A COUPLED SCHRÖDINGER EQUATIONS WITH TIME-OSCILLATING NONLINEARITY

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Abstract. This paper is concerned with the coupled system of supercritical nonlinear Schrödinger equations, which has applications in many physical problems, especially in nonlinear optics,

$$\begin{cases} iu_t + \Delta u + h(\omega t)(|u|^{2p} + \beta |u|^{p-1}|v|^{p+1})u &= 0, \\ iv_t + \Delta v + z(\omega t)(|v|^{2p} + \beta |v|^{p-1}|u|^{p+1})v &= 0, \end{cases} \tag{0.1}$$

where h and z are periodic functions. We prove that, for given initial data $\varphi, \psi \in H^1(\mathbb{R}^n)$, as $|\omega| \to \infty$, the solution (u_ω, v_ω) converges to the solution (U, V) of the value problem associated to.

$$\begin{cases} iU_t + \Delta U + I(h)(|U|^{2p} + \beta |U|^{p-1}|V|^{p+1})U &= 0, \\ iV_t + \Delta V + I(z)(|V|^{2p} + \beta |V|^{p-1}|U|^{p+1})V &= 0, \end{cases} \tag{0.2}$$

with the same initial data, where $I(g)$ is the average of the periodic function g. Moreover, if the solution (U, V) is global and bounded, then we prove that the solution (u_ω, v_ω) is also global provided as $|\omega| \to \infty$.

1. Introduction

In this work, we consider the following initial value problem (IVP) for two coupled nonlinear Schrödinger (NLS) equations:

$$\begin{cases} iu_t + \Delta u + h(\omega t)(|u|^{2p} + \beta |u|^{p-1}|v|^{p+1})u = 0 \\ iv_t + \Delta v + z(\omega t)(|v|^{2p} + \beta |v|^{p-1}|u|^{p+1})v = 0 \\ u(x, t_0) = \varphi(x) \qquad v(x, t_0) = \psi(x), \end{cases} \tag{1.1}$$

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in $\mathbb{R}^n$, where

$$1 < p < \frac{2}{(n-2)^+}, \tag{1.2}$$

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$t_0, \omega \in \mathbb{R}$ and $\varphi, \psi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$, and h, $z \in C(\mathbb{R}, \mathbb{R})$ are periodic functions with period $\tau > 0$. Moreover, β is real positive constant. To simplify the analysis, we translate the initial time t_0 to 0 and consider the following IVP

$$\begin{cases} iu_t + \Delta u + \theta_1(\omega(t+t_0)(|u|^{2p} + \beta|u|^{p-1}|v|^{p+1})u = 0 \\ iv_t + \Delta v + \theta_2(\omega(t+t_0)(|v|^{2p} + \beta|v|^{p-1}|u|^{p+1})v = 0 \\ u(x,0) = \varphi(x) \qquad v(x,0) = \psi(x). \end{cases} \tag{1.3}$$

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For $h = z = 1$ this kind of problem arises as a model for propagation of polarized laser beams in birefringent Kerr medium in nonlinear optics (see, for example, [2, 8, 10, 12, 16, 15] and the references therein for a complete discussion of the physics of the problem). The two functions u and v are the components of the slowly varying envelope of the electrical field, t is the distance in the direction of propagation, x are orthogonal variables and Δ is the diffraction operator. The case $n = 1$ corresponds to propagation in a planar geometry, $n = 2$ is the propagation in a bulk medium and $n = 3$ is the propagation of pulses in a bulk medium with time dispersion. The focusing nonlinear terms in (1.1) describes the dependence of the refraction index of material on the electric field intensity and the birefringence effects. The parameter $\beta > 0$ has to be interpreted as the birefringence intensity and describes the coupling between the two components of the electric-field envelope. This article is motivated by the papers Abdullaev et al. [1] and Konotop and Pacciani [9] where the authors investigate the effect of a time-oscillating term in factor of the nonlinear Schrödinger equations and Carvajal, Panthee and Scialom [3] where the authors considered a critical Korteweg-de Vries (KdV) equation.

We consider the system (1.1) when $h = z = 1$, $t_0 = 0$ i.e.

$$\begin{cases} iu_t + \Delta u + (|u|^{2p} + \beta|u|^{p-1}|v|^{p+1})u = 0 \\ iv_t + \Delta v + (|v|^{2p} + \beta|v|^{p-1}|u|^{p+1})v = 0 \\ u(x,t_0) = \varphi(x) \qquad v(x,t_0) = \psi(x), \end{cases} \tag{1.4}$$

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tthe system (1.4), admits the mass and the energy conservation in the space $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$. Namely, mass ($L^2$ norm):

$$M[u(t),v(t)] := \|\varphi\|^2_{L^2(\mathbb{R}^n)} + \|\psi\|^2_{L^2(\mathbb{R}^n)}, \tag{1.5}$$

and energy

$$\begin{aligned} E[u(t),v(t)] :=& \frac{1}{2}(\|\nabla u(t)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla v(t)\|^2_{L^2(\mathbb{R}^n)}) \\ &- \frac{1}{2p+2}(\|u(t)\|^{2p+2}_{L^{2p*2}(\mathbb{R}^n)} + 2\beta\|u(t)\,v(t)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} + \|v(t)\|^{2p+2}_{L^{2p+2}(\mathbb{R}^n)}) \\ =& E[\varphi,\psi]. \end{aligned} \tag{1.6}$$

The following results have been established:
1) When $1 < p < 2/n$, the solutions of the Cauchy problem (1.4), exist globally in time (see [8]).
2) When $p \geq 2/n$, the solutions of the Cauchy problem (1.4), blow up in a finite time for some initial data ($E[\varphi,\psi] < 0$), especially for a class of sufficiently large data (see [6, 8, 11, 13]). On the other hand, the solutions of the Cauchy problem (1.4), globally exist for other initial data, especially for a class of sufficiently small data (see [4, 8, 12]).

In the sections 2, 3 and 4 we will study the blow-up and the Cauchy problem for the problem (1.1) with $h, z \in L^\infty(\mathbb{R})$.

In [16] Xiaoguang et al. they obtain a sharp threshold of blow-up for (1.4), to study the blow-up threshold, the following stationary system associated with (1.4) was considered

$$\begin{cases} \Delta Q - \dfrac{(2-n)p+2}{2}Q + (|Q|^{2p} + \beta|Q|^{p-1}|R|^{p+1})Q = 0 \\ \Delta R - \dfrac{(2-n)p+2}{2}R + (|R|^{2p} + \beta|R|^{p-1}|Q|^{p+1})R = 0. \end{cases} \tag{1.7}$$ stat

Let, $s_c = n/2 - 1/p$, $\sigma_{p,n,\beta} := (\frac{pn}{2})^{1/4(1-1/p)}\sqrt{\|Q\|^2_{L^2(\mathbb{R}^n)} + \|R\|^2_{L^2(\mathbb{R}^n)}}$,

$$\Gamma[u,v] := E^{s_c}[u,v]M^{1-s_c}[u,v],$$

and

$$\vartheta[u,v] := (\|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \|\nabla v\|^2_{L^2(\mathbb{R}^n)})^{s_c/2}(\|u\|^2_{L^2(\mathbb{R}^n)} + \|v\|^2_{L^2(\mathbb{R}^n)})^{(1-s_c)/2}.$$

Xiaoguang et al. proved the following result:

Theorem 1.1. *Let $2 \leq p < A_n$, where $A_n = \infty$ if $n = 1, 2$, $A_n = 2/(n-2)$ if $n \geq 3$ and $(|x|\varphi, |x|\psi) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Assume that*

$$\Gamma[\varphi, \psi] < \Gamma[Q, R] \equiv \frac{s_c}{n}^{s_c} (\sigma_{p,n,\beta})^2,$$

then the following two conclusions are valid.
1) If $\vartheta[\varphi, \psi] < \vartheta[Q, R]$, then the solution exist globally in time.
2) If $\vartheta[\varphi, \psi] > \vartheta[Q, R]$, then the solution blow-up in finite time.

Note that the system (1.3) its equivalent to

$$\begin{cases} u(t) = & S(t)\varphi + i \int_0^t S(t-s) h(\omega(s+t_0)) F(u,v)(s) ds, \\ v(t) = & S(t)\psi + i \int_0^t S(t-s) z(\omega(s+t_0)) F(v,u)(s) ds, \end{cases} \tag{1.8}$$ intsis01

where $S(t) = e^{it\Delta}$ the group of Schrödinger equation. Using standard ideas we can see that the system (1.8) is locally well-posed in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.

1a **Proposition 1.1.** *Suppose p be as in (1.2). Given any $(\varphi, \psi) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, $h, z \in L^\infty(\mathbb{R})$ and $t_0 \in \mathbb{R}$, there exists a unique, maximal solution*

$$(u, v) \in \mathbb{C}([0, T_{max}), H^1),$$

of (1.8). Also the solution satisfies the blowup alternative, i.e. if $T_{max} < \infty$ then $|||(u(t), v(t))|||_{1,2} \to \infty$ as $t \to T_{max}$. Moreover,

$$(u, v) \in \mathbb{L}^q((0, T), W^{1,r}) \quad \textit{for} \quad 0 < T < T_{max},$$

for all admissible pairs (q, r).

