Compact embeddings of p-Sobolev-like cones of nuclear operators and Gagliardo-Nirenberg inequalities

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Abstract. Let $p \geq 2$, $\Omega \subseteq \mathbb{R}^N$ smooth bounded domain, $V \in \mathrm{L}^\infty(\Omega)$ non-negative, and $\mathscr{S}_1$ the space of self-adjoint trace-class operators on $\mathrm{L}^2(\Omega)$. We prove that $\mathscr{W}^{1,p}$, the p-Sobolev-like cone of operators $T \in \mathscr{S}_1$ having eigenvalues ν_i, $i \in \mathbb{N}$, and an eigenbasis $B = \{\psi_i \,/\, i \in \mathbb{N}\}$ of $\mathrm{L}^2(\Omega)$ such that

$$\langle\!\langle T \rangle\!\rangle_{p,B} = \sum_{i\in\mathbb{N}} |\nu_i| \int_\Omega [|\nabla\psi_i|^p + V(x)|\psi_i|^p]dx < +\infty,$$

is compactly embedded in $\mathscr{S}_1$. In the path we prove regularity properties for the density function associated to T as well as Gagliardo-Niremberg type inequalities departing from Lieb-Thirring type conditions. We apply the compactness property to minimize free energy functionals where the entropy term is generated by a Cassimir-class function related to the eigenvalue problem of the Schrödinger operator $-\alpha\Delta + V$, $\alpha > 0$, with Dirichlet condition.

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1. Introduction

Let's quickly introduce the concept of p-Sobolev-like cone of nuclear operators and our main result, a compactness property analogous to that of the classical Sobolev embedding but at operators level. After this we shall present a short state of the art and some relevant comments.

Let $\Omega \subseteq \mathbb{R}^N$ be a smooth bounded domain, $H = \mathrm{L}^2(\Omega)$ and $T : H \longrightarrow H$ a compact self-adjoint linear operator. By the Hilbert-Schmidt and Riesz-Schauder theorems, there exist a Hilbert basis of H, $B = \{\psi_i \,/\, i \in \mathbb{N}\}$, and $(\nu_i)_{i\in\mathbb{N}} \subseteq \mathbb{R}$ such that, for each $i \in \mathbb{N}$,

$$T\psi_i = \nu_i \psi_i. \tag{1.1}$$

We say that B is an *eigenbasis* of T, denoted $B \in \mathscr{B}_T$, as the previous relation implies that, for each $i \in \mathbb{N}$, ψ_i is an eigenfunction associated to the eigenvalue ν_i. We shall say that T is a *nuclear operator*, denoted $T \in \mathscr{S}_1$, if, in addition, $(\nu_i)_{i\in\mathbb{N}} \in l^1(\mathbb{R})$ and, in this case, the values

$$\mathrm{Tr}\,(T) = \sum_{i\in\mathbb{N}} \nu_i \quad \text{and} \quad \|T\|_1 = \mathrm{Tr}\,(|T|) = \sum_{i\in\mathbb{N}} |\nu_i|$$

are referred to as the *trace* and the *trace norm* of the operator T, respectively, [11]. Whenever $T \geq 0$ the numbers ν_i, $i \in \mathbb{N}$, are usually referred to as *occupation numbers* and the sequence $(\nu_i, \psi_i)_{i\in\mathbb{N}} \subseteq \mathbb{R}_* \times H$ is called a *mixed state.*

Let's assume that $p \geq 1$ and that $V \in \mathrm{L}^\infty(\Omega)$ is non-negative. Let's consider the Banach space $\mathrm{W}_V^{1,p}(\Omega) = (\mathrm{W}_0^{1,p}(\Omega), \|\cdot\|_{V,p})$, where

$$\|u\|_{V,p}^p = \|u\|_{\mathrm{W}_0^{1,p}(\Omega)}^p + \int_\Omega V(x)|u(x)|^p dx.$$

For $T \in \mathscr{S}_1$, we write $\mathscr{B}_T^p = \{B \in \mathscr{B}_T / \; B \subseteq \mathrm{W}_V^{1,p}(\Omega)\}$. Now we can introduce the main concept of this work. The *p-Sobolev-like cone*, $\mathscr{W}^{1,p}$, is the set of operators $T \in \mathscr{S}_1$ such that there exists $B = \{\psi_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}_T^p$ such that the *pB-energetic value of* T is finite:

$$\langle\!\langle T \rangle\!\rangle_{p,B} = \sum_{i\in\mathbb{N}} |\nu_i| \; \|\psi_i\|_{V,p}^p < +\infty.$$

The value

$$\langle\!\langle T \rangle\!\rangle_p = \inf_{B\in\mathscr{B}_T^p} \langle\!\langle T \rangle\!\rangle_{p,B}$$

shall be referred to as the *p-energy of* T. Our main result is the following.

Theorem 1.1. *Let $p \geq 2$. The embedding $\mathscr{W}^{1,p} \subseteq \mathscr{S}_1$ is compact, i.e., if $(T_n)_{n\in\mathbb{N}} \subseteq \mathscr{W}^{1,p}$ is $\langle\!\langle \cdot \rangle\!\rangle_p$-bounded, then there are a subsequence $(T_{n_k})_{k\in\mathbb{N}}$ and an operator $T \in \mathscr{W}^{1,p}$ such that $\|T - T_{n_k}\|_1 \longrightarrow 0$, as $k \longrightarrow +\infty$.*

In the context of quantum mechanics, [13], a physical state of a system is represented by a positive nuclear operator $T \in \mathscr{S}_1$, also referred to as a density matrix operator because of the decomposition

$$T\phi = \sum_{i\in\mathbb{N}} \nu_i (\phi, \psi_i)\, \psi_i, \quad \phi \in H,$$

where (1.1) is assumed and $(\cdot,\cdot)$ denotes the inner product of H.

In a number of works free energy functionals have been considered, dealing with nuclear operators or mixed states, to obtain insights in quantum mechanics situations (see e.g. [2], [6] and [8]). These and many other works deal with $p = 2$ because that is a natural setting for modeling quantum systems.

From the mathematical point of view is quite interesting to extend the study of [5], [6] and [9] to nuclear operators having eigenfunctions with a different kind of regularity, say $\mathrm{W}^{1,p}_V(\Omega)$, $\mathrm{W}^{1,p}_0(\Omega) \cap \mathrm{W}^{2,p}(\Omega)$, etc. In [8] it was considered $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$-regularity to prove existence and nonlinear stability of stationary states of the Schrödinger-Poisson system, while in [5] it was necessary only $\mathrm{H}^1_0(\Omega)$-regularity to produce, for $p = 2$, Theorem 1.1 and other results. In other direction, it's also interesting to replace the pivot space $H = \mathrm{L}^2(\Omega)$ with other separable Hilbert spaces as a good part of the machinery we develop seems to be extendable. In this work we undertake the first path.

It's well known (see e.g. [6] and [5]) that fixed the potential V, the minimization of a given free energy functional is related to Lieb-Thirring type inequalities while, in other hand, the optimization on V produces interpolation inequalities. Inequalities of these types are interesting by themselves, in particular to study the stability of many-particle quantum systems (see e.g. [14] and [4]), and are building blocks in our context. See the description below.

The paper is organized in the following way.

