdot\vec{\nabla})(\theta_r(\vec{U}-\vec{V}))\right\|_{L^2}\\ \leq & c(\alpha,\beta,\nu)\left(\left\|\theta_r(\vec{U}-\vec{V})\right\|_{L^\infty}\left\|\vec{\nabla}\otimes(\theta_r\vec{U})\right\|_{L^2}+\left\|\theta_r\vec{V}\right\|_{L^\infty}\left\|\vec{\nabla}\otimes(\theta_r(\vec{U}-\vec{V}))\right\|_{L^2}\right)\\ \leq & c(\alpha,\beta,\nu,\theta_r)\left(\left\|\vec{U}-\vec{V}\right\|_{L^\infty}\left\|\vec{U}\right\|_{H^1}+\left\|\vec{V}\right\|_{L^\infty}\left\|\vec{U}-\vec{V}\right\|_{H^1}\right)\\ \leq & c(\alpha,\beta,\nu,\theta_r)\left(\left\|\vec{U}-\vec{V}\right\|_{H^2}\left\|\vec{U}\right\|_{H^2}+\left\|\vec{V}\right\|_{H^2}\left\|\vec{U}-\vec{V}\right\|_{H^2}\right).\end{aligned}$$

Thus, we are able to write

$$\left\|T_r(\vec{U})-T_r(\vec{V})\right\|_{H^2}\leq c(\alpha,\beta,\nu,\theta_r)\left(\left\|\vec{U}\right\|_{H^2}+\left\|\vec{V}\right\|_{H^2}\right)\left\|\vec{U}-\vec{V}\right\|_{H^2},$$

hence we have the continuity of the operator $T_r:E\to E$.

We verify now the point 2). Let $(\vec{V}_n)_{n\in\mathbb{N}}$ be a bounded sequence in the space $E$. Then, this sequence converges to a limit $\vec{V}\in H^2(\mathbb{R}^3)$ in the weak topology of the space $H^2(\mathbb{R}^3)$. Moreover, as we have $div(\vec{V}_n)=0$, for all $n\in\mathbb{N}$, then we get that $div(\vec{V})=0$, and thus we have $\vec{V}\in E$.

We must show that the sequence $(T_r(\vec{V}_n))_{n\in\mathbb{N}}$ strongly converges (via a sub-sequence) to $T_r(\vec{V})$ in the space $E$. For this, by (61), and moreover, by the well-known properties of operators $\dfrac{1}{-\nu\Delta+\beta I_d}$, $\mathbb{P}$ and $(\cdot)_\alpha=\dfrac{1}{-\alpha^2\Delta+I_d}$, we can write

$$\begin{aligned}\|T_r(\vec{V}_n)-T_r(\vec{V})\|_{H^2} = & \left\|\frac{1}{-\nu\Delta+\beta I_d}\left[\mathbb{P}\left(((\theta_r(\vec{V}_n-\vec{V}))\vec{\nabla})(\theta_r\vec{V}_n)+((\theta_r\vec{V})\cdot\vec{\nabla})(\theta_r(\vec{V}_n-\vec{V}))\right)_\alpha\right]\right\|_{H^2}\\ \leq & c_{\alpha,\beta,\nu}\left\|((\theta_r(\vec{V}_n-\vec{V}))\vec{\nabla})(\theta_r\vec{V}_n)+((\theta_r\vec{V})\cdot\vec{\nabla})(\theta_r(\vec{V}_n-\vec{V}))\right\|_{H^{-2}}\\ \leq & c_{\alpha,\beta,\nu}\left\|((\theta_r(\vec{V}_n-\vec{V}))\vec{\nabla})(\theta_r\vec{V}_n)+((\theta_r\vec{V})\cdot\vec{\nabla})(\theta_r(\vec{V}_n-\vec{V}))\right\|_{H^{-1}}\\ \leq & c_{\alpha,\beta,\nu}\left\|((\theta_r(\vec{V}_n-\vec{V}))\vec{\nabla})(\theta_r\vec{V}_n)+((\theta_r\vec{V})\cdot\vec{\nabla})(\theta_r(\vec{V}_n-\vec{V}))\right\|_{\dot{H}^{-1}}\\ \leq & c_{\alpha,\beta,\nu}\left(\underbrace{\left\|((\theta_r(\vec{V}_n-\vec{V}))\vec{\nabla})(\theta_r\vec{V}_n)\right\|_{\dot{H}^{-1}}}_{(a)}+\underbrace{\left\|((\theta_r\vec{V})\cdot\vec{\nabla})(\theta_r(\vec{V}_n-\vec{V}))\right\|_{\dot{H}^{-1}}}_{(b)}\right),\end{aligned}\tag{62}$$

where, we will prove that the terms $(a)$ and $(b)$ converge to zero when $n$ goes to $+\infty$.

For the term $(a)$, by the Hardy-Littlewood-Sobolev inequalities, by the Hölder inequalities, and moreover, recalling that the sequence $(\vec{V}_n)_{n\in\mathbb{N}}$ is bounded in $H^2(\mathbb{R}^3)$, we can write

$$\begin{aligned}(a)\leq & c\left\|((\theta_r(\vec{V}_n-\vec{V}))\vec{\nabla})(\theta_r\vec{V}_n)\right\|_{L^{6/5}}\leq c\left\|(\theta_r(\vec{V}_n-\vec{V})\right\|_{L^3}\left\|\vec{\nabla}\otimes(\theta_r\vec{V}_n)\right\|_{L^2}\\ \leq & c(\theta_r)\left\|(\theta_r(\vec{V}_n-\vec{V})\right\|_{L^3}\|\vec{V}_n\|_{H^1}\leq c(\theta_r)\left\|(\theta_r(\vec{V}_n-\vec{V})\right\|_{L^3}\|\vec{V}_n\|_{H^2}\leq c(\theta_r)\left\|(\theta_r(\vec{V}_n-\vec{V})\right\|_{L^3}.\end{aligned}$$

We will prove now the term in the right side converges to zero when $n$ goes to $+\infty$. For this we have the following technical result.

**Lemma 6.1** *For $r>0$ fixed, the sequence $(\theta_r \vec{V}_n)_{n\in\mathbb{N}}$ strongly converges (through a sub-sequence) to $\theta_r\vec{V}$ in the space $L^p(\mathbb{R}^3)$ with $2\leq p<+\infty$.*

**Proof.** We observe first that the sequence $(\theta_{2r}\vec{V}_n)_{n\in\mathbb{N}}$ is also bounded in the space $H^2(\mathbb{R}^3)$, and moreover, by definition of the cut-off function $\theta_{2r}$ we have $supp\left(\theta_{2r}\vec{V}_n\right)\subset B(0,4r)$, for all $n\in\mathbb{N}$. Then, by the Rellich-Lions lemma there exists $\vec{W}\in H^2(\mathbb{R}^3)$ such that the sequence $(\theta_{2r}\vec{V}_n)_{n\in\mathbb{N}}$ strongly converges (through a sub-sequence) to $\vec{W}$ in $L^2(\mathbb{R}^3)$. Moreover, the sequence $(\theta_{2r}\vec{V}_n)_{n\in\mathbb{N}}$ strongly converges to $\vec{W}$ in $L^p(\mathbb{R}^3)$ with $2\leq p\leq+\infty$. On the other hand, as we have $\theta_{2r}=1$ on $supp(\theta_r)$ then we obtain that the sequence $(\theta_r\vec{V}_n)_{n\in\mathbb{N}}$ strongly converges to $\theta_r\vec{W}$ in $L^p(\mathbb{R}^3)$ with $2\leq p<+\infty$. Finally, we will prove the identity $\theta_r\vec{W}=\theta_r\vec{V}$. For this, just recall that $\vec{V}_n$ converges to $\vec{V}$ in the weak topology of $H^2(\mathbb{R}^3)$ and then it also converges to $\vec{V}$ in the weak topology of $L^2(\mathbb{R}^3)$. Hence we get that the sequence $\theta_r\vec{V}_n$ weakly converges to $\theta_r\vec{V}$ in $L^2(\mathbb{R}^3)$. Thus, since $\theta_r\vec{V}_n$ strongly converges $\theta_r\vec{W}$ in $L^2(\mathbb{R}^3)$ then we have $\theta_r\vec{W}=\theta_r\vec{V}$. ■

We study now the term $(b)$. We will need here the following identity.