Proof. See [5]. □

The purpose of this paper is to study how the solution (u, v) behaves as $|\omega| \to \infty$. It is natural to expect that the nonlinearity averages to $\{I(h)(|U|^{2p} + \beta |U|^{p-1}|V|^{p+1})U, I(z)(|V|^{2p} + \beta |V|^{p-1}|U|^{p+1})V\}$ as $|\omega| \to \infty$, where $I(h)$ is the average of h, i.e.

$$I(h) := \frac{1}{\tau} \int_0^\tau h(s) ds, \tag{1.9}$$ med

and that the solution (u, v) of system (1.8) converges locally in time as $|\omega| \to \infty$ to solution (U, V) of

$$\begin{cases} iU_t + \Delta U + I(h)(|U|^{2p} + \beta |U|^{p-1}|V|^{p+1})U = 0, \\ iV_t + \Delta V + I(z)(|V|^{2p} + \beta |V|^{p-1}|U|^{p+1})V = 0, \\ U(x,0) = \varphi(x) \qquad V(x,0) = \psi(x), \end{cases} \tag{1.10}$$ trans01

or equivalently

$$\begin{cases} U(t) = S(t)\varphi + iI(h)\int_0^t S(t-s)F(U,V)(s)ds, \\ V(t) = S(t)\psi + iI(z)\int_0^t S(t-s)F(V,U)(s)ds. \end{cases} \tag{1.11}$$ intrans01

This is indeed what the following result shows.

the1b **Theorem 1.2.** *Assume (1.2). Fix an initial value* $(\varphi, \psi) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. *Given* $t_0, \omega \in \mathbb{R}$, *denote by* $(u_{t_0,\omega}, v_{t_0,\omega})$ *the maximal solution of (1.8). Let* (U, V) *be the solution of (1.11) defined on the maximal interval* $[0, S_{max})$.

- *Given any* $0 < T < S_{max}$, *the solution* $(u_{t_0,\omega}, v_{t_0,\omega})$ *exists on* $[0, T]$ *for all* $t_0 \in \mathbb{R}$ *provided* $|\omega|$ *is sufficient large.*
- *We have that* $(u_{t_0,\omega}, v_{t_0,\omega}) \to (U, V)$ *in* $\mathbb{L}^\gamma((0,T), W^{1,\rho})$ *as* $|\omega| \to \infty$, *uniformly in* $t_0 \in \mathbb{R}$, *for all admissible pairs* (γ, ρ) *and all* $0 < T < S_{max}$. *In particular, convergence holds in* $\mathbb{C}([0,T], H^1)$ *for all* $0 < T < S_{max}$.

Whenever $S_{max} = \infty$, one may wonder whether or not $(u_{t_0,\omega}, v_{t_0,\omega})$ is global when $|\omega|$ is sufficiently large. The following result shows that the answer is positive provided (U, V) has sufficient decay as $t \to \infty$.

the1c **Theorem 1.3.** *Assume (1.2). Set*

$$r = 2(\alpha + 1) \qquad a = \frac{4p(p+1)}{2 - p(n-2)}. \tag{1.12}$$ dacons

Fix the initial data $(\varphi, \psi) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. *For* $t_0, \omega \in \mathbb{R}$ *denote by* $(u_{t_0,\omega}, v_{t_0,\omega})$ *the maximal solutions of (1.8). Suppose* (U, V) *be the maximal solution of (1.11) defined on the maximal interval* $[0, S_{max})$. *If* $S_{max} = \infty$ *and*

$$(U, V) \in \mathbb{L}^a((0,\infty), L^r(\mathbb{R}^n)), \tag{1.13}$$ retra

then it follows that $(u_{t_0,\omega}, v_{t_0,\omega})$ *is global for all* $t_0 \in \mathbb{R}$ *if* $|\omega|$ *is sufficiently large. Moreover,* $(u_{t_0,\omega}, v_{t_0,\omega}) \rightarrow (U, V)$ *in* $\mathbb{L}^\gamma((0,\infty), W^{1,\rho})$ *as* $|\omega| \rightarrow \infty$, $t_0 \in \mathbb{R}$ *for all admissible pairs* (γ, ρ). *In particular, the convergence holds in* $\mathbb{L}^\infty((0,\infty), H^1)$.

The rest of the paper is organized as follows. In Section 2, we obtain some preliminary results. In Section 3, we prove Lemma 3.1 and in the Section 4, we prove Theorems 1.2 and 1.3.

Notation

The L^2-based Sobolev space of order s will be denoted by H^s with norm

$$||f||_{H^s} := \left(\int_{\mathbb{R}^n} (1+\xi^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

For $f : [0,T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ we define the mixed $L_t^p L_x^q-$norm by

$$||f||_{L_T^p L_x^q} := \left\{ \int_0^T \left[\int_{\mathbb{R}^n} |f(x,t)|^q dt \right]^{p/q} dx \right\}^{1/p},$$

with usual modifications when $p = \infty$. We use the letter C to denote various constants whose exact values are immaterial and which may vary from one line to the next. We use the notation

$\mathbb{L}^r(\mathbb{R}^n) = L^r(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$

$\mathbb{H}^s(\mathbb{R}^n) = H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$

$||.||_\infty = ||.||_{L^\infty}$

$||.||_{1,2} = ||.||_{H^1(\mathbb{R}^n)}$

$|||(.,.)|||_{1,2} = ||.||_{1,2} + ||.||_{1,2}$

$L_\infty^a L_x^r := L^a((0,\infty), L^r(\mathbb{R}^n))$

$L_T^a L_x^r := L^a((0,T), L^r(\mathbb{R}^n))$

$\mathbb{L}^a((0,T), L^r) := L_T^a L_x^r \times L_T^a L_x^r$

$F(u,v) := (|u|^{2p} + \beta |u|^{p-1}|v|^{p+1})u.$

$\mathbb{C}([a,b], H^1) := C([a,b], H^1(\mathbb{R}^n)) \times C([a,b], H^1(\mathbb{R}^n)).$

$\mathbb{L}^p((a,b), W^{1,q}) := L^p((a,b), W^{1,q}(\mathbb{R}^n)) \times L^p((a,b), W^{1,q}(\mathbb{R}^n)).$

sec-2

2. Preliminary results

2.1. **Useful estimates.** Given $1 \leq p \leq \infty$, we denote by p' its conjugate given by $\frac{1}{p'} := 1 - \frac{1}{p}$. We use the standard Sobolev spaces and their embedding. We consider the standard notion of a (non-endpoint) admissible pair (q, r), i.e.

$$\frac{2}{q} := \frac{n}{2} - \frac{n}{r} \qquad 2 \leq r < \frac{2n}{(n-2)^+}. \tag{2.1}$$

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We will use the Strichartz estimates. More precisely, given any two admissible pairs (q, r) and (γ, ρ), there exists a constant C such that if

$$u(t) = S(t)\varphi + i \int_0^t S(t-s) f(s) ds,$$

then

$$||u||_{\mathbb{L}^q(\mathbb{R}, L^r(\mathbb{R}^n))} \leq C(\, ||\varphi||_{L^2(\mathbb{R}^n)} + ||f||_{L^{\gamma'}(\mathbb{R}, L^{\rho'}(\mathbb{R}^n))}). \tag{2.2}$$

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In this section, we first recall some results concerning the local and global well-posedness for (1.8). Next, we study the effect of the oscillating term $\theta(\omega\, t)$ as $|\omega| \to \infty$ on the linear, nonhomogeneus Schrödinger equation. Given $h, z \in L^\infty(\mathbb{R})$, we consider the equation

$$\begin{cases} u(t) = S(t)\varphi + i \int_0^t S(t-s) h(s) F(u, v)(s) ds, \\ v(t) = S(t)\psi + i \int_0^t S(t-s) z(s) F(v, u)(s) ds, \end{cases} \tag{2.3}$$

inth01

which is slightly more general than (1.3).

bba **Proposition 2.1.** *(Local Existence) Assume (1.2). Given $A, M > 0$, there exists $\delta = \delta(A, M)$ such that*

$$||h||_\infty + ||z||_\infty \leq A \quad \textit{for all} \quad h, z \in L^\infty(\mathbb{R}^n) \tag{2.4}$$

$$||\varphi||_{1,2} + ||\psi||_{1,2} \leq M \quad \textit{for all} \quad \varphi, \psi \in H^1(\mathbb{R}^n), \tag{2.5}$$

then there exists a unique solution $(u, v) \in \mathbb{C}([0, \delta], H^1)$ of (2.3). In addition

$$||(u, v)||_{\mathbb{L}^\infty((0,\delta), H^1)} \leq C\{ \, ||\varphi||_{1,2} + ||\psi||_{1,2} \, \}. \tag{2.6}$$

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Moreover, $(u, v) \in \mathbb{L}^\gamma((0, \delta), W^{1,\rho})$ for all admissible pairs (γ, ρ).