1. In Section 2.1 we shall present the relevant operators setting.
2. In Section 2.2 we introduce $\|\cdot\|_{V,p}$, an equivalent norm for $\mathrm{W}^{1,p}_0(\Omega)$, and state some easy but useful inequalities that help to build up our the results.
3. The cone $\mathscr{W}^{1,p}$ and initial properties of its elements are stated and proved in Section 2.3. For example, given $T \in \mathscr{W}^{1,p}$ and a suitable continuous function β such that $\beta(0) = 0$, it holds $\beta(T) \in \mathscr{W}^{1,p}$. A Poincaré type inequality, $\|T\|_1 \leq K^p_p \cdot C_p \langle\!\langle T \rangle\!\rangle_p$, is also proved here.
4. In addition to the p-energy, the concepts of p-kinetic energy, $\mathscr{K}_p(\cdot)$, and p-potential energy, $\mathscr{P}_p(\cdot)$, are introduced in Section 2.4, showing an intrinsic imbalance of energy, $\langle\!\langle T \rangle\!\rangle_p \geq \mathscr{K}_p(T) + \mathscr{P}_p(T)$, on $\mathscr{W}^{1,p}_+ = \{T \in \mathscr{W}^{1,p} \,/\, T \geq 0\}$, which dissapears when $p = 2$.
5. In Section 2.5 it's proved a regularity result for the *density function of* $T \in \mathscr{W}^{1,p}_+$, $\rho_T(x) = \sum_{i\in\mathbb{N}} \nu_{i,T} |\psi_{i,T}(x)|^2$, for a.e. $x \in \Omega$, that is, $\rho_T \in \mathrm{W}^{1,\gamma}(\Omega) \cap \mathrm{L}^s(\Omega)$, for every $\gamma \in [1,p]$ and every $s \in [1, pN/(2N-2-p)]$.
6. In Section 3.1 we will introduce free energy functionals like
$$\mathscr{F}_{\beta,p}(T) = \mathrm{Tr}\,(\beta(T)) + \langle\!\langle T \rangle\!\rangle_p, \quad T \in \mathscr{W}^{1,p}_+, \tag{1.2}$$
where the *entropy seed* β, will be generated by a Casimir-class function F by $\beta(s) = F^*(-s)$, $s \in \mathbb{R}$.
7. Under a suitable condition on the Schrödinger operator $-\alpha\Delta + V$, $\alpha > 0$, in Section 3.2 we prove that $-\mathrm{Tr}\left(F(\hat{C}^{-1}(-\alpha\Delta + V)^{p/2})\right)$ is a lower bound for the free energy functional.
8. Assuming that $p \geq 2$ and that V bounded away from zero, in Section 3.3 we obtain more useful estimates for free energy functionals. By assuming

that a Lieb-Thirring type inequality holds,

$$\operatorname{Tr}\left(F(\hat{C}^{-1}(-\alpha\Delta + V)^{p/2})\right) \leq \int_\Omega G(V(x))dx,$$

we produce a Gagliardo-Niremberg type inequality

$$\operatorname{Tr}(\beta(T)) + \mathscr{K}_p(T) \geq \inf_{B\in\mathscr{B}_T^p} \int_\Omega \tau(\rho_{p,B}(x))dx,$$

where, for $B \in \mathscr{B}_T^p$, the *pB-density function associated to T* is given by $\rho_{p,B}(x) = \sum_{i\in\mathbb{N}} \nu_{i,T}|\eta_{i,T}(x)|^p$, for a.e. $x \in \Omega$.

9. In Section 4 we prove Theorem 1.1 for $p \geq 2$. This extends the results of [5] and [9] were it was considered the much simpler case of $p = 2$.
10. By using Theorem 1.1, in Section 5 we shall prove that several kinds of free energy functionals, including but not restricting to (1.2), have operator ground states.

2. Definitions and preliminary results

2.1. General operators setting

Let's introduce the global setting for our work. For completeness we repeat some concepts that were already mentioned in Section 1.

Let $\Omega \subseteq \mathbb{R}^N$ a smooth bounded domain and $H = \mathrm{L}^2(\Omega)$. As a separable Hilbert space, H has a Hilbert basis, i.e., there exists $D = \{\varphi_i \,/\, i \in \mathbb{N}\} \subseteq H$ orthonormal and such that it's span is dense in H.

We denote by $\mathscr{L}$ and $\mathscr{I}_\infty$ the spaces of bounded and compact linear operators on H, respectively. We also write $\mathscr{L}_S = \{T \in \mathscr{L} \,/\, T \text{ is self-adjoint}\}$ and $\mathscr{S}_\infty = \mathscr{I}_\infty \cap \mathscr{L}_S$. An operator $T \in \mathscr{L}$ belongs to the trace class $\mathscr{I}_1$ iff

$$\sum_{i\in\mathbb{N}} |(T\varphi_i, \varphi_i)| < +\infty.$$

In this case, [11], the values

$$\operatorname{Tr}(T) = \sum_{i\in\mathbb{N}} (T\varphi_i, \varphi_i) \quad \text{and} \quad \|T\|_1 = \operatorname{Tr}(|T|), \tag{2.1}$$

are actually basis-independent; they are referred to as the *trace* and the *trace norm* of the operator T, respectively. The space $(\mathscr{I}_1, \|\cdot\|_1)$ is Banach and contained in $\mathscr{I}_\infty$. The elements of the Banach space $(\mathscr{S}_1, \|\cdot\|_1)$,

$$\mathscr{S}_1 = \mathscr{I}_1 \cap \mathscr{L}_S \subseteq \mathscr{S}_\infty,$$

are referred to as *nuclear operators*.

By the Hilbert-Schmidt and Riesz-Schauder theorems (see e.g. [11]), for an operator $T \in \mathscr{S}_\infty$ there exist $B = \{\psi_{i,T} \,/\, i \in \mathbb{N}\} \subseteq H$, a Hilbert basis of H, and $\tilde{\sigma}(T) = \{\nu_{i,T} \,/\, i \in \mathbb{N}\} \subseteq \mathbb{R}$ such that, for each $i \in \mathbb{N}$,

$$T\psi_{i,T} = \nu_{i,T}\,\psi_{i,T}. \tag{2.2}$$

Because of (2.2), we say that B is an *eigenbasis* of T, and denote $B \in \mathscr{B}_T$. In this case, T belongs to $\mathscr{S}_1$ iff $(\nu_{i,T})_{i\in\mathbb{N}} \in l^1(\mathbb{R})$. In this case the formulas

$$\mathrm{Tr}\,(T) = \sum_{i\in\mathbb{N}} \nu_{i,T} \quad \text{and} \quad \|T\|_1 = \mathrm{Tr}\,(|T|) = \sum_{i\in\mathbb{N}} |\nu_{i,T}| \tag{2.3}$$

are equivalent to those in (2.1). The *density function associated to* T is $\rho_T \in \mathrm{L}^1(\Omega)$, given by

$$\rho_T(x) = \sum_{i\in\mathbb{N}} \nu_{i,T} |\psi_{i,T}(x)|^2, \quad \text{for a.e. } x \in \Omega. \tag{2.4}$$

We shall always assume that the sequence of eigenvalues $(\nu_{i,T})_{i\in\mathbb{N}} \subseteq \mathbb{R}$ is ordered: $|\nu_{i,T}| \geq |\nu_{j,T}|$ if $i \leq j$, and if both ν and $-\nu$ are eigenvalues, $-\nu$ comes first.

Remark 2.1. If $T \in \mathscr{S}_1$ is a positive operator, which is denoted $T \geq 0$ and means that $(T\psi, \psi) \geq 0$, for every $\psi \in H$, then every eigenvalue $\nu_{i,T}$ is non-negative and both values in (2.3) coincide.

Remark 2.2. Since $l^1(\mathbb{R}) \subseteq l^2(\mathbb{R})$, it follows that $\mathscr{S}_1$ is contained in the space of Hilbert-Schmidt operators, [13],

$$\mathscr{S}_2 = \left\{ T \in \mathscr{S}_\infty \,/ \quad \|T\|_2 = \left(\sum_{i\in\mathbb{N}} |\nu_{i,T}|^2 \right)^{1/2} < +\infty \right\}.$$

$\mathscr{S}_2$ is a Hilbert space as its norm is induced by the inner-product given by $(L, R)_2 = \mathrm{Tr}\,(R^* L)$. The action of an operator $T \in \mathscr{S}_2$ is characterized by a kernel function $K_T \in \mathrm{L}^2(\Omega \times \Omega)$ such that $K_T(x, y) = K_T(y, x)$, for a.e. $x, y \in \Omega$. In fact, it holds

$$(T\phi)(x) = \int_\Omega K_T(x, y)\, \phi(y) dy, \quad \text{for a.e. } x \in \Omega,$$

for $\phi \in H$, and $\|T\|_2 = \|K_T\|_{\mathrm{L}^2(\Omega\times\Omega)}$.