**Lemma 6.2** *Let $\vec{A}=(A_1,A_2,A_3),\vec{B}=(B_1,B_2,B_3)$ be two vector fields. If $div(\vec{A})=0$ then we have*

$$((\theta_r\vec{A})\cdot\vec{\nabla})(\theta_r\vec{B})=div(\theta_r^2\vec{A}\otimes\vec{B})-\theta_r(\vec{\nabla}\theta_r\cdot\vec{A})\vec{B}.$$

**Proof.** We write

$$\begin{aligned}((\theta_r\vec{A})\cdot\vec{\nabla})(\theta_r\vec{B})=&\sum_{i=1}^{3}\theta_rA_i\partial_i(\theta_r\vec{B})=\sum_{i=1}^{3}\partial_i\left(\theta_r^2A_i\vec{B}\right)-\sum_{i=1}^{3}\theta_r\partial_i(\theta_rA_i)\vec{B}=div(\theta_r^2\vec{A}\otimes\vec{B})-\sum_{i=1}^{3}\theta_r\partial_i(\theta_rA_i)\vec{B}\\ =&div(\theta_r^2\vec{A}\otimes\vec{B})-\sum_{i=1}^{3}\theta_r(\partial_i\theta_r)A_i-\sum_{i=1}^{3}\theta_r^2(\partial_iA_i)\vec{B}=div(\theta_r^2\vec{A}\otimes\vec{B})-\theta_r(\vec{\nabla}\theta_r\cdot\vec{A})\vec{B}.\end{aligned}$$ ■

By this lemma, setting $\vec{A}=\vec{V}$ and $\vec{B}=\vec{V}_n-\vec{V}$, in the term $(b)$ we have

$$(b)=\left\|div(\theta_r^2\vec{V}\otimes(\vec{V}_n-\vec{V}))\right\|_{\dot{H}^{-1}}+\left\|\theta_r(\vec{\nabla}\theta_r\cdot\vec{V})(\vec{V}_n-\vec{V})\right\|_{\dot{H}^{-1}}=(b_1)+(b_2),$$

and then, we must study the terms $(b_1)$ and $(b_2)$ separately. For the term $(b_1)$ we write

$$(b_1)\leq c\left\|\theta_r^2\vec{V}\otimes(\vec{V}_n-\vec{V}))\right\|_{L^2}\leq\left\|\theta_r\vec{V}\right\|_{L^\infty}\left\|\theta_r(\vec{V}_n-\vec{V})\right\|_{L^2}\leq c\left\|\vec{V}\right\|_{H^2}\left\|\theta_r(\vec{V}_n-\vec{V})\right\|_{L^2}$$

where, by Lemma 6.1 we conclude that this term tends to zero when $n$ goes to $+\infty$. On the other hand, to estimate the term $(b_2)$, we use again the Hardy-Littlewood-Sobolev inequalities and the Hölder inequalities to write

$$(b_2)\leq c\left\|\theta_r(\vec{\nabla}\theta_r\cdot\vec{V})(\vec{V}_n-\vec{V})\right\|_{L^{6/5}}\leq\left\|\vec{\nabla}\theta_r\cdot\vec{V}\right\|_{L^3}\left\|\theta_r(\vec{V}_n-\vec{V})\right\|_{L^2},$$

where, always by Lemma 6.1, we also conclude that this term tends to zero when $n$ goes to $+\infty$.

Once we have shown that the $(a)$ and $(b)$ converge to zero, by (62) we obtain that $T_r(\vec{V}_n)$ strongly converges to $T_r(\vec{V})$ in the space $E$, and then, the operator $T_r$ is compact.

We verify now the point 3). Let $\lambda \in [0,1]$, and $\vec{U} \in E$ such that it verifies $\vec{U} = \lambda T_r(\vec{U})$. Then, by definition of the operator $T_r$ given in (60), we get that $\vec{U}$ solves the equation:

$$-\nu\Delta\vec{U} + \beta\vec{U} + \lambda\,\mathbb{P}\left((\theta_r\vec{U}\cdot\vec{\nabla})(\theta_r\vec{U})\right)_\alpha = \lambda\,\vec{f}.$$

Moreover, applying the operator $(-\alpha^2\Delta + Id)$ to each term in this equation, and recalling that we have $(\cdot)_\alpha = (-\alpha^2\Delta + Id)^{-1}$, we obtain that $\vec{U}$ also solves the equation:

$$\nu\alpha^2\Delta^2\vec{U} - (\nu+\beta\alpha^2)\Delta\vec{U} + \beta\vec{U} + \lambda\,\mathbb{P}(\theta_r\vec{U}\cdot\vec{\nabla})(\theta_r\vec{U}) = -\alpha^2\lambda\,\Delta\vec{f} + \lambda\,\vec{f}.$$

In this equation, we multiply first by $\vec{U}$ and then we integrate on the whole space $\mathbb{R}^3$. After some integration by parts, and moreover, as we have $div(\vec{U}) = 0$, we finally get

$$\nu\alpha^2\|\vec{U}\|^2_{\dot{H}^2} + (\nu+\beta\alpha^2)\|\vec{U}\|^2_{\dot{H}^1} + \beta\|\vec{U}\|^2_{L^2} = \alpha^2\lambda\int_{\mathbb{R}^3}\vec{\nabla}\otimes\vec{f}\cdot\vec{\nabla}\otimes\vec{U}\,dx + \lambda\int_{\mathbb{R}^3}\vec{f}\cdot\vec{U}\,dx.$$

In the term in the right side, by the Cauchy-Schwarz inequalities, using the fact that $\lambda \in [0,1]$, and moreover, by the Young inequalities, we write

$$\begin{aligned}-\alpha^2\lambda\int_{\mathbb{R}^3}\vec{f}\cdot\Delta\vec{U}\,dx + \lambda\int_{\mathbb{R}^3}\vec{f}\cdot\vec{U}\,dx &\leq \alpha^2\lambda\,\|\vec{f}\|_{\dot{H}^1}\,\|\vec{U}\|_{\dot{H}^1} + \lambda\,\|\vec{f}\|_{L^2}\,\|\vec{U}\|_{L^2}\\ &\leq \alpha^2\|\vec{f}\|_{\dot{H}^1}\,\|\vec{U}\|_{\dot{H}^1} + \|\vec{f}\|_{L^2}\,\|\vec{U}\|_{L^2}\\ &\leq \frac{\alpha^2}{2\beta}\|\vec{f}\|^2_{\dot{H}^1} + \frac{\beta\alpha^2}{2}\|\vec{U}\|^2_{\dot{H}^1} + \frac{1}{2\beta}\|\vec{f}\|^2_{L^2} + \frac{\beta}{2}\|\vec{U}\|^2_{L^2}.\end{aligned} \tag{63}$$

Getting back to the last identity, we obtain

$$\frac{\nu\alpha^2}{2}\|\vec{U}\|^2_{\dot{H}^2} + (\nu + \frac{\beta\alpha^2}{2})\|\vec{U}\|^2_{\dot{H}^1} + \frac{\beta}{2}\beta\|\vec{U}\|^2_{L^2} \leq \frac{1}{2\beta}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right), \tag{64}$$

hence we have

$$\beta\|\vec{U}\|^2_{L^2} + \nu\alpha^2\|\vec{U}\|^2_{\dot{H}^2} \leq \frac{1}{\beta}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right).$$

Finally, we can write

$$\|\vec{U}\|^2_{H^2} \leq \frac{1}{\min(\beta,\nu\alpha^2)\beta}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right), \tag{65}$$

where, setting the constant $M^2 = \dfrac{1}{\min(\beta,\nu\alpha^2)\beta}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right)$, the point 3) is now verified.