Proof. See [5] or [14]. □

We will also use the following result.

bbb **Proposition 2.2.** *Assume (1.2). Let r, q, a be defined by*

$$r := 2(p+1), \qquad q := \frac{4(p+1)}{pn}, \qquad a := \frac{4p(p+1)}{2-p(n-2)}. \tag{2.7}$$
const

Note that $a > \frac{q}{2}$. Given any $A > 0$, there exists $\varepsilon = \varepsilon(A)$ and Λ such that if

$$||h||_\infty + ||z||_\infty \leq A, \tag{2.8}$$

$$||S(\cdot)(\varphi,\psi)||_{\mathbb{L}^a((0,\infty),L^r)} \leq \varepsilon, \qquad \textit{for all} \quad \varphi,\psi \in H^1(\mathbb{R}^n), \tag{2.9}$$

then the corresponding solution (u,v) of (2.3) is global and satisfies

$$||(u,v)||_{\mathbb{L}^a((0,\infty),L^r)} \leq C||S(\cdot)(\varphi,\psi)||_{\mathbb{L}^a((0,\infty),L^r)} \quad , \tag{2.10}$$

$$||(u,v)||_{\mathbb{L}^q((0,\infty),W^{1,r})} + ||(u,v)||_{\mathbb{L}^\infty((0,\infty),H^1)} \leq \Lambda|||(\varphi,\psi)|||_{1,2}. \tag{2.11}$$

Conversely, if solution (u,v) of (2.3) is global and satisfies

$$||(u,v)||_{\mathbb{L}^a((0,\infty),L^r)} \leq \epsilon, \tag{2.12}$$
hip-c

then

$$||S(\cdot)(\varphi,\psi)||_{\mathbb{L}^a((0,\infty),L^r)} \leq c||(u,v)||_{\mathbb{L}^a((0,\infty),L^r)}. \tag{2.13}$$
xm-13

Proof. Let $G_{h,u,v}(t) := i\int_0^t S(t-s)h(s)F(u,v)(s)ds$. From (2.3), we obtain that

$$\begin{cases} u(t) = S(t) + G_{h,u,v}(t), \\ v(t) = S(t) + G_{z,v,u}(t). \end{cases} \tag{2.14}$$

Then

$$|\,||u||_{L^a_T L^r_x} - ||S(\cdot)\varphi||_{L^a_T L^r_x}\,| \leq ||G_{h,u,v}||_{L^a_T L^r_x}, \quad \forall\, 0 < T < T_{max} \tag{2.15}$$
eq-a

and

$$|\,||v||_{L^a_T L^r_x} - ||S(.)\psi||_{L^a_T L^r_x}\,| \leq ||G_{z,v,u}||_{L^a_T L^r_x}, \quad \forall\, 0 < T < T_{max}. \tag{2.16}$$
eq-b

From ()-(), (2.15) and (2.16), we obtain

$$||u||_{L^a_T L^r_x} \leq ||S(\cdot)\varphi||_{L^a_T L^r_x} + CA\{\, ||u||^{2p+1}_{L^a_T L^r_x} + \beta\, ||v||^{p+1}_{L^a_T L^r_x} ||u||^p_{L^a_T L^r_x},\, \} \tag{2.17}$$
eq-f

and

$$||v||_{L^a_T L^r_x} \leq ||S(\cdot)\psi||_{L^a_T L^r_x} + CA\{\, ||v||^{2p+1}_{L^a_T L^r_x} + \beta\, ||u||^{p+1}_{L^a_T L^r_x} ||v||^p_{L^a_T L^r_x},\, \}. \tag{2.18}$$
eq-g

From estimates (2.17) and (2.18), we conclude

$$||(u,v)||_{\mathbb{L}^a((0,T),L^r)} \leq ||S(\cdot)\{\varphi,\psi\}||_{\mathbb{L}^a((0,T),L^r)} + CA\{\, X_T(u,v) + X_T(v,u)\,\} \tag{2.19}$$
eq-h

where

$$X_T(u,v) := ||u||^{2p+1}_{L^a_T L^r_x} + \beta\, ||v||^{p+1}_{L^a_T L^r_x} ||u||^{p}_{L^a_T L^r_x}.$$

Let $\epsilon = \epsilon(A)$ be small enough so that

$$(1+2\beta)2^{2p+2}\epsilon^{2p} CA \,<\, 1. \tag{2.20}$$

ep-on

From (2.9) and (2.19), we can say that

$$||(u,v)||_{\mathbb{L}^a((0,T),L^r)} \leq \epsilon + CA\{\, X_T(u,v) + X_T(v,u)\,\} \qquad \forall\, 0 < T < T_{max}. \tag{2.21}$$

eq-i

We will show that

$$||(u,v)||_{\mathbb{L}^a((0,T),L^r)} \leq 2\epsilon \qquad \forall 0 \leq T \leq T_{max}. \tag{2.22}$$

eq-j

The idea is to use contradiction method to get it. Suppose that there is $T_* \in [0, T_{max}]$ such that

$$f(T_*) > 2\epsilon \qquad \text{where} \qquad f(T) := ||(u,v)||_{\mathbb{L}^a((0,T),L^r)} \qquad 0 < T < T_{max}. \tag{2.23}$$

cont-a

As $f(t)$ is a continuous function and increasing in $0 < T < T_{max}$, there is $0 < T_0 < T_*$ such that

$$f(T_0) := ||(u,v)||_{\mathbb{L}^a((0,T_0),L^r)} = 2\epsilon.$$

We observe that, from (2.21)

$$\begin{aligned} f(T_0) \;&\leq \epsilon + CA\{\, X_{T_0}(u,v) + X_{T_0}(v,u)\,\},\\ &\leq \epsilon + CA\{\,(2\epsilon)^{2p+1} + \beta(2\epsilon)^{2p+1} + (2\epsilon)^{2p+1} + \beta(2\epsilon)^{2p+1}\,\},\\ &= \epsilon + CA\, 2(2\epsilon)^{2p+1}(1+\beta). \end{aligned} \tag{2.24}$$

Therefore,

$$\begin{aligned} 2\epsilon \;&\leq \epsilon + 2^{2p+1} CA \epsilon^{2p+1}(1+\beta),\\ 1 \;&\leq 2^{2p+1} CA \epsilon^{2p}(1+2\beta), \end{aligned} \tag{2.25}$$

which is a contradiction.

We now show that

$$||(u,v)||_{\mathbb{L}^a((0,T_{max}),L^r)} \leq\; 2\{\, ||S(.)(\varphi,\psi)||_{\mathbb{L}^a((0,T_0),L^r)}.$$

If possible, assume that

$$||(u,v)||_{\mathbb{L}^a((0,T_{max}),L^r)} >\; 2\{\, ||S(\cdot)(\varphi,\psi)||_{\mathbb{L}^a((0,T_0),L^r)}. \tag{2.26}$$

des-a

From (2.19) and the argument of continuity, we have

$$\|(u,v)\|_{\mathbb{L}^a((0,T),L^r)} \le \|S(\cdot)(\varphi,\psi)\|_{\mathbb{L}^a((0,T),L^r)} + \{\, \|G_{h,u,v}\|_{L^a_T L^r_x} + \|G_{z,v,u}\|_{L^a_T L^r_x} \,\}. \quad (2.27)$$ des-b

From (2.26) and (2.27), we obtain

$$\|(u,v)\|_{\mathbb{L}^a((0,T_{max}),L^r)} \le 2\{\, \|G_{h,u,v}\|_{L^a_T L^r_x} + \|G_{z,v,u}\|_{L^a_T L^r_x} \,\}. \quad (2.28)$$ des-c

From the above inequality, we get

$$X_{T_{max}} \le 2\{\, \|u\|^{2p}_{L^a_{T_{max}} L^r_x} + \beta \|v\|^p_{L^a_{T_{max}} L^r_x} \|u\|^p_{L^a_{T_{max}} L^r_x} \,\} G_{h,z,T}, \quad (2.29)$$ des-d

where $G_{h,z,T} = \|G_{h,u,v}\|_{L^a_T L^r_x} + \|G_{z,v,u}\|_{L^a_T L^r_x}$.

Now, from (2.22) and (2.29)

$$C A\, X_{T_{max}} \le 2CA(\, (2\epsilon)^{2p} + \beta (2\epsilon)^{2p}\,) G_{h,z,T} = (1+\beta) 2^{2p+1} C A \epsilon^{2p} G_{h,z,T} < \, G_{h,z,T},$$

which is a contradiction.