2.2. An equivalent norm for $\mathrm{W}_0^{1,p}(\Omega)$

Let $p \geq 1$. Let's introduce a norm for $\mathrm{W}_0^{1,p}(\Omega)$ which is equivalent to the classical one,

$$\|u\|_{\mathrm{W}_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u(x)|^p dx \right)^{1/p},$$

and state some simple but useful properties. From now on we shall assume that

$(V\Omega)$ $V \in \mathrm{L}^\infty(\Omega)$ is a non-negative potential.

Remark 2.3. Given $p \geq 1$ we shall denote $p^* = pN/(N-p)$ if $p < N$, and $p^* = +\infty$ if $p \geq N$. We write $p' \geq 1$ the conjugate of p, $1/p + 1/p' = 1$.

We write

$$\mathrm{W}_V^{1,p}(\Omega) = (\mathrm{W}_0^{1,p}(\Omega), \|\cdot\|_{V,p}),$$

where

$$\|u\|_{V,p}^p = \|u\|_{\mathrm{W}_0^{1,p}(\Omega)}^p + \int_\Omega V(x) |u(x)|^p dx. \tag{2.5}$$

Remark 2.4. Let $1 \le p < N$ and $q \in [1, p^*[$. By Rellich-Kondrachov's theorem, [3, Th.9.16], the embedding $\mathrm{W}^{1,p}_V(\Omega) \subseteq \mathrm{L}^q(\Omega)$ is compact. Then, in particular, there exists $S_{p,q} > 0$ such that

$$\forall u \in \mathrm{W}^{1,p}_0(\Omega): \quad \|u\|_{\mathrm{L}^q(\Omega)} \le S_{p,q} \|u\|_{V,p} . \tag{2.6}$$

Remark 2.5. Let $1 \le p < N$ and $q \in [1, p^*]$. By [7, Sec.5.6.1, Th.3], there exists $C_{p,q} > 0$ such that

$$\forall u \in \mathrm{W}^{1,p}_0(\Omega): \quad \|u\|_{\mathrm{L}^q(\Omega)} \le C_{p,q} \|\nabla u\|_{\mathrm{L}^p(\Omega)} . \tag{2.7}$$

Also, for all $1 \le p \le +\infty$, we have Poincaré's inequality,

$$\forall u \in \mathrm{W}^{1,p}_0(\Omega): \quad \|u\|_{\mathrm{L}^p(\Omega)} \le C_p \|\nabla u\|_{\mathrm{L}^p(\Omega)} . \tag{2.8}$$

Remark 2.6. Let $1 \le q \le p \le +\infty$. Let's recall that by Hölder inequality,

$$\forall u \in \mathrm{L}^p(\Omega): \quad \|u\|_{\mathrm{L}^q(\Omega)} \le K_{p,q} \|u\|_{\mathrm{L}^p(\Omega)} , \tag{2.9}$$

where $K_{p,q} = |\Omega|^{(p-q)/pq}$. For future reference we denote $K_p = K_{p,2}$ if $p \ge 2$, i.e.,

$$\forall u \in \mathrm{L}^p(\Omega): \quad \|u\|_{\mathrm{L}^2(\Omega)} \le K_p \|u\|_{\mathrm{L}^p(\Omega)} . \tag{2.10}$$

Proposition 2.7. *Let $1 \le q \le p < +\infty$. Then*

$$\forall \psi \in \mathrm{W}^{1,p}_0(\Omega): \quad \|\psi\|_{V,q} \le \hat{C}_{p,q} \|\psi\|_{V,p}, \tag{2.11}$$

where $\hat{C}^q_{p,q} = K^q_{p,q}(1 + \|V\|_{\mathrm{L}^\infty(\Omega)} C^q_p)$. So, if $p \ge 2$, denoting $\hat{C}_p = \hat{C}_{p,2}$,

$$\forall \psi \in \mathrm{W}^{1,p}_0(\Omega): \quad \|\psi\|_{V,2} \le \hat{C}_p \|\psi\|_{V,p}. \tag{2.12}$$

Proof. Point (2.11) is obtained by a direct and simple computation using Hölder inequality and (2.8). □

2.3. The p-Sobolev-like cone

Let $p \ge 1$. For $T \in \mathscr{S}_1$, we write $\mathscr{B}^p_T = \{B \in \mathscr{B}_T / \ B \subseteq \mathrm{W}^{1,p}_V(\Omega)\}$.

Definition 2.8. The *p-Sobolev-like cone*, $\mathscr{W}^{1,p}$, is the set of operators $T \in \mathscr{S}_1$ for which there exists $B = \{\psi_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}^p_T$ such that the *pB-energetic value of T* is finite:

$$\langle\!\langle T \rangle\!\rangle_{p,B} = \sum_{i \in \mathbb{N}} |\nu_{i,T}| \ \|\psi_i\|^p_{V,p} < +\infty. \tag{2.13}$$

The value

$$\langle\!\langle T \rangle\!\rangle_p = \inf_{B \in \mathscr{B}^p_T} \langle\!\langle T \rangle\!\rangle_{p,B} \tag{2.14}$$

shall be referred to as the *p-energy* (or *p-total energy*) *of T*.

As it was already mentioned, the case $p = 2$ comes in a natural way for most of the applications to quantum mechanics and, moreover, has the advantage that the 2-energy is basis-independent, i.e., on $T \in \mathscr{W}^{1,2} = \mathscr{H}^1$ is unnecessary to take the infimum in (2.14). The term *cone* in Definition 2.8 is justified by point i) in the following result.

Proposition 2.9. *The following properties hold.*

i) *For every* $\alpha \in \mathbb{R}$, $\alpha \mathscr{W}^{1,p} \subseteq \mathscr{W}^{1,p}$ *and*

$$\forall T \in \mathscr{W}^{1,p}: \quad \langle\!\langle T \rangle\!\rangle_p = |\alpha| \langle\!\langle T \rangle\!\rangle_p.$$

ii) *Given* $\alpha \in \mathbb{R}$ *and* $T \in \mathscr{W}^{1,p}$,

$$\langle\!\langle \alpha T \rangle\!\rangle_p = 0 \quad \text{iff} \quad \alpha = 0 \vee T = 0.$$

iii) *Let* $T \in \mathscr{W}^{1,p}$, $M > 0$ *and* $\beta \in \mathrm{C}([-M, M])$ *such that* $\beta(0) = 0$ *and* $\tilde{\sigma}(T) \subseteq [-M, M]$. *Assume that there exist* $c_1 > 0$, $\alpha \geq 1$ *and* $t_0 \in]0, M]$ *such that*

$$\forall t \in [-t_0, t_0]: \quad |\beta(t)| \leq c_1 |t|^\alpha. \tag{2.15}$$

Then, for every $B \in \mathscr{B}_T^p$ *such that* $\langle\!\langle T \rangle\!\rangle_{p,B} < +\infty$, *there exists* $C_B > 0$ *such that*

$$\langle\!\langle \beta(T) \rangle\!\rangle_p \leq C_B \langle\!\langle T \rangle\!\rangle_{p,B}. \tag{2.16}$$

Consequently, $\beta(T) \in \mathscr{W}^{1,p}$.