We may apply the Theorem 6.1 to obtain a solution $\vec{U}_r \in E$ of the problem $\vec{U}_r = T_r(\vec{U}_r)$, for all $r > 0$ fixed. We will show now that the family $(\vec{U}_r)_{r>0}$ converges (in the distributional sense) to a solution of the equation (12) when the parameter $r$ goes (via a sub-sequence) to infinity.

We observe first that by estimate (65) the family $(\vec{U}_r)_{r>0}$ in uniformly bounded in the space $H^2(\mathbb{R}^3)$. Thus, for all $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^3)$ we have: $\sup_{r>0}\|\varphi\,\vec{U}_r\|_{H^2} < +\infty$, and then, by the Rellich-Lions lemma there exists $\vec{U} \in H^2_{loc}(\mathbb{R}^3)$ and a sub-sequence $(r_n)_{n\in\mathbb{N}}$, such that $r_n \to +\infty$ when $n \to +\infty$, and such that $\vec{U}_{r_n}$ strongly converges to $\vec{U}$ in the space $L^p_{loc}(\mathbb{R}^3)$ for $2 \leq p < +\infty$. Moreover, as $div(\vec{U}_{r_n}) = 0$ for all $n \in \mathbb{N}$ we have $div(\vec{U}) = 0$, and always by the uniform boundness of the sequence $\vec{U}_{r_n}$ in the space $H^2(\mathbb{R}^3)$, by the Banach-Alaoglu theorem we obtain that $\vec{U} \in H^2(\mathbb{R}^3)$.

In order to prove that $\vec{U} \in H^2(\mathbb{R}^3)$ is a solution of the equation (12), we recall that as we have $\vec{U}_{r_n} = T_{r_n}(\vec{U}_{r_n})$, then $\vec{U}_{r_n}$ solves the equation

$$\nu\alpha^2\Delta^2\vec{U}_{r_n} - (\nu+\beta\alpha^2)\Delta\vec{U}_{r_n} + \beta\vec{U}_{r_n} + \mathbb{P}(\theta_{r_n}\vec{U}_{r_n}\cdot\vec{\nabla})(\theta_{r_n}\vec{U}_{r_n}) = -\alpha^2\Delta\vec{f} + \vec{f}.$$

Moreover, since we have $div(\vec{U}_{r_n}) = 0$ and $div(\vec{f}) = 0$, then $\vec{U}_{r_n}$ verifies the equation

$$\mathbb{P}\left(\nu\alpha^2\Delta^2\vec{U}_{r_n} - (\nu+\beta\alpha^2)\Delta\vec{U}_{r_n} + \beta\vec{U}_{r_n} + (\theta_{r_n}\vec{U}_{r_n}\cdot\vec{\nabla})(\theta_{r_n}\vec{U}_{r_n}) + \alpha^2\Delta\vec{f} - \vec{f}\right) = 0,$$

where, by well-known properties of the Leray's projector, for all $n \in \mathbb{N}$ we have

$$\vec{\nabla}\wedge\left(\nu\alpha^2\Delta^2\vec{U}_{r_n} - (\nu+\beta\alpha^2)\Delta\vec{U}_{r_n} + \beta\vec{U}_{r_n} + (\theta_{r_n}\vec{U}_{r_n}\cdot\vec{\nabla})(\theta_{r_n}\vec{U}_{r_n}) + \alpha^2\Delta\vec{f} - \vec{f}\right) = 0. \tag{66}$$

Here, we will prove that the non-linear term $(\theta_{r_n}\vec{U}_{r_n}\cdot\vec{\nabla})(\theta_{r_n}\vec{U}_{r_n})$ converges in the distributional sense to the non-linear term $(\vec{U}\cdot\vec{\nabla})\vec{U}$ when $n \to +\infty$. Indeed, we observe first that as $div(\vec{U}_{r_n}) = 0$, then we can write

$$(\theta_{r_n}\vec{U}_{r_n}\cdot\vec{\nabla})(\theta_{r_n}\vec{U}_{r_n}) = \theta_{r_n}\left((\vec{U}_{r_n}\cdot\vec{\nabla})(\theta_{r_n}\vec{U}_{r_n})\right) = \theta_{r_n}div\left(\vec{U}_{r_n}\otimes(\theta_{r_n}\vec{U}_{r_n})\right).$$

Thereafter, as $\vec{U}_{r_n}$ strongly converges to $\vec{U}$ in the space $L^4_{loc}(\mathbb{R}^3)$, and moreover, as we have $\theta_{r_n}(x) = 1$ when $|x| < r_n$, then $\theta_{r_n}\vec{U}_{r_n}$ also strongly converges to $\vec{U}$ in $L^4_{loc}(\mathbb{R}^3)$. We get that $\vec{U}_{r_n}\otimes(\theta_{r_n}\vec{U}_{r_n})$ converges to $\vec{U}\otimes\vec{U}$ in the strong topology of the space $L^2_{loc}(\mathbb{R}^3)$, hence we conclude the desired convergence. We thus have the following limit in the distributional sense:

$$\begin{aligned}\lim_{r_n\to+\infty}&\left(\nu\alpha^2\Delta^2\vec{U}_{r_n} - (\nu+\beta\alpha^2)\Delta\vec{U}_{r_n} + \beta\vec{U}_{r_n} + (\theta_{r_n}\vec{U}_{r_n}\cdot\vec{\nabla})(\theta_{r_n}\vec{U}_{r_n}) + \alpha^2\Delta\vec{f} - \vec{f}\right)\\ &= \nu\alpha^2\Delta^2\vec{U} - (\nu+\beta\alpha^2)\Delta\vec{U} + \beta\vec{U} + (\vec{U}\cdot\vec{\nabla})\vec{U} + \alpha^2\Delta\vec{f} - \vec{f}.\end{aligned}$$

Thus, by (66) we get

$$\vec{\nabla}\wedge\left(\nu\alpha^2\Delta^2\vec{U} - (\nu+\beta\alpha^2)\Delta\vec{U} + \beta\vec{U} + (\vec{U}\cdot\vec{\nabla})\vec{U} + \alpha^2\Delta\vec{f} - \vec{f}\right) = 0,$$

and then, there exists $Q \in \mathcal{D}'(\mathbb{R}^3)$ such that

$$\nu\alpha^2\Delta^2\vec{U} - (\nu+\beta\alpha^2)\Delta\vec{U} + \beta\vec{U} + (\vec{U}\cdot\vec{\nabla})\vec{U} + \alpha^2\Delta\vec{f} - \vec{f} = \vec{\nabla}Q.$$

Moreover, as $\vec{U} \in H^2(\mathbb{R}^3)$ and $\vec{f} \in H^1(\mathbb{R}^3)$, we observe that the term in the left side of this equation belongs to the space $H^{-2}(\mathbb{R}^3)$, hence $\vec{\nabla}Q \in H^{-2}(\mathbb{R}^3)$.