Us note that, applying Holder's inequality

$$\||u(t)|^{2p} u(t)\|_{W^{1,r'}} \le C_p \|u(t)\|^{2p}_{L^r} \|u(t)\|_{W^{1,r}}. \quad (2.30)$$ desw-a

Moreover,

$$\||u|^{p-1}|v|^{p+1} u\|_{W^{1,r'}} \le C'_p\{\, \|u\|^p_{L^r}\|v\|^p_{L^r} + \|u\|^{p-1}_{L^r}\|v\|^{p+1}_{L^r} \,\}(\, \|u\|_{W^{1,r}} + \|v\|_{W^{1,r}}\,), \quad (2.31)$$ desw-b

where r' is the conjugate of r and that $\nabla(|u|^{p+1}) = (\frac{p+1}{2})|u|^{p-1}(\overline{u}\nabla u + u\overline{\nabla u})$. We observe that

$$\|\, \|u\|^{2p}_{L^r}\|u\|_{W^{1,r}}\, \|_{L^{q'}_x} \le \|u\|^{2p}_{L^a_T L^r_x} \|u\|_{L^q_T W^{1,r}}, \quad (2.32)$$ desw-c

and

$$\int_0^T \|u\|^{(p-1)q'}_{L^r_x} \|v\|^{(p+1)q'}_{L^r_x} \|u\|^{q'}_{W^{1,r}} dt \le \|u\|^{(p-1)q'}_{L^a_T L^r} \|v\|^{(p+1)q'}_{L^a_T L^r} \|u\|^{q'}_{L^q_T W^{1,r}}, \quad (2.33)$$ desw-d

where q' is the conjugate of q. Thus,

$$\|hF(u,v)\|_{L^{q'}((0,T),W^{1,r'})} \le B_p(\|u\|^{2p}_{L^a_T L^r_x} + \beta \|u\|^{(p-1)q'}_{L^a_T L^r} \|v\|^{(p+1)q'}_{L^a_T L^r})\{\|u\|_{L^q_T W^{1,r}} + \|v\|_{L^q_T W^{1,r}}\}. \quad (2.34)$$ desw-e

Consequently

$$\|(u,v)\|_{\mathbb{L}^q((0,T),W^{1,r})} + \|(u,v)\|_{\mathbb{L}^\infty((0,T),H^1)} \le \Lambda\, |||(\varphi,\psi)|||_{1,2} \qquad \forall 0 < T < T_{max}. \quad (2.35)$$ des-m

Moreover by continuity

$$\|(u,v)\|_{\mathbb{L}^q((0,T_{max}),W^{1,r})} \le \Lambda\, |||(\varphi,\psi)|||_{1,2},$$

in particular, $(u,v) \in \mathbb{L}^\infty((0,T_{max}),H^1)$ so that $T_{max} = \infty$ by the blowup alternative. $\square$

cor-i1 **Corollary 2.1.** *Assume (1.2) e $p > 2/n$ and a, q, r be as defined in (2.7). Suppose that $h, z \in L^\infty(\mathbb{R})$ be such that $\|h\|_{L^\infty} + \|z\|_{L^\infty} \le A$ for some $A > 0$. Also let $\epsilon = \epsilon(A)$ and Λ be as in Proposition 2.2. For given $(\phi,\psi) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, let (u,v) be the corresponding solution of (2.3) defined on the maximal interval of existence $[0, T_{\max})$. If there is $0 < T < T_{\max}$ such that $\|(e^{it\Delta}u(T), e^{it\Delta}v(T))\|_{\mathbb{L}^a((0,\infty);L^r)} \le \epsilon$, then the solution (u,v) is global, i.e. $T_{\max} = \infty$. Moreover*

$$\|(u,v)\|_{\mathbb{L}^a((T,\infty),L^r)} \le 2\epsilon, \qquad \|(u,v)\|_{\mathbb{L}^q((T,\infty),W^{1,r})} \le \Lambda |||(u(T),v(T))|||_{1,2}. \tag{2.36}$$ eq-c1

Proof. If we apply Proposition 2.2 with (φ,ψ) replaced by $(u(T),v(T))$ and $h(t), z(t)$ replaced by $h(t+T)$ and $z(t+T)$, it can be infered that the solution (w_1,w_2) of the system

$$\begin{cases} w_1(t) = e^{it\Delta}u(T) + i\int_0^t e^{i(t-s)\Delta} h(s+T)F(w_1,w_2)(s)ds, \\ w_2(t) = e^{it\Delta}v(T) + i\int_0^t e^{i(t-s)\Delta} z(s+T)F(w_2,w_1)(s)ds, \end{cases} \tag{2.37}$$ cor-eq1

is global and satisfies

$$\|(w_1,w_2)\|_{\mathbb{L}^a((0,\infty);L^r)} \le 2\epsilon, \qquad \|(w_1,w_2)\|_{\mathbb{L}^q((0,\infty);W^{1,r})} \le \Lambda |||(u(T),v(T))|||_{1,2}. \tag{2.38}$$

Now, if we define

$$\tilde{u} = \begin{cases} u(t), & 0 \le t \le T, \\ w_1(t-T), & T < t < \infty, \end{cases} \tag{2.39}$$

$$\tilde{v} = \begin{cases} v(t), & 0 \le t \le T, \\ w_2(t-T), & T < t < \infty, \end{cases} \tag{2.40}$$

then it can be seen that $(\tilde{u},\tilde{v})$ solves (2.3) in $[0,\infty)$, thereby completing the proof. $\square$

In what follows we prove some more estimates that will be used in sequel.

prop-m1 **Proposition 2.3.** *Assume $h \in C(\mathbb{R},\mathbb{R})$ is a periodic function with period $\tau > 0$ whose average is given by (1.9). Set (q,r) be an admissible pair. Given $f \in L^{q'}(\mathbb{R}, L^{r'}(\mathbb{R}^n))$, it follows that for every admissible pair (γ,ρ)*

$$\int_0^t h(\omega(s+t_0))S(t-s)f(s)ds \quad \to \quad I(h)\int_0^t S(t-s)f(s)ds \tag{2.41}$$ eq-p1

in $L^\gamma(\mathbb{R}, L^\rho(\mathbb{R}^n))$*, uniformly in* $t_0 \in \mathbb{R}$*.*

Proof. A detailed proof of this result has been presented in [7]. For the sake of clarity, we just give a sketch here. Using the Strichartz estimate (2.2), we have that

$$\| \int_0^t h(\omega(s+t_0)S(t-s)f(s)ds\|_{L^\gamma(\mathbb{R};L^\rho(\mathbb{R}^n))} \leq C\|h\|_{L^\infty}\|f\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^n))}. \tag{2.42}$$

eq-p2

So, by the density argument, it is enough to prove (2.41) for $f \in C_c^1(\mathbb{R}; \mathcal{S}(\mathbb{R}^N))$. Defining, $\lambda(t) := h(t) - I(h)$ and $\Lambda(t) = \int_0^t \lambda(s)ds$, one has

$$\frac{d}{ds}\Lambda(\omega(s+t_0)) = \omega\lambda(\omega(s+t_0)).$$

Now, integrating by parts and using the Strichartz estimate (2.2), it is easy to obtain

$$\begin{aligned} &\| \int_0^t \lambda(\omega(s+t_0))S(t-s)f(s)ds\|_{L^\gamma(\mathbb{R};L^\rho(\mathbb{R}^n))} \leq \\ &\quad \frac{C}{|\omega|}\|\Lambda\|_{L^\infty}\big[\|f\|_{L^\gamma(\mathbb{R};L^\rho(\mathbb{R}^n))} + \|f(0)\|_{L^2} + \|f_t - i\Delta f\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^N))}\big]. \end{aligned} \tag{2.43}$$

eq-p3

Taking $|\omega| \to \infty$, the result of the proposition follows. □

3. Proof of the main Lemma

The following Lemma plays a crucial role in the proof of the main result.

lem-a **Lemma 3.1.** *Assume (1.2). Set the initial data* $\varphi, \psi \in H^1(\mathbb{R}^n)$ *and* $t_0, \omega \in \mathbb{R}$*, denote by* $(u_{t_0,\omega}, v_{t_0,\omega})$ *the maximal solution of (2.3). Suppose* (U, V) *be the maximal solution of (1.11) defined in* $[0, S_{max})$*. Let* $0 < T < S_{max}$ *and assume that* $(u_{t_0,\omega}, v_{t_0,\omega})$ *exists on* $[0, T]$ *for* $|\omega|$ *is sufficiently large and that*

$$\limsup_{|\omega|\to\infty} \sup_{t_0\in\mathbb{R}} ||(u_{t_0,\omega}, v_{t_0,\omega})||_{\mathbb{L}^\infty((0,T),H^1)} < \infty. \tag{3.1}$$

equam

It follows that

$$\sup_{t_0\in\mathbb{R}} ||\, u_{t_0,\omega} \; - \; U||_{\mathbb{L}^\gamma((0,T),W^{1,\rho})} \; \to \; 0, \quad \textit{when} \quad |\omega| \to \infty, \tag{3.2}$$

for all admissible pair (γ, ρ)*. In particular,* $(u_{t_0,\omega}, v_{t_0,\omega}) \; \to \; (U, V)$ *in* $\mathbb{L}^\infty((0,T), H^1)$*.*