Proof. Points i) and ii) follow from simple computations. Let's prove iii). Let $B = \{\eta_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}_T^p$ such that $\langle\!\langle T \rangle\!\rangle_{p,B} < +\infty$, and $t_1 = \min\{1, t_0\}$. Then $\#\{i \in \mathbb{N} \,/\, |\nu_{i,T}| > t_1\} < +\infty$ and we can choose $c_B > 0$ such that

$$|\beta(\nu_j)| \leq c_B |\nu_j|,$$

for each $j \in \{i \in \mathbb{N} \,/\, |\nu_i| > t_1\}$. Therefore, by the Spectral Theorem, [11, Th.VII.2], and (2.15), we get

$$\begin{aligned}
\langle\!\langle \beta(T) \rangle\!\rangle_{B,p} &= \sum_{i \in \mathbb{N}} |\beta(\nu_i)| \|\eta_i\|_{V,p}^p = \sum_{\substack{i \in \mathbb{N} \\ |\nu_i| \leq t_1}} |\beta(\nu_i)| \|\eta_i\|_{V,p}^p + \sum_{\substack{i \in \mathbb{N} \\ |\nu_i| > t_1}} |\beta(\nu_i)| \|\eta_i\|_{V,p}^p \\
&\leq c_1 \sum_{\substack{i \in \mathbb{N} \\ |\nu_i| \leq t_1}} |\nu_i|^\alpha \|\eta_i\|_{V,p}^p + c_B \sum_{\substack{i \in \mathbb{N} \\ |\nu_i| > t_1}} |\nu_i| \|\eta_i\|_{V,p}^p \\
&\leq C_B \sum_{i \in \mathbb{N}} |\nu_i| \|\eta_i\|_{V,p}^p = C_B \langle\!\langle T \rangle\!\rangle_{p,B},
\end{aligned}$$

where $C_B = \max\{c_B, c_1\}$. □

The following estimate of the trace norm in terms of its p-energy is a kind of Poincaré inequality at operators level. We use the constants provided in Remarks 2.4, 2.5 and 2.6.

Proposition 2.10. *Let* $p \geq 2$. *Then*

$$\forall T \in \mathscr{W}^{1,p}: \quad \|T\|_1 \leq K_p^p \cdot C_p \langle\!\langle T \rangle\!\rangle_p. \tag{2.17}$$

Proof. Let $T \in \mathscr{W}^{1,p}$ and $B = \{\eta_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}_T^p$ such that $\langle\!\langle T \rangle\!\rangle_{p,B} < +\infty$. Then, by (2.9), (2.8) and (2.5), we have, for $i \in \mathbb{N}$,

$$|\nu_{i,T}| = |\nu_{i,T}| \, \|\eta_i\|_{\mathrm{L}^2(\Omega)}^p \leq K_p^p |\nu_{i,T}| \, \|\eta_i\|_{\mathrm{L}^p(\Omega)}^p \leq K_p^p C_p |\nu_{i,T}| \, \|\eta_i\|_{V,p}^p,$$

whence, by summing over i,

$$\|T\|_1 = \sum_{i \in \mathbb{N}} |\nu_{i,T}| \leq \sum_{i \in \mathbb{N}} K_p^p C_p |\nu_{i,T}| \|\eta_i\|_{V,p}^p = K_p^p C_p \langle\!\langle T \rangle\!\rangle_{p,B},$$

which, by the arbitrariness of B and T, gives (2.17). □

In general, $1 \leq q \leq s < +\infty$ does not imply that $\mathscr{W}^{1,s} \subseteq \mathscr{W}^{1,q}$. However we can prove something of the kind, by using sets of the form

$$\mathscr{W}_b^{1,p} = \left\{ T \in \mathscr{W}^{1,p} \,/\, \exists B \in \mathscr{B}_T^p : B \text{ is bounded in } \|\cdot\|_{V,p} \right\}.$$

Proposition 2.11. *Let $1 \leq q \leq p < +\infty$. Then, $\mathscr{W}_b^{1,p} \subseteq \mathscr{W}^{1,q}$.*

Proof. Let $T \in \mathscr{W}_b^{1,p}$ and $B = \{\eta_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}_T^p$, bounded in the norm $\|\cdot\|_{V,p}$. Then there exists $\beta > 0$ such that $\|\eta_i\|_{V,p} \leq \beta$, for every $i \in \mathbb{N}$. By (2.14), (2.11) and (2.3) we have that

$$\begin{aligned} \langle\!\langle T \rangle\!\rangle_q &\leq \sum_{i \in \mathbb{N}} |\nu_{i,T}| \, \|\eta_i\|_{V,q}^q \leq \hat{C}_{p,q}^q \sum_{i \in \mathbb{N}} |\nu_{i,T}| \, \|\eta_i\|_{V,p}^q \\ &\leq (\beta \hat{C}_{p,q})^q \sum_{i \in \mathbb{N}} |\nu_{i,T}| = (\beta \hat{C}_{p,q})^q \|T\|_1 < +\infty, \end{aligned}$$

so that $T \in \mathscr{W}^{1,q}$. Since T was chosen arbitrarily, we are done. □

2.4. Imbalance of the p-energy

Now we shall introduce the concepts of p-kinetic and p-potential energy for operators that belong to the positive cone

$$\mathscr{W}_+^{1,p} = \{ T \in \mathscr{W}^{1,p} \,/\, T \geq 0 \}.$$

By Remark 2.1 we have $\|T\|_1 = \mathrm{Tr}\,(T) = \sum_{i \in \mathbb{N}} \nu_{i,T}$, for every $T \in \mathscr{W}_+^{1,p}$.

Let $T \in \mathscr{W}_+^{1,p}$ and $B = \{\eta_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}_T^p$. The *pB-kinetic energetic value* and the *pB-potential energetic value of T* are given, respectively, by

$$\mathscr{K}_{p,B}(T) = \sum_{i \in \mathbb{N}} \nu_{i,T} \int_\Omega |\nabla \eta_i(x)|^p dx, \quad \mathscr{P}_{p,B}(T) = \sum_{i \in \mathbb{N}} \nu_{i,T} \int_\Omega V(x) |\eta_i(x)|^p dx.$$

Then, as it was done in (2.14), the *p-kinetic energy* and the *p-potential energy of T* are given, respectively, by

$$\mathscr{K}_p(T) = \inf_{B \in \mathscr{B}_T^p} \mathscr{K}_{p,B}(T), \quad \mathscr{P}_p(T) = \inf_{B \in \mathscr{B}_T^p} \mathscr{P}_{p,B}(T). \tag{2.18}$$

Remark 2.12. Let $T \in \mathscr{W}_+^{1,p}$. Let's assume that $B = \{\eta_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}_T^p$ is an eigenbasis of T and that $\langle\!\langle T \rangle\!\rangle_{p,B} < +\infty$. Even though $-\Delta_p$ is not a linear operator, by using (2.1) and integration by parts, we formally have that

$$\begin{aligned} \mathrm{Tr}_B[-\Delta_p T] &= \sum_{i \in \mathbb{N}} (\eta_i, -\Delta_p T \eta_i)_{\mathrm{L}^2(\Omega)} = \sum_{i \in \mathbb{N}} \nu_{i,T} \int_\Omega \eta_i(x) \cdot [-\Delta_p \eta_i(x)] dx \\ &= \sum_{i \in \mathbb{N}} \nu_{i,T} \int_\Omega |\nabla \eta_i(x)|^p dx = \mathscr{K}_{p,B}(T). \end{aligned} \tag{2.19}$$

Let's assume now that $p = 2$. Then (see e.g. [13]) the following equality is valid and basis-independent

$$\mathscr{P}_{2,V}(T) = \mathrm{Tr}\,(VT) = \int_\Omega V(x) \rho_T(x) dx. \tag{2.20}$$

Here V is interpreted both as a function and as a multiplication operator on $\mathrm{L}^2(\Omega)$ and we have used (2.4). Moreover, if $B \in \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega)$, as it was assumed e.g. in [8], then the computations (2.19) and (2.14) become valid and basis-independent:

$$\mathscr{K}_2(T) = \mathrm{Tr}(-\Delta T) \quad \text{and} \quad \langle\!\langle T \rangle\!\rangle_2 = \mathrm{Tr}((-\Delta + V)T).$$

From (2.14) and (2.18) the p-total energy is imbalanced:

$$\forall T \in \mathscr{W}_+^{1,p}: \quad \langle\!\langle T \rangle\!\rangle_p \geq \mathscr{K}_p(T) + \mathscr{P}_p(T). \tag{2.21}$$

For the reasons mentioned in Remark 2.12, only the 2-energy is preserved: $\langle\!\langle T \rangle\!\rangle_2 = \mathscr{K}_2(T) + \mathscr{P}_2(T)$, for every $T \in \mathscr{W}_+^{1,2}$.