From this equation we can write

$$-\nu(-\alpha^2\Delta + I_d)\Delta\vec{U} + \beta(-\alpha^2\Delta + I_d)\vec{U} + (\vec{U}\cdot\vec{\nabla})\vec{U} - \vec{\nabla}Q = (-\alpha^2\Delta^2 + I_d)\vec{f}, \tag{67}$$

then, applying the filtering operator $(\cdot)_\alpha = (-\alpha^2\Delta + I_d)^{-1}$ to each term, and defining the pressure $P$ as $P = -(-\alpha^2\Delta^2 + I_d)^{-1}Q \in H^1(\mathbb{R}^3)$, we finally obtain that the couple $(\vec{U}, P)$ is a solution of equation (12).

**2) The energy estimate**

As we have $\vec{U} \in H^2(\mathbb{R}^3)$ and $P \in H^1(\mathbb{R}^3)$, then we can multiply each term in the equation (67) by $\vec{U}$. Thereafter, we integrate on $\mathbb{R}^3$ and after some integration by parts we obtain

$$\nu\alpha^2\|\vec{U}\|^2_{\dot{H}^2} + (\nu+\beta\alpha^2)\|\vec{U}\|^2_{\dot{H}^1} + \beta\|\vec{U}\|^2_{L^2} = \alpha^2\int_{\mathbb{R}^3}\vec{\nabla}\otimes\vec{f}\cdot\vec{\nabla}\otimes\vec{U}\,dx + \int_{\mathbb{R}^3}\vec{f}\cdot\vec{U}\,dx.$$

Moreover, the term in the right side was estimated in (63) (where we set $\lambda = 1$) and we have (64). From this estimate we can write now

$$\frac{\beta}{2}\|\vec{U}\|^2_{L^2} + \frac{\beta\alpha^2}{2}\|\vec{U}\|^2_{\dot{H}^1} + \frac{\nu\alpha^2}{2}\|\vec{U}\|^2_{\dot{H}^2} \leq \frac{1}{2\beta}\left(\|\vec{f}\|^2_{L^2} + \alpha^2\|\vec{f}\|^2_{\dot{H}^1}\right),$$

hence, recalling that $\|\cdot\|^2_{\mathcal{H}^1_\alpha} = \|\cdot\|^2_{L^2} + \alpha^2\|\cdot\|^2_{\dot{H}^1}$, we finally obtain the desired energy estimate

$$\|\vec{U}\|^2_{\mathcal{H}^1_\alpha} \leq \nu\alpha^2\|\vec{U}\|^2_{\dot{H}^2} \leq \frac{2}{\beta^2}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha}.$$

**3) All the stationary solutions belong to the global attractor**

To prove this point, we just recall that by (11) the global attractor $\mathcal{A}_{\vec{f}} \subset \mathcal{H}^1_\alpha(\mathbb{R}^3)$ is the set of functions $\vec{v}(0,\cdot)$, where $\vec{v} \in L^\infty\Big(\mathbb{R}, \mathcal{H}^1_\alpha(\mathbb{R}^3)\Big)$ is an eternal solution of the equation (10). On the other hand, by the energy estimate proven in the point 2), we know that all the solutions $\vec{U} \in H^2(\mathbb{R}^3)$ of the stationary problem (12) verify $\vec{U} \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$, and moreover, as they do not depend on the time variable, they also are a particular case of bounded in time eternal solutions, and we can write $\vec{U}(0,\cdot) = \vec{U}$. Consequently, all the solutions $\vec{U} \in H^2(\mathbb{R}^3)$ of the equation (12) belong to the global attractor $\mathcal{A}_{\vec{f}}$. Theorem 4 is now proven. ■

# 7 Additional properties of the global attractor driven by the damping parameter

## Proof of Theorem 5

The proof of this theorem bases on the following result concerning the long-time behavior of two solutions of the equation (4).

**Proposition 7.1** *Let $\vec{f}_1, \vec{f}_2 \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$ be two divergence-free external forces and let $\vec{u}_{0,1}, \vec{u}_{0,2} \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$ be two initial data. Moreover, let $(\vec{u}_1, p_1)\, (\vec{u}_2, p_2) \in L^\infty_t \mathcal{H}^1_\alpha \cap (L^2_{loc})_t \dot{H}^2_x \times (L^2_{loc})_t H^1_x$, be two global in time solutions of the equation (4) arising from the data $(\vec{u}_{0,1}, \vec{f}_1)$ and $(\vec{u}_{0,2}, \vec{f}_2)$ respectively.*

*Moreover, for a numerical constant $c > 0$, and the parameters $\alpha > 0$ and $\beta > 0$, we define the quantity $\eta(\beta)$ as:*

$$\eta(\beta) = 2\left(-\frac{\beta}{2} + \frac{c}{\alpha^{5/2}\beta}\|\vec{f}_1\|_{\mathcal{H}^1_\alpha}\right),$$

*Then, for all time $t \geq 0$, the following estimate holds:*

$$\|\vec{u}_1(t,\cdot) - \vec{u}_2(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} \leq \|\vec{u}_{0,1} - \vec{u}_{0,2}\|^2_{\mathcal{H}^1_\alpha} e^{\eta(\beta)\, t} + \frac{1}{\beta}\|\vec{f}_1 - \vec{f}_2\|^2_{\mathcal{H}^1_\alpha} \frac{1}{\eta(\beta)}\Big(e^{\eta(\beta)\, t} - 1\Big). \tag{68}$$

**Proof.** From the solutions $(\vec{u}_1, p_1) \in L^\infty_t \mathcal{H}^1_\alpha \cap (L^2_t)_{loc} \dot{H}^2_x \times (L^2_t)_{loc} H^1_x$ and $(\vec{u}_2, p_2) \in L^\infty_t \mathcal{H}^1_\alpha \cap (L^2_t)_{loc} \dot{H}^2_x \times (L^2_t)_{loc} H^1_x$, we define $\vec{w} = \vec{u}_1 - \vec{u}_2$ and $q = p_1 - p_2$. We observe that $(\vec{w}, q) \in L^\infty_t \mathcal{H}^1_\alpha \cap (L^2_t)_{loc} \dot{H}^2_x \times (L^2_t)_{loc} H^1_x$, solves the equation:

$$\partial_t \vec{w} + \Big((\vec{w}\cdot\vec{\nabla})\vec{u}_1 + (\vec{u}_2\cdot\vec{\nabla})\vec{w}\Big)_\alpha - \nu\Delta\vec{w} + \vec{\nabla} q = \vec{f}_1 - \vec{f}_2 - \beta\vec{w}, \quad div(\vec{w}) = 0, \quad \vec{w}(0,\cdot) = \vec{u}_1(0,\cdot) - \vec{u}_2(0,\cdot).$$

Moreover, performing the same computations done in (27), for $t \geq 0$, we have the following energy equality:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} =& -\nu\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^2} - \beta\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} - \Big\langle \vec{f}_1 - \vec{f}_2, \vec{w}\Big\rangle_{L^2\times L^2} \\ &+ \alpha^2\Big\langle \vec{\nabla}\otimes(\vec{f}_1 - \vec{f}_2), \vec{\nabla}\otimes\vec{w}\Big\rangle_{L^2\times L^2} - \Big\langle (\vec{w}\cdot\vec{\nabla})\vec{u}_1(t,\cdot), \vec{w}(t,\cdot)\Big\rangle_{\dot{H}^{-1}\times\dot{H}^1}. \end{aligned} \tag{69}$$