Proof. Symmetry of the system allows us to work for a single component. The estimates for the other component will be similar. We consider $|\omega| \geq L$, where L is chosen sufficiently large so that

$$\sup_{|\omega|\geq L} \sup_{t_0\in\mathbb{R}} ||(u_{t_0,\omega}, v_{t_0,\omega})||_{\mathbb{L}^\infty((0,T),H^1)} < \infty. \tag{3.3}$$

Let $r = 2(p+1)$ and $q = \frac{4(p+1)}{np}$ so that (q,r) is an admissible pair. As the initial data for $u_{t_0,\omega}$ and U are the same, we have,

$$\begin{aligned} u_{t_0,\omega} - U &= i\int_0^t S(t-s)\big[\theta_1 F(u_{t_0,\omega}, v_{t_0,\omega})(s) - I(\theta_1)F(U,V)(s)\big]ds \\ &= i\int_0^t S(t-s)\big[\theta_1 |u_{t_0,\omega}|^{2p}u_{t_0,\omega}(s) - I(\theta_1)|U|^{2p}U(s)\big]ds \\ &\quad + i\beta\int_0^t S(t-s)\big[\theta_1 |u_{t_0,\omega}|^{p-1}|v_{t_0,\omega}|^{p+1}u_{t_0,\omega}(s) - I(\theta_1)|U|^{p-1}|V|^{p+1}U(s)\big]ds \\ &=: A + B. \end{aligned} \tag{3.4}$$

eq-m5

The estimates for A follow from [7]. In fact, from [7] we have

$$\|A\|_{L^q((0,t);L^r)} + \|A\|_{L^\gamma((0,t);L^\rho)} \leq C_\omega + C\|u_{t_0,\omega} - U\|_{L^{q'}((0,t);L^r)}, \tag{3.5}$$

eq-m5.1

for all $0 < t \leq l$, where $C_\omega \to 0$ as $|\omega| \to \infty$.

We move to estimate B by writing it as

$$\begin{aligned} B &= i\beta\int_0^t S(t-s)\theta_1\big[|u_{t_0,\omega}|^{p-1}|v_{t_0,\omega}|^{p+1}u_{t_0,\omega}(s) - |U|^{p-1}|V|^{p+1}U(s)\big]ds \\ &\quad + i\beta\int_0^t S(t-s)\big[\theta_1 - I(\theta_1)\big]|U|^{p-1}|V|^{p+1}U(s)ds \\ &:= B_1 + B_2. \end{aligned} \tag{3.6}$$

eq-m6

We note that $|U|^{p-1}|V|^{p+1}U \in L^{q'}((0,l);L^{r'}(\mathbb{R}^n))$, because for $r = 2p+2$, $r' = \frac{r}{2p+1}$, using Hölder's inequality, one has

$$\begin{aligned} \int_{\mathbb{R}^n} |U|^{pr'}|V|^{(p+1)r'}dx &= \int_{\mathbb{R}^n} |U|^{\frac{pr}{2p+1}}|V|^{\frac{(p+1)r}{2p+1}}dx \\ &\leq \Big(\int_{\mathbb{R}^n} |U|^r dx\Big)^{\frac{p}{2p+1}}\Big(\int_{\mathbb{R}^n} |V|^r dx\Big)^{\frac{p+1}{2p+1}} \\ &= \|U\|_{L^r}^{\frac{pr}{2p+1}}\|V\|_{L^r}^{\frac{(p+1)r}{2p+1}}. \end{aligned} \tag{3.7}$$

eq-m7

From Sobolev embedding, we have

$$\|U\|_{L^r} \leq C\|U\|_{H^{\frac{N}{2}-\frac{N}{r}}} \leq \|U\|_{H^1}, \tag{3.8}$$
eq-m8

whenever $p < \frac{2}{(N-2)^+}$. Hence

$$\||U|^{p-1}|V|^{p+1}U\|^{q'}_{L^{q'}((0,l);L^{r'})} \leq \int_0^l \|U\|_{L^r}^{\frac{prq'}{(2p+1)r'}}\|V\|_{L^r}^{\frac{(p+1)rq'}{(2p+1)r'}}dt = \int_0^l \|U\|_{L^r}^{pq'}\|V\|_{L^r}^{(p+1)q'}dt < \infty,$$

as required. Therefore, from Proposition 2.3 we conclude that

$$\sup_{t_0\in\mathbb{R}} \|B_2\|_{L^q((0,l);L^r)} + \|B_2\|_{L^\gamma((0,l);L^\rho)} \to 0, \quad \text{as } |\omega| \to \infty. \tag{3.9}$$
eq-m9

To estimate B_1 we proceed as follows. For $p > 0$, we have that

$$\begin{aligned}\big||u_{t_0,\omega}|^{p-1}|v_{t_0,\omega}|^{p+1}u_{t_0,\omega} - |U|^{p-1}|V_{t_0,\omega}|^{p+1}U\big| &\leq \big(|u_{t_0,\omega}|^p|v_{t_0,\omega}|^p + |U|^p|V|^p\big)\big||v_{t_0,\omega}| - |V|\big|\\ &\leq \big(|u_{t_0,\omega}|^p|v_{t_0,\omega}|^p + |U|^p|V|^p\big)|v_{t_0,\omega} - V|.\end{aligned} \tag{3.10}$$
eq-m10

Using Strichartz estimate, one obtains

$$\|B_1\|_{L^\gamma((0,t);L^\rho)} \leq C\|(|u_{t_0,\omega}|^p|v_{t_0,\omega}|^p + |U|^p|V|^p)|v_{t_0,\omega} - V|\|_{L^{q'}((0,t);L^{r'})}. \tag{3.11}$$
eq-m11

Using Hölder's inequality, and the fact that $r = 2p+2$, we get

$$\|(|u_{t_0,\omega}|^p|v_{t_0,\omega}|^p+|U|^p|V|^p)|v_{t_0,\omega}-V|\|_{L^{r'}} \leq \Big(\|u_{t_0,\omega}\|_{L^r}\|v_{t_0,\omega}\|_{L^r}+\|U\|_{L^r}\|V\|_{L^r}\Big)\|v_{t_0,\omega}-V\|_{L^r}. \tag{3.12}$$
eq-m12

Inserting (3.12) in (3.11) and using Hölder's inequality in time variable, yields

$$\begin{aligned}&\|B_1\|_{L^\gamma((0,t);L^\rho)}\\ &\leq C\Big(\|u_{t_0,\omega}\|_{L^\infty((0,t);L^r)}\|v_{t_0,\omega}\|_{L^\infty((0,t);L^r)} + \|U\|_{L^\infty((0,t);L^r)}\|V\|_{L^\infty((0,t);L^r)}\Big)\|v_{t_0,\omega} - V\|_{L^{q'}((0,t);L^r)}.\end{aligned} \tag{3.13}$$
eq-m13

The estimate (3.13) and Sobolev embedding $H^1(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$ imply

$$\|B_1\|_{L^\gamma((0,t);L^\rho)} \leq C\|v_{t_0,\omega} - V\|_{L^{q'}((0,t);L^r)}. \tag{3.14}$$
eq-m14

From (3.6), (3.9) and (3.14), one obtains that

$$\|B\|_{L^\gamma((0,t);L^\rho)} \leq C_\omega + C\|v_{t_0,\omega} - V\|_{L^{q'}((0,t);L^r)}, \tag{3.15}$$
eq-m15.1

for all $0 < t \leq l$, where $C_\omega \to 0$ as $|\omega| \to \infty$.

With the similar procedure for the admissible pair (q, r) we get estimates analogous to (3.15), to have

$$\|B\|_{L^q((0,t);L^r)} + \|B\|_{L^\gamma((0,t);L^\rho)} \leq C_\omega + C\|v_{t_0,\omega} - V\|_{L^{q'}((0,t);L^r)}, \tag{3.16}$$

eq-m15

for all $0 < t \leq l$, where $C_\omega \to 0$ as $|\omega| \to \infty$.

Now from (3.4), combining the estimates (3.5) and (3.16), we get

$$\|u_{t_0,\omega}-U\|_{L^q((0,t);L^r)}+\|u_{t_0,\omega}-U\|_{L^\gamma((0,t);L^\rho)} \leq C_\omega+C\|u_{t_0,\omega}-U\|_{L^{q'}((0,t);L^r)}+C\|v_{t_0,\omega}-V\|_{L^{q'}((0,t);L^r)}. \tag{3.17}$$

eq-m18

With the analogous argument we get the similar estimate for the second component too, i.e.,

$$\|v_{t_0,\omega}-V\|_{L^q((0,t);L^r)}+\|v_{t_0,\omega}-V\|_{L^\gamma((0,t);L^\rho)} \leq C_\omega+C\|v_{t_0,\omega}-V\|_{L^{q'}((0,t);L^r)}+C\|u_{t_0,\omega}-U\|_{L^{q'}((0,t);L^r)}, \tag{3.18}$$

eq-m19

From (3.17) and (3.18), we conclude that

$$\begin{aligned}\|(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)\|_{\mathbb{L}^q((0,t);L^r)}&+\|(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)\|_{\mathbb{L}^\gamma((0,t);L^\rho)} \\ &\leq C_\omega + C\|(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)\|_{\mathbb{L}^{q'}((0,t);L^r)},\end{aligned} \tag{3.19}$$

eq-m20

for all $0 < t \leq l$, where $C_\omega \to 0$ as $|\omega| \to \infty$.