2.5. Regularity of the density function

In this section we shall prove that the density function associated to an element of $\mathscr{W}_+^{1,p}$, given by (2.4), belongs to a range of Lebesgue and Sobolev spaces.

Lemma 2.13. *Let $N \geq 3$, $2 \leq p < N$ and $T \in \mathscr{W}_+^{1,p}$. Let's denote $I_1 = [p/2, pN/(2N-2)]$ and $I_2 = [1, pN/(2N-2-p)]$.*

i) *For every $r \in I_1$ and every $B \in \mathscr{B}_T^p$ such that $\langle\!\langle T \rangle\!\rangle_{B,p} < +\infty$, there exists $Z = Z(B,p,N,r) > 0$ such that*

$$\|\nabla \rho_L\|_{\mathrm{L}^r(\Omega)} \leq Z \langle\!\langle T \rangle\!\rangle_{p,B}^{2(r-1)/p+1/r},$$

where $L = T^{2r/p}$.

ii) *For every $r \in I_1$ and every $s \in I_2$, $\rho_L \in \mathrm{W}^{1,r}(\Omega) \cap \mathrm{L}^s(\Omega)$.*

Proof. Let's assume that $T \neq 0$, $r \in I_1$ and that $B = \{\eta_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}_T^p$ is such that $\langle\!\langle T \rangle\!\rangle_{p,B} < +\infty$. Since $2r/p \geq 1$, it follows, by Proposition 2.9, that $L = T^{2r/p} \in \mathscr{W}_+^{1,p}$. By the Spectral Theorem, $\nu_{i,L} = \nu_{i,T}^{2r/p}$, for each $i \in \mathbb{N}$.

Since $r \leq pN/2(N-1)$ it follows that

$$P = \frac{p}{r} \geq 2 - \frac{2}{N} \geq 1 \quad \text{and} \quad P' = \frac{p}{p-r}.$$

Then, by the convexity of $\mathbb{R} \ni y \mapsto |y|^r \in \mathbb{R}$, Hölder inequality, (2.16), (2.17) and (2.7) with $q = pr/(p-r)$, we get

$$\begin{aligned}\int_\Omega |\nabla \rho_L(x)|^r dx &\leq 2^r \int_\Omega \left(\sum_{i\in\mathbb{N}} |\nu_{i,L} \eta_i(x) \nabla \eta_i(x)| \right)^r dx \\ &\leq \left(2 \sum_{j\in\mathbb{N}} \nu_{j,L} \right)^r \int_\Omega \left[\sum_{i\in\mathbb{N}} \left(\frac{\nu_{i,L}}{\sum_{j\in\mathbb{N}} \nu_{j,L}} \right) |\eta_i(x)| |\nabla \eta_i(x)| \right]^r dx\end{aligned}$$

$$\le 2^r \|L\|_1^r \int_\Omega \sum_{i\in\mathbb{N}} \left(\frac{\nu_{i,L}}{\|L\|_1}\right) |\eta_i(x)|^r |\nabla\eta_i(x)|^r dx$$

$$\le 2^r \|L\|_1^{r-1} \sum_{i\in\mathbb{N}} \nu_{i,L} \left(\int_\Omega (|\nabla\eta_i|^r)^P dx\right)^{1/P} \left(\int_\Omega (|\eta_i|^r)^{P'} dx\right)^{1/P'}$$

$$\le 2^r \|T\|_1^{2r(r-1)/p} \sum_{i\in\mathbb{N}} \nu_{i,L} \left(\int_\Omega |\nabla\eta_i|^p dx\right)^{r/p} \left(\int_\Omega |\eta_i|^{pr/(p-r)} dx\right)^{(p-r)/p}$$

$$\le Z_1 \langle\!\langle T\rangle\!\rangle_p^{2r(r-1)/p} \sum_{i\in\mathbb{N}} \nu_{i,L} \left(\int_\Omega |\nabla\eta_i|^p dx\right)^{2r/p}$$

$$= Z_1 \langle\!\langle T\rangle\!\rangle_p^{2r(r-1)/p} \sum_{i\in\mathbb{N}} \left(\nu_{i,T} \int_\Omega |\nabla\eta_i|^p dx\right)^{2r/p}, \tag{2.22}$$

where $Z_1 = 2^r K_p^{2r(r-1)} C_p^{2r(r-1)/p} C_{p,q}^r$. Since $(\nu_{i,T} \|\eta_i\|^p_{\mathrm{W}_0^{1,p}(\Omega)})_{i\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded, we take $A = A(B) = \sup_{i\in\mathbb{N}} \nu_{i,T} \|\eta_i\|^p_{\mathrm{W}_0^{1,p}(\Omega)} < +\infty$, so that

$$(A^{-1} \nu_{i,T} \|\eta_i\|^p_{\mathrm{W}_0^{1,p}(\Omega)})^{2r/p} \le A^{-1} \nu_{i,T} \|\eta_i\|^p_{\mathrm{W}_0^{1,p}(\Omega)}$$

as $2r/p \ge 1$. Therefore, from (2.22) we get

$$\int_\Omega |\nabla\rho_L(x)|^r dx \le Z_1 A^{2r/p-1} \langle\!\langle T\rangle\!\rangle_p^{2r(r-1)/p} \sum_{i\in\mathbb{N}} \nu_{i,T} \int_\Omega |\nabla\eta_i|^p dx$$

$$= Z_1 A^{2r/p-1} \langle\!\langle T\rangle\!\rangle_{p,B}^{2r(r-1)/p+1},$$

whence

$$\|\nabla\rho_{T^{2r/p}}\|_{\mathrm{L}^r(\Omega)} \le Z_1^{1/r} A^{2/p-1/r} \langle\!\langle T\rangle\!\rangle_{p,B}^{2(r-1)/p+1/r},$$

i.e., point i) with $Z = Z_1^{1/r} A^{2r/p-1}$.

Let's prove point ii). Observe that we already have that $\rho_L \in \mathrm{L}^1(\Omega) \cap \mathrm{W}^{1,r}(\Omega)$, for every $r \in I_1$. Now, by choosing $\tilde{P} = pN/(N-2) \ge 1$ and $\tilde{P}^* = pN/(2N-2-p)$ in Remark 2.5, it follows that $\rho_L \in \mathrm{L}^{\tilde{P}^*}(\Omega)$ and we conclude by interpolation in Lebesgue spaces, [3, pp.93]. □

Theorem 2.14. *Let $N \ge 3$, $2 \le p < N$ and $T \in \mathscr{W}_+^{1,p}$. Then*

$$\|\nabla\rho_T\|_{\mathrm{L}^{p/2}(\Omega)} \le \tilde{Z} \langle\!\langle T\rangle\!\rangle_{p,B}$$

with $\tilde{Z} = \tilde{Z}(B,p,N) > 0$. Consequently, $\rho_T \in \mathrm{W}^{1,\gamma}(\Omega) \cap \mathrm{L}^s(\Omega)$, for every $\gamma \in [1, p/2]$ and every $s \in [1, pN/(2N-2-p)]$.