We study now the term $\Big\langle (\vec{w}\cdot\vec{\nabla})\vec{u}_1(t,\cdot), \vec{w}(t,\cdot)\Big\rangle_{\dot{H}^{-1}\times\dot{H}^1}$. More precisely, as $div(\vec{w}) = 0$ and integrating by parts, we can write

$$\begin{aligned} \Big\langle (\vec{w}\cdot\vec{\nabla})\vec{u}_1(t,\cdot), \vec{w}(t,\cdot)\Big\rangle_{\dot{H}^{-1}\times\dot{H}^1} &= \sum_{i,j=1}^{3}\langle w_j(\partial_j u_{1,i}), w_i)\rangle_{\dot{H}^{-1}\times\dot{H}^1} = \sum_{i,j=1}^{3}\langle(\partial_j w_j u_{1,i}), w_i)\rangle_{\dot{H}^{-1}\times\dot{H}^1} \\ &= -\sum_{i,j=1}^{3}\langle w_j u_{1,i}, \partial_j w_i)\rangle_{L^2\times L^2} = -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} w_j\, u_{1,i}\, \partial_j w_i\, dx. \end{aligned}$$

By the Parseval's identity and the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
-\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} w_j u_{1,i}\,\partial_j w_i\,dx = &-\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\widehat{w_j}\;\widehat{u_{1,i}\partial_j w_i}\,d\xi = -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}|\xi|\widehat{w_j}\;|\xi|^{-1}(\widehat{u_{1,i}\partial_j w_i})\,d\xi\\
&\leq \|\vec{w}(t,\cdot)\|_{\dot{H}^1}\|\vec{u}_1(t,\cdot)(\vec{\nabla}\otimes\vec{w})(t,\cdot)\|_{\dot{H}^{-1}}.
\end{aligned}
$$

Then, applying the Hardy-Littlewood-Sobolev inequalities, the Hölder inequalities (with $5/6 = 1/3 + 1/2$), and moreover, recalling that $\|\cdot\|^2_{\mathcal{H}^1_\alpha} = \|\cdot\|^2_{L^2} + \alpha^2\|\cdot\|_{\dot{H}^1}$, we have

$$
\begin{aligned}
\|\vec{w}(t,\cdot)\|_{\dot{H}^1}\|\vec{u}_1(\vec{\nabla}\otimes\vec{w})\|_{\dot{H}^{-1}} \leq & c\|\vec{w}(t,\cdot)\|_{\dot{H}^1}\|\vec{u}_1(\vec{\nabla}\otimes\vec{w})\|_{L^{6/5}} \leq c\|\vec{w}(t,\cdot)\|_{\dot{H}^1}\|\vec{u}_1(t,\cdot)\|_{L^3}\|\vec{\nabla}\otimes\vec{w}(t,\cdot)\|_{L^2}\\
\leq & c\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}\|\vec{u}_1(t,\cdot)\|_{L^3} \leq c\,\alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}\frac{1}{\alpha^2}\|\vec{u}_1(t,\cdot)\|_{L^3}\\
\leq & c\,\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha}\frac{1}{\alpha^2}\|\vec{u}_1(t,\cdot)\|_{L^3}.
\end{aligned}
$$

We still need to estimate the term $\frac{1}{\alpha^2}\|\vec{u}_1(t,\cdot)\|_{L^3}$. By the interpolation inequalities (with $1/3 = \theta/2 + (1-\theta)/6$, and $\theta = 1/2$) and applying again the Hardy-Littlewood-Sobolev inequalities, we can write

$$
\begin{aligned}
\frac{1}{\alpha^2}\|\vec{u}_1(t,\cdot)\|_{L^3} \leq & \frac{c}{\alpha^2}\|\vec{u}_1(t,\cdot)\|^{1/2}_{L^2}\,\|\vec{u}_1(t,\cdot)\|^{1/2}_{L^6} \leq \frac{c}{\alpha^2}\|\vec{u}_1(t,\cdot)\|^{1/2}_{L^2}\,\|\vec{u}_1(t,\cdot)\|^{1/2}_{\dot{H}^1}\\
\leq & \frac{c}{\alpha^{5/2}}\|\vec{u}_1(t,\cdot)\|^{1/2}_{L^2}\,\alpha^{1/2}\|\vec{u}_1(t,\cdot)\|^{1/2}_{\dot{H}^1} \leq \frac{c}{\alpha^{5/2}}\|\vec{u}_1(t,\cdot)\|_{\mathcal{H}^1_\alpha}.
\end{aligned}
$$

Now, by the point 1) in the Proposition 1 we obtain

$$
\frac{c}{\alpha^{5/2}}\|\vec{u}_1(t,\cdot)\|_{\mathcal{H}^1_\alpha} \leq \frac{c}{\alpha^{5/2}}\left(\|\vec{u}_{0,1}\|_{\mathcal{H}^1_\alpha}e^{-\frac{\beta}{2}t} + \frac{2}{\beta}\|\vec{f}_1\|_{\mathcal{H}^1_\alpha}\right).
$$

On the other hand, we recall that for any external force $\vec{f}\in\mathcal{H}^1_\alpha(\mathbb{R}^3)$, by Lemma 4.1 we have that the set $\mathcal{B} = \left\{\vec{u}\in\mathcal{H}^1_\alpha(\mathbb{R}^3) : \|\vec{u}\|^2_{\mathcal{H}^1_\alpha} \leq \frac{8}{\beta^2}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha}\right\}$, is a absorbing set in the sense of Definition 4.1. Then, for any initial datum $\vec{u}_0\in\mathcal{H}^1_\alpha(\mathbb{R}^3)$, the solution $\vec{u}(t,x)$ arising from $(\vec{u}_0,\vec{f})$ always lies in the set $\mathcal{B}$ from a time large enough. Thus, getting back to the solution $\vec{u}_1(t,x)$, without lost of generality we may suppose that the initial datum $\vec{u}_{0,1}$ belongs to the absorbing set $\mathcal{B} = \left\{\vec{u}\in\mathcal{H}^1_\alpha(\mathbb{R}^3) : \|\vec{u}\|^2_{\mathcal{H}^1_\alpha} \leq \frac{8}{\beta^2}\|\vec{f}_1\|^2_{\mathcal{H}^1_\alpha}\right\}$, and we can write the estimate $\|\vec{u}_{0,1}\|_{\mathcal{H}^1_\alpha} \leq \frac{\sqrt{8}}{\beta}\|\vec{f}_1\|_{\mathcal{H}^1_\alpha}$.

We thus have

$$
\frac{c}{\alpha^{5/2}}\|\vec{u}_1(t,\cdot)\|_{\mathcal{H}^1_\alpha} \leq \frac{c}{\alpha^{5/2}}\left(\frac{\sqrt{8}}{\beta}\|\vec{f}_1\|_{\mathcal{H}^1_\alpha}e^{-\frac{\beta}{2}t} + \frac{2}{\beta}\|\vec{f}_1\|_{\mathcal{H}^1_\alpha}\right) \leq \frac{c}{\alpha^{5/2}\beta^2}\|\vec{f}_1\|_{\mathcal{H}^1_\alpha},
$$

and then, we can write

$$
\frac{1}{\alpha^2}\|\vec{u}_1(t,\cdot)\|_{L^3} \leq \frac{c}{\alpha^{5/2}\beta}\|\vec{f}_1\|_{\mathcal{H}^1_\alpha}.
$$

Finally, gathering these inequalities we get the following estimate

$$
\left|\left\langle(\vec{w}\cdot\vec{\nabla})\vec{u}_1(t,\cdot),\vec{w}(t,\cdot)\right\rangle_{\dot{H}^{-1}\times\dot{H}^1}\right| \leq \|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha}\frac{c}{\alpha^{5/2}\beta}\|\vec{f}_1\|_{\mathcal{H}^1_\alpha}.
$$