From (3.19), we have that

$$\|(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)\|_{\mathbb{L}^q((0,t);L^r)} \leq C_\omega + C\|(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)\|_{\mathbb{L}^{q'}((0,t);L^r)}. \tag{3.20}$$

eq-m21

Since, $q > q'$, we have

$$\|(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)\|_{\mathbb{L}^q((0,t);L^r)} \leq CC_\omega \to 0, \quad |\omega| \to \infty. \tag{3.21}$$

eq-m22

Therefore, from (3.21) and (3.19) one can conclude that

$$\sup_{t_0\in\mathbb{R}} \|(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)\|_{\mathbb{L}^\gamma((0,t);L^\rho)} \to 0, \qquad |\omega| \to \infty, \tag{3.22}$$

eq-m23

for all admissible pairs (γ, ρ).

Next, we move to prove convergence in the space $\mathbb{L}^\gamma((0, l); W^{1,\rho})$. In other words, we prove the following

$$\|\nabla[(u, v) - (U, V)]\|_{\mathbb{L}^\gamma((0,l);L^\rho)} \to 0, \quad |\omega| \to \infty. \tag{3.23}$$

eq-m24

Note that

$$\|\nabla[(u, v) - (U, V)\|_{\mathbb{L}^\gamma((0,l);L^\rho)} = \|\nabla(u - U)\|_{\mathbb{L}^\gamma((0,l);L^\rho)} + \|\nabla(v - V)\|_{\mathbb{L}^\gamma((0,l);L^\rho)}. \tag{3.24}$$

eq-m25

We have,

$$
\begin{aligned}
\nabla(u-U) &= i\nabla\int_0^t S(t-s)[\theta_1|u|^{2p}u_{t_0,\omega}(s) - I(\theta_1)|U|^{2p}U(s)]ds \\
&\quad + i\beta\nabla\int_0^t S(t-s)[\theta_1|u|^{p-1}|v|^{p+1}u(s) - I(\theta_1)|U|^{p-1}|V|^{p+1}U(s)]ds \\
&=: I_1 + I_2.
\end{aligned}
\tag{3.25}
$$

`eq-d1`

With the same technique as in [7], we obtain

$$
\|I_1\|_{L^\gamma((0,l);L^\rho)} \to 0 \qquad |\omega| \to \infty. \tag{3.26}
$$

`eq-d2`

To estimate I_2, let us define $g(u,v) = |u|^{p-1}u|v|^{p+1}$, so that

$$
\begin{aligned}
\nabla g(u,v) &= \begin{pmatrix} \frac{p+1}{2}|u|^{p-1}|v|^{p+1} \\ \frac{p-1}{2}|u|^{p-3}|v|^{p-1}u^2 \end{pmatrix}\begin{pmatrix} \nabla u \\ \nabla \bar{u} \end{pmatrix} + \begin{pmatrix} \frac{p+1}{2}|u|^{p-1}|v|^{p-1}u\bar{v} \\ \frac{p-1}{2}|u|^{p-1}|v|^{p+1}uv \end{pmatrix}\begin{pmatrix} \nabla v \\ \nabla \bar{v} \end{pmatrix} \\
&=: g_1'(u,v)\cdot Du + g_2'(u,v)\cdot Dv.
\end{aligned}
\tag{3.27}
$$

`eq-d3`

Now, using (3.27), we get

$$
\begin{aligned}
I_2 &= i\beta\int_0^t S(t-s)\big[\theta_1\big(g_1'(u,v)\cdot Du + g_2'(u,v)\cdot Dv\big) - I(\theta_1)\big(g_1'(U,V)\cdot DU + g_2'(U,V)\cdot DV\big)\big]ds \\
&= i\beta\int_0^t \theta_1 S(t-s)\big[g_1'(u,v)\cdot(Du-DU) + g_2'(u,v)\cdot(Dv-DV)\big]ds \\
&\quad + i\beta\int_0^t \theta_1 S(t-s)\big[(g_1'(u,v)-g_1'(U,V))\cdot DU + (g_2'(u,v)-g_2'(U,V))\cdot DV\big]ds \\
&\quad + i\beta\int_0^t (\theta_1 - I(\theta_1))S(t-s)\big[g_1'(U,V)\cdot DU + g_2'(U,V)\cdot DV\big]ds \\
&=: J_1 + J_2 + J_3.
\end{aligned}
\tag{3.28}
$$

`eq-d4`

Now, using Strichartz inequality, for any admissible pair (γ,ρ) we get

$$
\begin{aligned}
\|J_1\|_{L^\gamma((0,l);L^\rho)} &\le C\big\{\|g_1'(u,v)\cdot(Du-DU)\|_{L^{q'}((0,l);L^{r'})} + \|g_2'(u,v)\cdot(Dv-DV)\|_{L^{q'}((0,l);L^{r'})}\big\} \\
&\le C\big\{\||g_1'(u,v)||(Du-DU)|\|_{L^{q'}((0,l);L^{r'})} + \||g_2'(u,v)||(Dv-DV)|\|_{L^{q'}((0,l);L^{r'})}\big\}.
\end{aligned}
\tag{3.29}
$$

`eq-d5`

Since

$$
\begin{aligned}
&|g_1'(u,v)| \le C_p |u|^{p-1}|v|^{p+1}, \qquad\qquad |g_2'(u,v)| \le \hat{C}_p |u|^p |v|^p \\
&|Du - DU| = \left| \begin{pmatrix} \nabla(u-U) \\ \overline{\nabla(u-U)} \end{pmatrix} \right| \le 2|\nabla(u-U)|, \\
&|Dv - DV| = \left| \begin{pmatrix} \nabla(v-V) \\ \overline{\nabla(v-V)} \end{pmatrix} \right| \le 2|\nabla(v-V)|,
\end{aligned}
\tag{3.30}
$$

`eq-d6`

we obatin from (3.29) that

$$
\begin{aligned}
\|J_1\|_{L^\gamma((0,l);L^\rho)} &\le C_p \| |u|^{p-1}|v|^{p+1} |\nabla(u-U)| \|_{L^{q'}((0,l);L^{r'})} + \hat{C}_p \| |u|^p |v|^p |\nabla(v-V)| \|_{L^{q'}((0,l);L^{r'})} \\
&\le C_p \| |u|^{p-1}|v|^{p+1} \|_{L^\infty((0,l);L^{\frac{p+1}{p}})} \|\nabla(u-U)\|_{L^{q'}((0,l);L^r)} \\
&\quad + \hat{C}_p \| |u|^p |v|^p \|_{L^\infty((0,l);L^{\frac{p+1}{p}})} \|\nabla(v-V)\|_{L^{q'}((0,l);L^r)}.
\end{aligned}
\tag{3.31}
$$

`eq-d7`

Using Hölders inequality, we have

$$
\begin{aligned}
&\| |u|^{p-1}|v|^{p+1} \|_{L^\infty((0,l);L^{\frac{p+1}{p}})} \le C_p \|u\|^{p-1}_{L^\infty((0,l);L^r)} \|v\|^{p+1}_{L^\infty((0,l);L^r)} \\
&\| |u|^p |v|^p \|_{L^\infty((0,l);L^{\frac{p+1}{p}})} \le C_p \|u\|^p_{L^\infty((0,l);L^r)} \|v\|^p_{L^\infty((0,l);L^r)}.
\end{aligned}
\tag{3.32}
$$

`eq-d8`

Now, using the Sobolev embedding $H^1 \hookrightarrow L^r$, we obtain from (3.32) and (3.31)

$$
\|J_1\|_{L^\gamma((0,l);L^\rho)} \le C_p \|(\nabla(u-U), \nabla(v-V))\|_{\mathbb{L}^{q'}((0,l),L^r)}. \tag{3.33}
$$

`eq-d9`

With the similar procedure, also for the admissible pair (q,r), we obtain

$$
\|J_1\|_{L^q((0,l);L^r)} \le C_p \|(\nabla(u-U), \nabla(v-V))\|_{\mathbb{L}^{q'}((0,l),L^r)}. \tag{3.34}
$$