Proof. Simply take $r = p/2$ in Lemma 2.13 and apply Hölder inequality. □

3. Free energy functionals

3.1. Casimir class. Free energy and entropy functionals.

As a first step to introduce free enery functionals we shall present the concept of entropy of a positive nuclear operator. Recall that the Legendre-Fenchel transform of $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $\varphi \not\equiv +\infty$, is given by

$$\varphi^*(y) = \sup_{\lambda \in \mathbb{R}} [y\lambda - \varphi(\lambda)], \quad y \in \mathbb{R}. \tag{3.1}$$

We say that $\beta : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is an *entropy seed* iff it's convex and such that $\beta(0) = 0$. Given $T \in \mathscr{S}_{1,+} = \{L \in \mathscr{S}_1 \,/\, L \geq 0\}$, the value

$$\mathcal{S}_\beta(T) = \operatorname{Tr}\left(\beta(T)\right),$$

is called the *β-entropy of T* provided $\mathcal{S}_\beta(T) \in]-\infty, +\infty]$. We say that the entropy seed β is generated by a function F if

$$\beta(y) = F^*(-y), \quad y \in \mathbb{R}.$$

Let $p \geq 1$ and consider an entropy seed β. The *βp-free energy functional*, $\mathscr{F}_{\beta,p} : \mathscr{W}_+^{1,p} \to \mathbb{R} \cup \{+\infty\}$, is given by

$$\mathscr{F}_{\beta,p}(T) = \mathcal{S}_\beta(T) + \langle\!\langle T \rangle\!\rangle_p. \tag{3.2}$$

Example 1. Let $\gamma > N/p$ so that $m = \gamma/(\gamma+1) \in]N/(N+p), 1[$. The function $\beta_m : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, given by

$$\beta_m(s) = \begin{cases} -(1-m)^{m-1} m^{-m} s^m & , \quad \text{if } s \geq 0, \\ +\infty & , \quad \text{if } s < 0, \end{cases}$$

is an entropy seed generated by the function $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, defined by

$$F_\gamma(s) = \begin{cases} s^{-\gamma} & , \quad \text{if } s \geq 0, \\ +\infty & , \quad \text{if } s < 0. \end{cases}$$

A nice class of functions that generate entropy seeds is the Casimir class that we are going to introduce. We shall use the following kind of assumptions for the operators $-\alpha\Delta + V$.

($\mathrm{G}_{V,\alpha}$) α is positive and the operator $-\alpha\Delta + V$ with Dirichlet boundary conditions has a sequence of eigenelements $(\lambda_{V,i}^{(\alpha)}, \phi_{V,i}^{(\alpha)})_{i \in \mathbb{N}} \subseteq \mathbb{R} \times \mathrm{H}_0^1(\Omega)$ such that $\{\phi_{V,i}^{(\alpha)} \,/\, i \in \mathbb{N}\}$ is a Hilbert basis of $\mathrm{L}^2(\Omega)$ and $\lambda_{V,i}^{(\alpha)} \to +\infty$, as $i \to +\infty$. We denote $\lambda_{V,i} = \lambda_{V,i}^{(1)}$ and $\phi_{V,i} = \phi_{V,i}^{(1)}$, $i \in \mathbb{N}$.

Remark 3.1. Condition ($\mathrm{G}_{0,1}$) holds by [3, Th. 9.31]. This corresponds to the classical eigenvalue problem of the Laplacian operator $-\Delta$. Moreover, since in our context $0 \leq V \in \mathrm{L}^\infty(\Omega)$, we have $\lambda_{0,i} \leq \lambda_{V,i}$, for every $i \in \mathbb{N}$, and, therefore, ($\mathrm{G}_{V,1}$) also holds.

Definition 3.2. Assume ($\mathrm{G}_{V,\alpha}$) and $p \geq 2$. A function $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ belongs to the Casimir class $\mathscr{C}_{p,V}^\alpha$ if it is convex, non-increasing on $]0, +\infty[$ and

$$\operatorname{Tr}\left(F\left(\hat{C}^{-1}(-\Delta + V)\right)^{p/2}\right) = \sum_{i \in \mathbb{N}} F([\hat{C}^{-1}\lambda_{V,i}^{(\alpha)}]^{p/2}) < +\infty, \tag{3.3}$$

where $\hat{C} = \hat{C}_p^p$ comes from (2.12). We write $\mathscr{C}_{V,p} = \mathscr{C}_{p,V}^1$.

Example 2. Let $p \geq 2$ and $\gamma > N/p$. The function F_γ defined in Example 1 belongs to $\mathscr{C}_{p,0} \cap \mathscr{C}_{p,V}$. In fact, by [10, Th.1.3.1], there are $c(\Omega), C(\Omega) > 0$ such that $c(\Omega) i^{2/N} \leq \lambda_{0,i} \leq C(\Omega) i^{2/N}$, for each $i \in \mathbb{N}$, and so, by Remark 3.1, we get

$$\sum_{i\in\mathbb{N}} (\hat{C}^{-1}\lambda_{V,i})^{-\gamma p/2} \leq \sum_{i\in\mathbb{N}} (\hat{C}^{-1}\lambda_{0,i})^{-\gamma p/2} < +\infty.$$

Example 3. A sufficient condition for (3.3) to hold is that for some $q > N/p$, $(1/\lambda^{(\alpha)}_{V,i})_{i\in\mathbb{N}} \subseteq l^q(\mathbb{R})$ and, for some $M' > 0$,

$$\forall y \in]M', +\infty[: \quad F(y) \leq M'|y|^{-2q/p}.$$

This is an extension of the situation in Example 2, where it was assumed $\alpha = 1$.

3.2. Lower bounds for basic free energy functionals

The following theorem gives a lower bound for $\mathscr{F}_{p,\beta}$. This result is very important as it allows us to prove some Gagliardo-Nirenberg type inequalities in the context of the cone $\mathscr{W}^{1,p}_+$ and, therefore, it's a building block to prove our main result, Theorem 1.1.

Remark 3.3 (Important). Keep in mind that by saying that a function F belongs to the class $\mathscr{C}^\alpha_{p,V}$ we are immediately implying that condition $(\mathrm{G}_{V,\alpha})$ holds, $p \geq 2$ and $\alpha > 0$.

Theorem 3.4. *Let β be an entropy seed generated by $F \in \mathscr{C}_{V,p}$. Then*

$$\forall T \in \mathscr{W}^{1,p}_+ : \quad \mathscr{F}_{\beta,p}(T) \geq -\mathrm{Tr}\left(F\left(\left(\frac{-\Delta + V}{\hat{C}} \right)^{p/2} \right) \right).$$

Proof. Let $T \in \mathscr{W}^{1,p}_+$ and $B = \{\eta_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}^p_T$. Let $i \in \mathbb{N}$. We have that

$$\eta_i = \sum_{j\in\mathbb{N}} \mu_{ij}\phi_{V,j}, \quad \sum_{j\in\mathbb{N}} \mu_{ij}^2 = 1,$$

where $\mu_{ij} = (\phi_{V,j}, \eta_i)$. Since F is non-increasing on $]0, +\infty[$ it follows from (2.12) that

$$F(\|\eta_i\|^p_{V,p}) \leq F(\hat{C}^{-1}\|\eta_i\|^p_{V,2}). \tag{3.4}$$

By the convexity of F and $\mathbb{R} \ni y \mapsto |y|^{p/2} \in \mathbb{R}$, it follows that

$$F(\hat{C}^{-1}\|\eta_i\|^p_{V,2}) = F\left(\left(\mu^2_{i,j}\hat{C}^{-1}\lambda_{V,j} \right)^{p/2} \right) \leq \sum_{j\in\mathbb{N}} \mu^2_{ij} F\left(\left(\hat{C}^{-1}\lambda_{V,j} \right)^{p/2} \right). \tag{3.5}$$

Then, since the Spectral Theorem provides

$$F\left((\hat{C}^{-1}(-\Delta + V))^{p/2} \right) \hat{\phi}_{V,j} = F\left(\left(\hat{C}^{-1}\hat{\lambda}_{V,j} \right)^{p/2} \right) \hat{\phi}_{V,j},$$

for each $j \in \mathbb{N}$, we get

$$\begin{aligned}\sum_{j\in\mathbb{N}} \mu_{ij}^2 F\left(\left(\hat{C}^{-1}\lambda_{V,j}\right)^{p/2}\right) &= \sum_{j\in\mathbb{N}} \mu_{ij}^2 F\left(\left(\hat{C}^{-1}\lambda_{V,j}\right)^{p/2}\right)(\phi_{V,j},\phi_{V,j}) \\ &= \left(\sum_{j\in\mathbb{N}} \mu_{ij}\phi_{V,j}, \sum_{j\in\mathbb{N}} \mu_{ij} F\left(\left(\hat{C}^{-1}\lambda_{V,j}\right)^{p/2}\right)\phi_{V,j}\right) \\ &= \left(\eta_i, F\left(\left(\frac{-\Delta+V}{\hat{C}}\right)^{p/2}\right)\eta_i\right). \end{aligned} \tag{3.6}$$