With this estimate at hand, we get back to the energy equality (69) to write

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} \leq & -\nu\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1} - \alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^2} - \beta\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} - \left\langle\vec{f}_1-\vec{f}_2,\vec{w}\right\rangle_{L^2\times L^2}\\
& + \alpha^2\left\langle\vec{\nabla}\otimes(\vec{f}_1-\vec{f}_2),\vec{\nabla}\otimes\vec{w}\right\rangle_{L^2\times L^2} + \|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha}\frac{c}{\alpha^{5/2}\beta}\|\vec{f}_1\|_{\mathcal{H}^1_\alpha}.
\end{aligned}
$$

As the first and second term in the right side are negatives, we have

$$\frac{1}{2}\frac{d}{dt}\,\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} \leq -\,\beta\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} - \left\langle \vec{f}_1-\vec{f}_2, \vec{w}\right\rangle_{L^2\times L^2} + \alpha^2\left\langle \vec{\nabla}\otimes(\vec{f}_1-\vec{f}_2), \vec{\nabla}\otimes\vec{w}\right\rangle_{L^2\times L^2}$$
$$+\,\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha}\,\frac{c}{\alpha^{5/2}\beta}\,\|\vec{f}_1\|_{\mathcal{H}^1_\alpha}.$$

Then, applying first the Cauchy-Schwarz inequalities and thereafter the Young inequalities, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\,\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} \leq & -\,\beta\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} + \frac{1}{2\beta}\|\vec{f}_1-\vec{f}_2\|^2_{L^2} + \frac{\beta}{2}\|\vec{w}(t,\cdot)\|^2_{L^2} + \frac{\alpha^2}{2\beta}\|\vec{f}_1-\vec{f}_2\|^2_{\dot{H}^1} + \frac{\alpha^2\beta}{2}\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1} \\
& + \|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha}\,\frac{c}{\alpha^{5/2}\beta}\,\|\vec{f}_1\|_{\mathcal{H}^1_\alpha} \\
\leq & -\,\beta\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} + \frac{\beta}{2}\left(\|\vec{w}(t,\cdot)\|^2_{L^2} + \alpha^2\|\vec{w}(t,\cdot)\|^2_{\dot{H}^1}\right) + \frac{1}{2\beta}\left(\|\vec{f}_1\|^2_{L^2} + \alpha^2\|\vec{f}_1\|^2_{\dot{H}^1}\right) \\
& + \|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha}\,\frac{c}{\alpha^{5/2}\beta}\,\|\vec{f}_1\|_{\mathcal{H}^1_\alpha} \\
\leq & \left(-\frac{\beta}{2} + \frac{c}{\alpha^{5/2}\beta}\|\vec{f}_1\|_{\mathcal{H}^1_\alpha}\right)\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} + \frac{1}{2\beta}\|\vec{f}_1-\vec{f}_2\|^2_{\mathcal{H}^1_\alpha},
\end{aligned}$$

hence, we write

$$\frac{d}{dt}\,\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} \leq 2\left(-\frac{\beta}{2} + \frac{c}{\alpha^{5/2}\beta}\|\vec{f}_1\|_{\mathcal{H}^1_\alpha}\right)\|\vec{w}(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} + \frac{1}{\beta}\|\vec{f}_1-\vec{f}_2\|^2_{\mathcal{H}^1_\alpha}.$$

We set now the quantity $\eta(\beta) = 2\left(-\frac{\beta}{2} + \frac{c}{\alpha^{5/2}\beta}\|\vec{f}_1\|_{\mathcal{H}^1_\alpha}\right)$, and using the Grönwall inequalities we finally obtain the desired estimate (68). Proposition 7.1 is now proven. ■

With Proposition 7.1 at hand, we are able to prove each point stated in Theorem 5. For this, we recall the definition of the expression $\eta(\beta)$ given in (13).

### 1) The orbital stability when $\eta(\beta) = 0$.

In the framework of Proposition 7.1, firs we set $\vec{f}_1 = \vec{f}_2 = \vec{f}$, and we get the estimate

$$\|\vec{u}_1(t,\cdot) - \vec{u}_2(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} \leq \|\vec{u}_{0,1} - \vec{u}_{0,2}\|^2_{\mathcal{H}^1_\alpha}\, e^{\eta(\beta)\,t}.$$

Then, we take $\eta(\beta) = 0$ to obtain the following control

$$\|\vec{u}_1(t,\cdot) - \vec{u}_2(t,\cdot)\|^2_{\mathcal{H}^1_\alpha} \leq \|\vec{u}_{0,1} - \vec{u}_{0,2}\|^2_{\mathcal{H}^1_\alpha},$$

hence, the result stated in this point follows directly.

### 2) The characterization of the global attractor when $\eta(\beta) < 0$.

In the first step, we will prove that the uniqueness of the stationary solution constructed in point 1) of the Theorem 4. Let $(\vec{U}_1, P_1)$, $(\vec{U}_2, P_2) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ be two solutions of the stationary problem (12) associated with the same external force $\vec{f}$. As $\vec{U}_1$ and $\vec{U}_2$ are time-independing functions we have $\partial_t \vec{U}_1 = 0$ and $\partial_t \vec{U}_2 = 0$, and thus, $(\vec{U}_1, P_1)$, $(\vec{U}_2, P_2)$ are also two solutions of the equation (4), arising from the initial data $\vec{U}_1$ and $\vec{U}_2$ respectively and with external force $\vec{f}$. Moreover, we also have $(\vec{U}_1, P_1)$, $(\vec{U}_2, P_2) \in L^\infty_t H^1_x \cap (L^2_{loc})_t \dot{H}^2_x \times (L^2_{loc})_t H^1_x$.

By Proposition 7.1 ( with $\vec{f}_1 = \vec{f}_2 = \vec{f}$) we can write: $\|\vec{U}_1 - \vec{U}_2\|_{\mathcal{H}^1_\alpha} \leq \|\vec{U}_1 - \vec{U}_2\|_{\mathcal{H}^1_\alpha} e^{\eta(\beta)\, t}$. Moreover, as we have $\eta(\beta) < 0$, for a time $t > 0$ large enough we can write $\|\vec{U}_1 - \vec{U}_2\|_{\mathcal{H}^1_\alpha} \leq \frac{1}{2}\|\vec{U}_1 - \vec{U}_2\|_{\mathcal{H}^1_\alpha}$, hence we obtain the identity $\vec{U}_1 = \vec{U}_2$. Finally, as the pressure term is always related to the velocity field by (29) we also obtain the identity $P_1 = P_2$.

In the second step, we prove now that the singleton $\{\vec{U}\}$ is also a global attractor for the equation (4) given in Definition 1. The first point in Definition 1 is evident, so we will focus on the second point. Let $t \geq 0$. We will prove the identity $S(t)\{\vec{U}\} = \{\vec{U}\}$. Let $\vec{u} \in S(t)\{\vec{U}\}$, *i.e.*, $\vec{u}(t, \cdot)$ is the solution of the equation (4), at the time $t$ and arising from the initial datum $\vec{U}$. But, as explained above, we also have that $\vec{U}$ is a solution of the equation (4) which arises from $\vec{U}$. Then, by Proposition 7.1 ( with $\vec{u}_{1,0} = \vec{u}_{0,2} = \vec{U}$ and $\vec{f}_1 = \vec{f}_2 = \vec{f}$) we can write $\|\vec{u}(t, \cdot) - \vec{U}\|_{\mathcal{H}^1_\alpha} \leq \|\vec{U} - \vec{U}\|_{\mathcal{H}^1_\alpha} e^{\eta(\beta)\, t}$, hence we get $\vec{u} = \vec{U}$. On the other hand, we also have $\vec{U} \in S(t)\{\vec{U}\}$, and it directly follows from the fact that taking $\vec{U}$ as an initial datum in equation (4) then, for all time $t \geq 0$, the same function $\vec{U}$ is the unique solution for this problem. We verify now the third point in Definition 1. Let $\vec{u}_0 \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$ an initial datum and let $\vec{u} \in L^\infty_t \mathcal{H}^1_\alpha \cap (L^2_{loc})_t \dot{H}^2_x$ be the unique solution of equation (4), arising from $\vec{u}_0$ and associated with the force $\vec{f}$, which is given by Theorem 1. On the other hand, let $\vec{U} \in H^2(\mathbb{R}^3)$ be the unique stationary solution of the problem (4) associated with the same force $\vec{f}$. Thus, always by Proposition 7.1 (with $\vec{f}_1 = \vec{f}_2 = \vec{f}$), for all time $t \geq 0$ we have the estimate

$$\|\vec{u}(t, \cdot) - \vec{U}\|_{\mathcal{H}^1_\alpha} \leq \|\vec{u}_0 - \vec{U}\|_{\mathcal{H}^1_\alpha} e^{\eta(\beta), t}, \quad \eta(\beta) < 0. \tag{70}$$