`eq-d10`

To estimate J_2, we use Strichartz estimate for the admissible pairs (γ, ρ) and (q, r), and the fact that the solution $(U, V) \in L^q W^{1,r}$ to obtain

$$\begin{aligned}
\|J_2\|_{L^\gamma((0,l);L^\rho)} + \|J_2\|_{L^q((0,l);L^r)} &\le C\|(g_1'(u,v) - g_1'(U,V)) \cdot DU\|_{L^{q'}((0,l);L^{r'})} \\
&\quad + C\|(g_2'(u,v) - g_2'(U,V)) \cdot DV\|_{L^{q'}((0,l);L^{r'})} \\
&\le C\|(g_1'(u,v) - g_1'(U,V))\|_{L^{\frac{q}{q-2}}((0,l);L^{\frac{r}{r-2}})} \|\nabla U\|_{L^q((0,l),L^r)} \\
&\quad + C\|(g_2'(u,v) - g_2'(U,V))\|_{L^{\frac{q}{q-2}}((0,l);L^{\frac{r}{r-2}})} \|\nabla V\|_{L^q((0,l),L^r)} \\
&\le C\|(g_1'(u,v) - g_1'(U,V))\|_{L^{\frac{q}{q-2}}((0,l);L^{\frac{r}{r-2}})} \\
&\quad + C\|(g_2'(u,v) - g_2'(U,V))\|_{L^{\frac{q}{q-2}}((0,l);L^{\frac{r}{r-2}})}.
\end{aligned} \tag{3.35}$$

eq-d11

We have that as $|\omega| \to \infty$, $\|u - U\|_{L_t^\infty L_x^2} \to 0$ and $\|v - V\|_{L_t^\infty L_x^2} \to 0$. Using the interpolation relation $\|u-U\|_{H^s} \le \|u-U\|_{L^2}^{1-s}\|u-U\|_{H^1}^s$, we can conclude that $u \to U$ and $v \to V$ in $C([0,l]; H^s(\mathbb{R}^n))$ as $|\omega| \to \infty$ for $0 \le s < 1$. If s is sufficiently close to 1 such that $s > \frac{n}{2} - \frac{n}{r}$, then using the Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$, we have that $u \to U$ and $v \to V$ in $C([0,l]; L^r(\mathbb{R}^n))$ as $|\omega| \to \infty$.

Note that $\|g_1'(u,v)\|_{\mathbb{L}^{\frac{r}{r-2}}} \le C_p \||u|^{p-1}|v|^{p+1}\|_{L^{\frac{r}{r-2}}} \le C\|u\|_{L^r}^{p-1}\|v\|_{L^r}^{p+1} < \infty$, and similar holds for $g_2'(u,v)$. Now, from dominated convergence theorem, we obtain that the mappings $(u,v) \to g_1'(u,v)$ and $(u,v) \to g_2'(u,v)$ are continuous from $\mathbb{L}^r(\mathbb{R}^n) \to \mathbb{L}^{\frac{r}{r-2}}(\mathbb{R}^n)$ and consequently,

$$\sup_{t_0 \in \mathbb{R}} \left[\|g_1'(u,v) - g_1'(U,V)\|_{\mathbb{L}^\infty((0,l);L^{\frac{r}{r-2}})} + \|g_2'(u,v) - g_2'(U,V)\|_{\mathbb{L}^\infty((0,l);L^{\frac{r}{r-2}})}\right] \to 0, \quad |\omega| \to \infty. \tag{3.36}$$

eq-d12

From (3.35) and (3.36) we conclude that

$$\sup_{t_o \in \mathbb{R}} [\|J_2\|_{L^\gamma((0,l);L^\rho)} + \|J_2\|_{L^q((0,l);L^r)}] =: C_\omega \to 0, \quad |\omega| \to \infty. \tag{3.37}$$

eq-d13

It is easy to see that $g_1'(U,V) \cdot DU + g_2'(U,V) \cdot DV \in L^{q'}((0,l); L^{r'}(\mathbb{R}^n))$, so by Proposition 2.3, we have

$$\sup_{t_o \in \mathbb{R}} [\|J_3\|_{L^\gamma((0,l);L^\rho)} + \|J_3\|_{L^q((0,l);L^r)}] =: C_\omega \to 0, \quad |\omega| \to \infty. \tag{3.38}$$

eq-d14

Combining the estimates (3.33), (3.34), (3.37) and (3.38), we can conclude as in (3.22) that

$$\sup_{t_0 \in \mathbb{R}} \|\nabla(u,v) - \nabla(U,V)\|_{\mathbb{L}^\gamma((0,l);L^\rho(\mathbb{R}^n))} \to 0, \quad |\omega| \to \infty, \tag{3.39}$$

eq-d15

for all admissible pairs (γ, ρ).

Hence the result of the leamma follows from (3.22) and (3.39). $\square$

sec-3

4. Proof of the main results.

Proof of Theorem 1.2. Let $T \in (0, S_{\max})$, $A = \max\{\|h\|_\infty, \|z\|_\infty\}$,

$$M = 2 \sup_{0 \le t \le T} |||(U(t), V(t))|||_{1,2}, \tag{4.1}$$ x1

and $\delta = \delta(A, M)$ be given by Proposition 2.1. It follows by this proposition that $(u_{t_0,\omega}(t), v_{t_0,\omega}(t))$ exists on $[0, \delta]$ and satisfies

$$||(u_{t_0,\omega}, v_{t_0,\omega})||_{\mathbb{L}^\infty((0,\delta),H^1)} \le C\, |||(\varphi, \psi)|||_{1,2}. \tag{4.2}$$ x2

Now Lemma 3.1 implies that

$$\sup_{t_0 \in \mathbb{R}} ||(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)||_{\mathbb{L}^\gamma((0,\delta),W^{1,\rho})} \to 0, \quad \text{when} \quad |w| \to \infty,$$

for all admissible pair (γ, ρ) and in particular

$$\sup_{t_0 \in \mathbb{R}} ||(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)||_{\mathbb{L}^\gamma((0,\delta),H^1)} \to 0, \quad \text{when} \quad |w| \to \infty. \tag{4.3}$$ x3

Combining (4.1) and (4.3) we obtain that for $|\omega|$ sufficiently large

$$\sup_{t_0 \in \mathbb{R}} |||(u_{t_0,\omega}(\delta), v_{t_0,\omega}(\delta))|||_{1,2} \le \sup_{t_0 \in \mathbb{R}} |||(U(\delta), V(\delta))|||_{1,2} + \frac{M}{2} \le M. \tag{4.4}$$

Applying again Proposition 2.1 translated by δ and using (4.2), we have that $u_{t_0,\omega}(t)$ exists on $[0, 2\delta]$ and that

$$\limsup_{|\omega| \to \infty} \sup_{t_0 \in \mathbb{R}} \|(u_{t_0,\omega}, v_{t_0,\omega})\|_{\mathbb{L}((0,2\delta),H^1)} < \infty.$$

If $2\delta < T$, iterating this argument, we deduce that

$$\limsup_{|\omega| \to \infty} \sup_{t_0 \in \mathbb{R}} \|(u_{t_0,\omega}, v_{t_0,\omega})\|_{\mathbb{L}((0,T),H^1)} < \infty.$$

The result then follows from Lemma 3.1. $\square$

Proof of Theorem 1.3. Let $\epsilon \in (0, \epsilon(A))$, where $\epsilon(A)$ is as in Proposition 2.2. For sufficiently large T, from (1.13), one gets that

$$\|(U, V)\|_{\mathbb{L}^a((T,\infty);L^r(\mathbb{R}^n))} \le \frac{\epsilon}{4}. \tag{4.5}$$ eq-z1

Applying the Proposition 2.2 to the global solution $(\tilde{U}(t), \tilde{V}(t)) = (U(t+T), V(t+T))$, the inequality (2.13) yields

$$\begin{aligned}\|S(\cdot)(U(T),V(T))\|_{\mathbb{L}^a((0,\infty);L^r)} &= \|S(\cdot)(\tilde{U}(0),\tilde{V}(0))\|_{\mathbb{L}^a((0,\infty);L^r)}\\ &\leq 2\|(\tilde{U},\tilde{V})\|_{\mathbb{L}^a((0,\infty);L^r)} = 2\|(U,V)\|_{\mathbb{L}^a((T,\infty);L^r)} \leq \frac{\epsilon}{2}.\end{aligned} \tag{4.6}$$
eq-z2

Now, in the light of this inequality, using Corollary 2.1, we get that

$$\|(U,V)\|_{\mathbb{L}^a((T,\infty);L^r)} \leq \Lambda|||(U(T),V(T))|||_{1,2}. \tag{4.7}$$
eq-z3

From Theorem 1.2, we have that

$$\sup_{t_0\in\mathbb{R}} \|(u_{t_0,\omega}, v_{t_0,\omega}) - (U,V)\|_{\mathbb{L}^\gamma((0,T);W^{1,\rho})} \to 0, \qquad |\omega| \to \infty, \tag{4.8}$$
eq-z4

for all $T < \infty$ and all admissible pairs (γ, ρ). So, in particular, we have

$$\sup_{t_0\in\mathbb{R}} |||(u_{t_0,\omega}(T), v_{t_0,\omega}(T)) - (U,V)|||_{1,2} \to 0, \qquad |\omega| \to \infty. \tag{4.9}$$
eq-z5

Therefore, we have, using (4.7) that

$$\begin{aligned}\|S(\cdot)(u_{t_0,\omega}(T), v_{t_0,\omega}(T))\|_{\mathbb{L}^a((0,\infty);L^r)} &\leq \|S(\cdot)(u_{t_0,\omega}(T), v_{t_0,\omega}(T)) - S(\cdot)(U(T),V(T))\|_{\mathbb{L}^a((0,\infty);L^r)}\\ &\quad + \|S(\cdot)(U(T),V(T))\|_{\mathbb{L}^a((0,\infty);L^r)}\\ &\leq |||(u_{t_0,\omega}(T), v_{t_0,\omega}(T)) - (U(T),V(T))|||_{1,2} + \frac{\epsilon}{2} \leq \epsilon,\end{aligned} \tag{4.10}$$
eq-z6

for sufficiently large $|\omega|$.