Therefore, by (3.4), (3.5), (3.6) and (2.1), adding over $i \in \mathbb{N}$ yields

$$\sum_{i\in\mathbb{N}} F(\|\eta_i\|_{V,p}^p) \le \operatorname{Tr}\left(F\left(\left(\frac{-\Delta+V}{\hat{C}}\right)^{p/2}\right)\right). \tag{3.7}$$

Since the entropy seed β is generated by F, we have that $\beta(\nu)+\nu y \ge -F(y)$, for all $\nu, y \in \mathbb{R}$. Therefore, using (3.7) with $\nu = \nu_{i,T}$ and $y = \|\eta_i\|_{V,p}^p$, and adding over $i \in \mathbb{N}$, we get

$$\sum_{i\in\mathbb{N}} \beta(\nu_{i,T}) + \sum_{i\in\mathbb{N}} \nu_{i,T}\|\eta_i\|_{V,p}^p \ge -\sum_{i\in\mathbb{N}} F(\|\eta_i\|_{V,p}^p) \ge -\operatorname{Tr}\left(F\left(\left(\frac{-\Delta+V}{\hat{C}}\right)^{p/2}\right)\right).$$

We conclude by (3.2) and the arbitrariness of B and T. □

Proposition 3.5. *Let β be an entropy seed generated by $F \in \mathscr{C}_{p,V}^{(\alpha)}$. Then, for every $T \in \mathscr{W}_+^{1,p}$,*

$$\mathcal{S}_\beta(T) + (\alpha+1)^{p/2}\langle\!\langle T\rangle\!\rangle_p \ge -\operatorname{Tr}\left(F\left(\left(\frac{-\alpha\Delta+V}{\hat{C}}\right)^{p/2}\right)\right). \tag{3.8}$$

Proof. The proof follows exactly the scheme of the proof of Theorem 3.4. We just have to observe that the elements of the Hilbert basis $\{\phi_{V,j}^{(\alpha)} \,/\, j \in \mathbb{N}\}$ verify

$$F\left(\left(\frac{-\alpha\Delta+V}{\hat{C}}\right)^{p/2}\right)\hat{\phi}_{V,j}^{\alpha} = F\left(\left(\hat{C}^{-1}\hat{\lambda}_{V,j}^{(\alpha)}\right)^{p/2}\right)\hat{\phi}_{V,j}^{(\alpha)}.$$

□

Remark 3.6. Let β be an entropy seed generated by $F \in \mathscr{C}_{p,0}^{(\alpha)}$, $V \equiv 0$. Then,

$$\mathcal{S}_\beta(T) + \alpha\mathscr{K}_p(T) \ge -\operatorname{Tr}\left(F\left(\left(\frac{-\alpha\Delta}{\hat{C}}\right)^{p/2}\right)\right). \tag{3.9}$$

3.3. Gagliardo-Niremberg type inequalities

Let $p \ge 2$. When we say that V is bounded away from zero, $\inf\{V(x) \,/\, x \in \Omega\} > 0$, we mean that there exists $\gamma_V^{(p)} > 0$ such that

$$V(x) \ge K_p^p \gamma_V^{(p)} > 0, \quad x \in \Omega.$$

In this case, for $\lambda \le \gamma_V^{(p)}$, we define a generalized free energy functional $\mathscr{F}_{\beta,p}^{(\lambda)}$: $\mathscr{W}_+^{1,p} \to \mathbb{R}\cup\{+\infty\}$ by

$$\mathscr{F}_{\beta,p}^{(\lambda)}(T) = \mathscr{F}_{\beta,p}(T) - \lambda\|T\|_1 = \mathcal{S}_\beta(T) + \langle\!\langle T\rangle\!\rangle_p - \lambda\|T\|_1.$$

The next result proves, given some conditions, that $\mathscr{F}^{(\lambda)}_{\beta,p}$ is bounded from below, i.e., the negative term $-\|T\|_1$ can be controled.

Theorem 3.7. *Assume that V is bounded away from zero and that $\lambda \leq \gamma_V^{(p)}$. Let β be an entropy seed generated by $F \in \mathscr{C}^{\epsilon/2}_{p,0}$ for some $\epsilon \in]0,1]$. Then, for every $T \in \mathscr{W}^{1,p}_+$,*

$$\mathscr{F}^{(\lambda)}_{\beta,p}(T) \geq -\mathrm{Tr}\left(F\left(\left(\frac{-\epsilon}{2\hat{C}}\Delta\right)^{p/2}\right)\right) + \frac{\epsilon}{2}\mathscr{K}_p(T). \tag{3.10}$$

Proof. Let $T \in \mathscr{W}^{1,p}_+$. Using (2.21) we get

$$\begin{aligned}\mathscr{F}^{(\lambda)}_{\beta,p}(T) &\geq \mathscr{K}_p(T) + \mathscr{P}_p(T) + \mathcal{S}_\beta(T) - \lambda\|T\|_1 \pm \epsilon\mathscr{K}_p(T)\\ &= \left[\frac{\epsilon}{2}\mathscr{K}_p(T) + \mathcal{S}_\beta(T)\right] + \frac{\epsilon}{2}\mathscr{K}_p(T) + (1-\epsilon)\mathscr{K}_p(T) + \\ &+ \mathscr{P}_p(T) - \lambda\|T\|_1.\end{aligned}$$

Since $F \in \mathscr{C}^{\epsilon/2}_{p,0}$, by Proposition 3.5, we have that

$$\frac{\epsilon}{2}\mathscr{K}_p(T) + \mathcal{S}_\beta(T) \geq -\mathrm{Tr}\left(F\left(\left(\frac{-\epsilon}{2\hat{C}}\Delta\right)^{p/2}\right)\right),$$

whence,

$$\begin{aligned}\mathscr{F}^{(\lambda)}_{\beta,p}(T) \geq -\mathrm{Tr}\left(F\left(\left(\frac{-\epsilon}{2\hat{C}}\Delta\right)^{p/2}\right)\right) + \frac{\epsilon}{2}\mathscr{K}_p(T)\\ + (1-\epsilon)\mathscr{K}_p(T) + \mathscr{P}_p(T) - \lambda\|T\|_1.\end{aligned} \tag{3.11}$$

Now we claim that

$$(1-\epsilon)\mathscr{K}_p(T) + \mathscr{P}_p(T) - \lambda\|T\|_1 \geq 0, \tag{3.12}$$

which, together with (3.11), imply (3.10).

If $\lambda \leq 0$, (3.12) is immediate. So let's assume that $\lambda > 0$. Let $B = \{\eta_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}^p_T$. Then, by using (2.10) and $\|\eta_i\|_{\mathrm{L}^2(\Omega)} = 1$, for each $i \in \mathbb{N}$, we have that

$$\begin{aligned}\mathscr{P}_{p,B}(T) - \lambda\|T\|_1 &= \sum_{i\in\mathbb{N}}\nu_{i,T}\int_\Omega V(x)|\eta_i(x)|^p dx - \lambda\sum_{i\in\mathbb{N}}\nu_{i,T}\\ &= \sum_{i\in\mathbb{N}}\nu_{i,T}\left(\int_\Omega V(x)|\eta_i(x)|^p dx - \frac{\lambda}{\|\eta_i\|^p_{\mathrm{L}^p(\Omega)}}\int_\Omega|\eta_i(x)|^p dx\right)\\ &\geq \sum_{i\in\mathbb{N}}\nu_{i,T}\left(\int_\Omega\left(V(x) - \lambda K_p^p\right)|\eta_i(x)|^p dx\right) \geq 0,\end{aligned}$$

which, by the arbitrariness of B, implies (3.12). ◻

A useful consequence of Theorem 3.5 is the following corollary which will be used in Section 5 to minimize free energy functionals.