By this estimate, we may observe that the unique stationary solution $\vec{U}$ is asymptotically stable. Actually we have a stronger stability property in the sense that, for any initial datum $\vec{u}_0$ the unique solution $\vec{u}(t, \cdot)$ arising from $\vec{u}_0$ strongly converges to the stationary solution $\vec{U}$ when $t \to +\infty$, and thus, the third point in Definition 1 holds.

In the third and last step, we will prove the identity $\{\vec{U}\} = \mathcal{A}_{\vec{f}}$. Indeed, on the one hand, we have that $\{\vec{U}\}$ is a global attractor of the equation (4). On the other hand, recall that by the Theorem 2 we also have the global attractor $\mathcal{A}_{\vec{f}}$ given by this theorem. But, by Lemma 2.18, page 16 in [24], we have the uniqueness of the global attractor, provided it satisfies all the points in Definition 1, hence we conclude that $\{\vec{U}\} = \mathcal{A}_{\vec{f}}$. Theorem 5 is now proven. ■

## Proof of Proposition 2

Let $\vec{U} \in \mathcal{A}_{\vec{f}}$ be the unique solution of the equation (12), and let $\vec{u} \in L^\infty_t (\mathcal{H}^1_\alpha)_x \cap (L^2_t)_{loc} \dot{H}^2_x$ be a solution of the equation (4) arising from an initial datum $\vec{u}_0 \in \mathcal{H}^1_\alpha(\mathbb{R}^3)$. We define the term $\mathcal{R}_{\vec{u}} = \vec{u} - \vec{U}$ which, to simplify the notation, we shall write as $\mathcal{R}$.

In order to prove (15), we observe first that as the stationary solution verifies $\partial_t \vec{U} = 0$, then it is also a solution of the equation (4) with initial datum $\vec{U}$. Thus, the term $\mathcal{R}$ solves the following equation:

$$\partial_t \mathcal{R} - \nu\Delta\mathcal{R} + \mathbb{P}\Big(((\vec{u} \cdot \vec{\nabla})\vec{u} - (\vec{U} \cdot \vec{\nabla})\vec{U})_\alpha\Big) + \beta\mathcal{R} = 0, \quad \mathcal{R}(0, \cdot) = \vec{u}_0 - \vec{U}, \tag{71}$$

and consequently $\mathcal{R}(t, x)$ can be written as the integral form:

$$\mathcal{R}(t, x) = e^{\nu t\Delta}(\vec{u}_0 - \vec{U}) - \beta \int_0^t e^{\nu(t-s)}\mathcal{R}(s, x)ds - \int_0^t e^{\nu(t-s)}\mathbb{P}\Big(((\vec{u} \cdot \vec{\nabla})\vec{u} - (\vec{U} \cdot \vec{\nabla})\vec{U})_\alpha\Big)(s, x)ds. \tag{72}$$

For all $t>0$, we write

$$
\begin{aligned}
\|\mathcal{R}(t,\cdot)\|_{L^\infty} \leq & \|e^{\nu t\Delta}(\vec{u}_0-\vec{U})\|_{L^\infty}+\beta\int_0^t \left\|e^{\nu(t-s)}\mathcal{R}(s,\cdot)\right\|_{L^\infty} ds \\
&+\int_0^t \left\|e^{\nu(t-s)}\mathbb{P}\Big(((\vec{u}\cdot\vec{\nabla})\vec{u}-(\vec{U}\cdot\vec{\nabla})\vec{U})_\alpha\Big)(s,\cdot)\right\|_{L^\infty} ds = I_1(t)+I_2(t)+I_3(t),
\end{aligned} \tag{73}
$$

where, we will prove now that each term above verify $\|I_i(t)\|_{L^\infty}\leq C\,t^{-3/4}$, with $t\gg 1$. We mention here $C>0$ is a generically constant which may change from one estimate to other, but it does not depend on the time variable.

For the first term $I_1(t)$, by well-known properties of the heat kernel $h_{\nu t}(x)$, and moreover, by the Young inequalities (with $1+1/\infty=1/2+1/2$), we directly have

$$
I_1(t)\leq \|h_{\nu t}\|_{L^2}\|\vec{u}_0-\vec{U}\|_{L^2}\leq \frac{c_\nu}{t^{3/4}}\|\vec{u}_0-\vec{U}\|_{\mathcal{H}^1_\alpha}\leq C\,t^{-3/4}. \tag{74}
$$

Thereafter, to study the second term $I_2(t)$ we write

$$
I_2(t)\leq \beta\int_0^{t/2}\left\|e^{\nu(t-s)}\mathcal{R}(s,\cdot)\right\|_{L^\infty} ds+\beta\int_{t/2}^t \left\|e^{\nu(t-s)}\mathcal{R}(s,\cdot)\right\|_{L^\infty} ds = I_{2,1}(t)+I_{2,2}(t).
$$

To estimate the term $I_{2,1}(t)$, using again the Young inequalities, using the identity $\mathcal{R}=\vec{u}-\vec{U}$, and moreover, by the estimate (70) (where we have $\eta(\beta)<0$), we can write

$$
\begin{aligned}
I_{2,1}(t)\leq & \beta\int_0^{t/2}\|h_{\nu(t-s)}\|_{L^2}\|\vec{u}(s,\cdot)-\vec{U}\|_{L^2}ds\leq c_{\beta,\nu}\int_0^{t/2}\frac{1}{(t-s)^{3/4}}\|\vec{u}(s,\cdot)-\vec{U}\|_{L^2}ds\\
\leq & \frac{c_{\beta,\nu}}{t^{3/4}}\int_0^{t/2}\|\vec{u}(s,\cdot)-\vec{U}\|_{L^2}ds\leq \frac{c_{\beta,\nu}}{t^{3/4}}\int_0^{t/2}\|\vec{u}(s,\cdot)-\vec{U}\|_{\mathcal{H}^1_\alpha}ds\leq \frac{c_{\beta,\nu}}{t^{3/4}}\|\vec{u}_0-\vec{U}\|_{\mathcal{H}^1_\alpha}\int_0^{t/2}e^{\eta(\beta)\,s}ds\leq C\,t^{-3/4}.
\end{aligned}
$$

These same facts also allow us to treat the term $I_{2,2}(t)$, where, for $t\gg 1$ we have

$$
\begin{aligned}
I_{2,2}(t)\leq & \beta\int_{t/2}^t\|h_{\nu(t-s)}\|_{L^2}\|\vec{u}(s,\cdot)-\vec{U}\|_{L^2}ds\leq c_{\beta,\nu}\int_{t/2}^t (t-s)^{-3/4}e^{\eta(\beta)s}ds\leq c_{\beta,\nu}\,e^{\eta(\beta)t/2}\int_{t/2}^t(t-s)^{-3/4}ds\\
\leq & Ce^{\eta(\beta)t/2}t^{1/4}\leq C\,t^{-3/4}.
\end{aligned}
$$