Hence, from Corollary 2.1, we conclude that the solution $(u_{t_0,\omega}, v_{t_0,\omega})$ is global and satisfies

$$\sup_{t_0\in\mathbb{R}} \|(u_{t_0,\omega}, v_{t_0,\omega})\|_{\mathbb{L}^a((T,\infty);L^r)} \leq 2\epsilon \tag{4.11}$$
eq-z7

and

$$\|(u_{t_0,\omega}, v_{t_0,\omega})\|_{\mathbb{L}^q((T,\infty);W^{1,r})} + \|(u_{t_0,\omega}, v_{t_0,\omega})\|_{\mathbb{L}^\infty((T,\infty);H^1)} \leq \Lambda|||(u_{t_0,\omega}(T), v_{t_0,\omega}(T))|||_{1,2}. \tag{4.12}$$
eq-z8

Also, in view of corollary 2.1 and (4.5), we have

$$\|(U,V)\|_{\mathbb{L}^a((T,\infty);L^r)} \leq \epsilon \tag{4.13}$$
eq-z9

and

$$\|(U,V)\|_{\mathbb{L}^q((T,\infty);W^{1,r})} + \|(U,V)\|_{\mathbb{L}^\infty((T,\infty);H^1)} \leq \Lambda|||(U(T),V(T))|||_{1,2}. \tag{4.14}$$
eq-z10

Let $M_0 = \sup_{0\le t\le T} |||(U(T), V(T))|||_{1,2}$. From (4.8), (4.10) and (4.12), it is easy to see that there exists $L > 0$ sufficiently large such that

$$\sup_{|\omega|\ge L}\sup_{t_0\in\mathbb{R}}\sup_{t\ge 0} |||(u_{t_0,\omega}(t), v_{t_0,\omega}(t))|||_{1,2} + \sup_{t\ge 0} |||(U(t), V(t))|||_{1,2} \le M_1 < \infty. \tag{4.15}$$

eq-z11

In what follows, we prove that, for all admissible pairs (γ, ρ), $(u_{t_0,\omega}, v_{t_0,\omega}) \to (U, V)$ in $\mathbb{L}^\gamma((0,\infty), W^{1,\rho})$ uniformly in $t_0 \in \mathbb{R}$.

Let $|\omega| \gg 1$ so that the solution $(u_{t_0,\omega}, v_{t_0,\omega})$ exists globally and fix $T > 0$ to be chosen later. Note that,

$$\begin{aligned} \|(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)\|_{\mathbb{L}^\gamma((0,\infty),W^{1,\rho})} &\le \|(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)\|_{\mathbb{L}^\gamma((0,T),W^{1,\rho})} \\ &\quad + \|(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)\|_{\mathbb{L}^\gamma((T,\infty),W^{1,\rho})}. \end{aligned} \tag{4.16}$$

eq-z12

From Theorem 1.2, we have that the first term in the right hand side of (4.16) converges to zero as $|\omega| \to \infty$. So, the convergence we are looking for would follow, if we can prove that, for every $\varepsilon > 0$, there exists $T > 0$ such that for $|\omega|$ sufficiently large

$$\|(u_{t_0,\omega}, v_{t_0,\omega}) - (U, V)\|_{\mathbb{L}^\gamma((T,\infty),W^{1,\rho})} \le \varepsilon, \tag{4.17}$$

eq-z13

holds true.

Our objective from here onwards is to prove (4.17). Looking at the symmetry of the model under consideration, the estimate (4.17) would follow if we prove it for a single component, i.e., if we prove that for every $\varepsilon > 0$, there exists $T > 0$ such that for $|\omega|$ sufficiently large, the following holds

$$\|u_{t_0,\omega} - U\|_{L^\gamma((T,\infty),W^{1,\rho})} \le \varepsilon. \tag{4.18}$$

eq-z14

Using Duhamel's formula, for all $t > 0$, we have

$$\begin{aligned} u_{t_0,\omega}(T+t) - U(T+t) &= S(t)(u_{t_0,\omega}(T) - U(T)) \\ &\quad + i\int_0^t S(t-s)h(\omega(T+s+t_0))F(u_{t_0,\omega}, v_{t_0,\omega})(T+s)ds \\ &\quad - iI(h)\int_0^t S(t-s)F(U,V)(T+s)ds \\ &=: Q_1(t) + Q_2(t) + Q_3(t). \end{aligned} \tag{4.19}$$

eq-z20

Using the Strichartz estimate, we obtain

$$\|Q_1(t)\|_{L^\gamma((0,\infty),W^{1,\rho})} \leq CA\|u_{t_0,\omega}(T) - U(T)\|_{H^1} \to 0, \quad |\omega| \to \infty, \tag{4.20}$$

eq-z21

$$\begin{aligned}\|Q_2(t)\|_{L^\gamma((0,\infty),W^{1,\rho})} \leq CA\Big[&\||u_{t_0,\omega}|^{2p}u_{t_0,\omega}\|_{L^{q'}((T,\infty);W^{1,r'})}\\ &+ \beta\||u_{t_0,\omega}|^{p-1}|v_{t_0,\omega}|^{p+1}u_{t_0,\omega}\|_{L^{q'}((T,\infty);W^{1,r'})}\Big],\end{aligned} \tag{4.21}$$

eq-z22

and

$$\begin{aligned}\|Q_3(t)\|_{L^\gamma((0,\infty),W^{1,\rho})} \leq CA\Big[&\||U|^{2p}U\|_{L^{q'}((T,\infty);W^{1,r'})}\\ &+ \beta\||U|^{p-1}|V|^{p+1}U\|_{L^{q'}((T,\infty);W^{1,r'})}\Big].\end{aligned} \tag{4.22}$$

eq-z23

We have that, proceeding as in the proof of (3.23) (see (3.31) and (3.32) in particular)

$$\||u_{t_0,\omega}|^{2p}u_{t_0,\omega}\|_{W^{1,r'}} \leq C\|u_{t_0,\omega}\|_{L^r}^{2p}\|u_{t_0,\omega}\|_{W^{1,r}}, \tag{4.23}$$

eq-z24

and

$$\||u_{t_0,\omega}|^{p-1}|v_{t_0,\omega}|^{p+1}u_{t_0,\omega}\|_{W^{1,r'}} \leq C\|u_{t_0,\omega}\|_{L^r}^{p-1}\|v_{t_0,\omega}\|_{L^r}^{p+1}\|u_{t_0,\omega}\|_{W^{1,r}}. \tag{4.24}$$

eq-z25

Now, applying Hölder's inequality in time variable and the definition of r and a in (1.12), we obtain that

$$\|Q_2\|_{L^\gamma((0,\infty),W^{1,\rho})} \leq CA\|(u_{t_0,\omega}, v_{t_0,\omega})\|_{\mathbb{L}^a((T,\infty);L^r)}^{2p}\|(u_{t_0,\omega}, v_{t_0,\omega})\|_{\mathbb{L}^q((T,\infty);W^{1,r})}. \tag{4.25}$$

eq-z26

Using (4.11), (4.14) and (4.15), we get

$$\|Q_2\|_{L^\gamma((0,\infty),W^{1,\rho})} \leq CA(2\epsilon)^{2p}\Lambda M_1. \tag{4.26}$$

eq-z27

With the similar argument, one obtains

$$\|Q_3\|_{L^\gamma((0,\infty),W^{1,\rho})} \leq CA(2\epsilon)^{2p}\Lambda M_0. \tag{4.27}$$

eq-z28

Now, given $\varepsilon > 0$, we choose sufficiently small $\epsilon > 0$ such that $CA(2\epsilon)^{2p}\Lambda(M_1+M_0) < \varepsilon/3$ and $|\omega| \gg 1$, so that (4.19), (4.20), (4.26) and (4.27) yield

$$\begin{aligned}\|u_{t_0,\omega}(t) - U(t)\|_{L^\gamma((T,\infty);W^{1,\rho})} &= \|u_{t_0,\omega}(T+t) - U(T+t)\|_{L^\gamma((0,\infty);W^{1,\rho})}\\ &\leq \|Q_1(t)\|_{L^\gamma((0,\infty);W^{1,\rho})} + \|Q_2(t)\|_{L^\gamma((0,\infty);W^{1,\rho})}\\ &\qquad + \|Q_3(t)\|_{L^\gamma((0,\infty);W^{1,\rho})}\\ &< \varepsilon,\end{aligned} \tag{4.28}$$

eq-z29

as required, and this completes the proof. □

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