We thus have

$$
I_2(t)\leq C\,t^{-3/4},\quad t\gg 1. \tag{75}
$$

Finally, to estimate the first term $I_3(t)$, we recall first that the filtering operator $(\cdot)_\alpha$ given in (2) can be also defined by a convolution product with a kernel $K_\alpha$ [14]. This kernel has good decaying properties in the spatial variable and moreover we have $\|K_\alpha\|_{L^p}<+\infty$, for $1\leq p<+\infty$. Thus, by the Young inequalities (with $1+1/\infty=5/6+1/6$), and by the boundness of the Leray's projector in the Lebesgue spaces, we have

$$
\begin{aligned}
I_3\leq & \int_0^t \left\|\Big(e^{\nu(t-s)}\mathbb{P}\Big((\vec{u}\cdot\vec{\nabla})\vec{u}-(\vec{U}\cdot\vec{\nabla})\vec{U}\Big)(s,\cdot)\Big)_\alpha\right\|_{L^\infty} ds\\
= & \int_0^t \left\|K_\alpha * \Big(e^{\nu(t-s)}\mathbb{P}\Big((\vec{u}\cdot\vec{\nabla})\vec{u}-(\vec{U}\cdot\vec{\nabla})\vec{U}\Big)(s,\cdot)\Big)\right\|_{L^\infty} ds\\
\leq & \|K_\alpha\|_{L^6}\int_0^t \left\|e^{\nu(t-s)}\mathbb{P}\Big((\vec{u}\cdot\vec{\nabla})\vec{u}-(\vec{U}\cdot\vec{\nabla})\vec{U}\Big)(s,\cdot)\right\|_{L^{6/5}} ds\\
\leq & c_\alpha\int_0^t \left\|e^{\nu(t-s)}\Big((\vec{u}\cdot\vec{\nabla})\vec{u}-(\vec{U}\cdot\vec{\nabla})\vec{U}\Big)(s,\cdot)\right\|_{L^{6/5}} ds=(a).
\end{aligned}
$$

Here, as we have $div(\vec{u})=0$ and $div(\vec{U})=0$, and moreover, applying again the Young inequalities (with $1+5/6=1+5/6$), we can write

$$(a)=c_\alpha\int_0^t \left\| e^{\nu(t-s)}div\Big(\vec{u}\otimes\vec{u}-\vec{U}\otimes\vec{U}\Big)(s,\cdot)\right\|_{L^{6/5}}ds \leq c_\alpha\int_0^t \|\vec{\nabla}h_{\nu(t-s)}\|_{L^{6/5}}\left\|\Big(\vec{u}\otimes\vec{u}-\vec{U}\otimes\vec{U}\Big)(s,\cdot)\right\|_{L^1}ds=(b).$$

Recalling that $\mathcal{R}=\vec{u}-\vec{U}$ then we have $\vec{u}\otimes\vec{u}-\vec{U}\otimes\vec{U}=\mathcal{R}\otimes\vec{u}+\vec{U}\otimes\mathcal{R}$, and moreover, by the Hölder inequalities (with $1=1/2+1/2$), we write

$$\begin{aligned}(b) &\leq c_{\alpha,\nu}\int_0^t (t-s)^{-3/4}\left\|\Big(\mathcal{R}\otimes\vec{u}+\vec{U}\otimes\mathcal{R}\Big)(s,\cdot)\right\|_{L^1}ds\\ &\leq c_{\alpha,\nu}\int_0^t (t-s)^{-3/4}\|\mathcal{R}(s,\cdot)\|_{L^2}\Big(\|\vec{u}(s,\cdot)\|_{L^2}+\|\vec{U}\|_{L^2}\Big)ds\\ &\leq c_{\alpha,\nu}\int_0^t (t-s)^{-3/4}\|(\vec{u}-\vec{U})(s,\cdot)\|_{L^2}\Big(\|\vec{u}(s,\cdot)\|_{L^2}+\|\vec{U}\|_{L^2}\Big)ds\\ &\leq c_{\alpha,\nu}\int_0^t (t-s)^{-3/4}\|(\vec{u}-\vec{U})(s,\cdot)\|_{\mathcal{H}^1_\alpha}\Big(\|\vec{u}(s,\cdot)\|_{\mathcal{H}^1_\alpha}+\|\vec{U}\|_{\mathcal{H}^1_\alpha}\Big)ds=(c).\end{aligned}$$

In this last expression, by (70) we have $\|(\vec{u}-\vec{U})(s,\cdot)\|_{\mathcal{H}^1_\alpha}\leq\|\vec{u}_0-\vec{U}\|_{\mathcal{H}^1_\alpha}e^{\eta(\beta)s}$ (with $\eta(\beta)<0$). Moreover, by point 1) of Proposition 1 we have $\|\vec{u}(s,\cdot)\|^2_{\mathcal{H}^1_\alpha}\leq\|\vec{u}_0\|^2_{\mathcal{H}^1_\alpha}e^{-\beta s}+\dfrac{1}{\beta^2}\|\vec{f}\|^2_{\mathcal{H}^1_\alpha}$, hence we can write $\|\vec{u}(s,\cdot)\|_{\mathcal{H}^1_\alpha}\leq\|\vec{u}_0\|_{\mathcal{H}^1_\alpha}+\|\vec{f}\|_{\mathcal{H}^1_\alpha}$. Thus, gathering these estimates, for $t\gg1$ we obtain

$$(c)\leq C\int_0^t (t-s)^{-3/4}e^{\eta(\beta)s}ds=C\int_0^{t/2}(t-s)^{-3/4}e^{\eta(\beta)s}ds+C\int_{t/2}^t (t-s)^{-3/4}e^{\eta(\beta)s}ds\leq C\,t^{-3/4}.$$

$$I_3(t)\leq C\,t^{-3/4},\quad t\gg1. \tag{76}$$

Once we have the estimates (74), (75) and (76), we get back to (73), hence we finally get the desired estimate (15). Proposition 2 is proven. ■

# 8 The damped Navier-Stokes-Bardina's model without external force

## Proof of Proposition 3

When $\vec{f}=0$, we observe first that $\vec{U}=0$ is a solution of the stationary problem (12), and moreover, by point 2) of Theorem 5 is the unique one since here we have $\eta(\beta)=-\beta<0$. We thus have $\mathcal{A}_{\vec{f}}=\{0\}$.

On the other hand, for any initial datum $\vec{u}_0\in\mathcal{H}^1_\alpha(\mathbb{R}^3)$, let $\vec{u}(t,\cdot)$ be the solution of the equation (4) given by Theorem 1. Then, by (70) we have $\|\vec{u}(t,\cdot)\|_{\mathcal{H}^1_\alpha}\leq\|\vec{u}_0\|_{\mathcal{H}^1_\alpha}e^{-\beta t}$, hence we can write $\|\vec{u}(t,\cdot)\|_{L^2}\leq\|\vec{u}_0\|_{\mathcal{H}^1_\alpha}e^{-\beta t}$. Moreover, by Proposition 2 (with $\vec{U}=0$) we also have $\|\vec{u}(t,\cdot)\|_{L^\infty}\leq C\,t^{-3/4}\leq C$, with $t\gg1$. Thus, for $2\leq p<+\infty$, by the interpolation inequalities we write

$$\|\vec{u}(t,\cdot)\|_{L^p}\leq c\|\vec{u}(t,\cdot)\|^{2/p}_{L^2}\|\vec{u}(t,\cdot)\|^{1-2/p}_{L^\infty}\leq C(p,\vec{u}_0)\,e^{-\frac{2\beta}{p}t}.$$

Proposition 3 is proven. ■

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