Corollary 3.8. *Assume that V is bounded away from zero and that $\lambda \leq \gamma_V^{(p)}$. Let β be an entropy seed generated by $F \in \mathscr{C}_{p,0}^{\epsilon/2}$ for some $\epsilon \in]0,1]$. Let's assume that $(T_\sigma)_{\sigma\in\Lambda} \subseteq \mathscr{W}_+^{1,p}$ is such that $(\mathscr{F}_{\beta,p}^{(\lambda)}(T_\sigma))_{\sigma\in\Lambda} \subseteq \mathbb{R}$ is bounded. Then the families $(\mathscr{K}_p(T_\sigma))_{\sigma\in\Lambda}$, $(\mathcal{S}_\beta(T_\sigma))_{\sigma\in\Lambda}$, $(\|T_\sigma\|_1)_{\sigma\in\Lambda}$, $(\langle\!\langle T_\sigma \rangle\!\rangle_p)_{\sigma\in\Lambda}$ and $(\mathscr{P}_p(T_\sigma))_{\sigma\in\Lambda}$ are also bounded in $\mathbb{R}$.*

Proof. By taking advantage of the boundedness of $(\mathscr{F}_{\beta,p}^{(\lambda)}(T_\sigma))_{\sigma\in\Lambda}$ and of the estimates appearing in the proof of Theorem 3.5 we first prove that there exists $A > 0$ such that

$$\operatorname{Tr}\left(F\left(\left(\frac{-\epsilon}{2\hat{C}}\Delta\right)^{p/2}\right)\right) \leq A.$$

With this it's proved the boundedness from above of $(\mathscr{K}_p(T_\sigma))_{\sigma\in\Lambda}$ which, with help of (3.12), allows to prove the same for $(\mathcal{S}_\beta(T_\sigma))_{\sigma\in\Lambda}$. Then with help of (2.10) and (2.8) it's proved that

$$\forall\sigma \in \Lambda : \quad \|T_\sigma\|_1 \leq C_p^p K_p^p \mathscr{K}_p(T_\sigma),$$

so that $(\|T_\sigma\|_1)_{\sigma\in\Lambda}$ is also bounded. This easily gives the boundedness of $(\langle\!\langle T_\sigma \rangle\!\rangle_p)_{\sigma\in\Lambda}$ and $(\mathscr{P}_p(T_\sigma))_{\sigma\in\Lambda}$.

□

Now that we have proved boundedness from below for the kind of free energy functionals under consideration, we shall obtain some Gagliardo-Nirenberg type inequalities for operators. The following result is an extension of [6, Th.15] and [5, Th.3.2] to the Sobolev-like cone $\mathscr{W}_+^{1,p}$, for $p \geq 2$.

Remark 3.9. Let $p \geq 1$, $T \in \mathscr{W}_+^{1,p}$ and $B = \{\eta_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}_T^p$. The *pB-density function associated to T* is formally given by

$$\rho_{p,B}(x) = \sum_{i\in\mathbb{N}} \nu_{i,T} |\eta_{i,T}(x)|^p, \qquad \text{for a.e. } x \in \Omega.$$

Theorem 3.10. *Let β be an entropy seed generated by $F \in \mathscr{C}_{p,V}$. Let's assume that the functions τ, G are such that $\tau(s) = -(-G)^*(s)$, $s \in \mathbb{R}$, and*

$$\operatorname{Tr}\left(F\left(\left(\frac{-\Delta + V}{\hat{C}}\right)^{p/2}\right)\right) \leq \int_\Omega G(V(x))dx. \tag{3.13}$$

Then, for every $T \in \mathscr{W}_+^{1,p}$,

$$\mathcal{S}_\beta(T) + \mathscr{K}_p(T) \geq \inf_{B\in\mathscr{B}_T^p} \int_\Omega \tau(\rho_{p,B}(x))dx.$$

Proof. Let $T \in \mathscr{W}_+^{1,p}$ and $B = \{\eta_i \,/\, i \in \mathbb{N}\} \in \mathscr{B}_T^p$. By (3.1), for $s, \lambda \in \mathbb{R}$,

$$-\lambda s - G(\lambda) \geq -(-G)^*(s) = \tau(s).$$

Therefore, by choosing $\lambda = V(x)$ and $s = \rho_{p,B}(x)$, we get, by (3.13) and Theorem 3.4, that

$$\begin{aligned}\mathcal{S}_\beta(T) + \mathscr{K}_{p,B}(T) &\geq -\mathscr{P}_{p,B}(T) - \int_\Omega G(V(x))dx \\ &= \int_\Omega \left[-V(x)\rho_{p,B}(x) - G(V(x))\right] dx \\ &\geq \int_\Omega \tau\left(\rho_{p,B}(x)\right) dx \geq \inf_{D\in\mathscr{B}_T^p} \int_\Omega \tau\left(\rho_{p,D}(x)\right) dx.\end{aligned}$$

We conclude by the arbitrariness of T and B. □

Example 4. Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_0^{+\infty} \frac{g(t)}{t}(1 + t^{-N/2})dt < +\infty.$$

Moreover, consider the convex non-increasing functions $F_g, G_g : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ given by

$$F_g(s) = \int_0^{+\infty} e^{-ts} g(t)\frac{dt}{t}, \qquad G_g(s) = \int_0^{+\infty} e^{-ts}(4\pi t)^{-N/2} g(t)\frac{dt}{t}.$$

For the case $p = 2$, it was proved in [6] that F_g, G_g satisfy the conditions of Theorem 3.10.

Example 5. Let's retake Example 1 for the case of $p = 2$. Then we consider $\gamma > N/2$ so that $m = \gamma/(\gamma+1) \in\,]N/(N+2), 1[$ and the entropy seed $\beta_m : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, given by $\beta_m(s) = -(1-m)^{m-1} m^{-m} s^m$ if $s \geq 0$, and $\beta_m(s) = +\infty$ if $s < 0$. Then, as in [6, Ex.2], there exists a constant $\mu = \mu(\gamma, N, \Omega) > 0$ such that the following interpolation inequality holds

$$\forall T \in \mathscr{W}_+^{1,p} : \quad \mathscr{K}_2(T) + \mathcal{S}_{\beta_m}(T) \geq -\mu \int_\Omega \rho_T^q(x)dx,$$

where $q = (2\gamma - N)/[2(\gamma+1) - N] \in]0, 1[$.

4. Compactness

In this section we prove our main result, Theorem 1.1. Given $p \geq 2$, it states that the embedding $\mathscr{W}^{1,p} \subseteq \mathscr{S}_1$ is compact.

Along this section we shall assume that a given sequence $(T_n)_{n\in\mathbb{N}} \subseteq \mathscr{W}_+^{1,p}$ is p-energetically bounded:

$$K_\infty = \sup_{n\in\mathbb{N}} \langle\!\langle T_n \rangle\!\rangle_p < +\infty. \tag{4.1}$$

Then we shall prove that there are a subsequence $(T_{n_k})_{k\in\mathbb{N}}$ and an operator $T \in \mathscr{W}_+^{1,p}$ such that $\|T - T_{n_k}\|_1 \longrightarrow 0$, as $k \longrightarrow +\infty$.

Remark 4.1. It's enough to deal with positive operators as any $T \in \mathscr{W}^{1,p}$ can be written as the diference of two operators $T_1, T_2 \in \mathscr{W}_+^{1,p}$:

$$\begin{aligned} T\phi &= \sum_{i\in\mathbb{N}} \nu_{i,T}(\phi,\psi_{i,T})\,\psi_{i,T} \\ &= \sum_{\substack{i\in\mathbb{N}\\ \nu_i\geq 0}} \nu_{i,T}(\phi,\psi_{i,T})\,\psi_{i,T} - \sum_{\substack{i\in\mathbb{N}\\ \nu_i< 0}} (-\nu_{i,T})(\phi,\psi_{i,T})\,\psi_{i,T} = T_1\phi - T_2\phi, \end{aligned}$$

where $\phi \in H$ and $\{\psi_{i,T} \,/\, i \in \mathbb{N}\}$ is an eigenbasis of T.

Let's introduce some necessary notation. For each $n \in \mathbb{N}$, we shall denote by $(\nu_i^{(n)})_{i\in\mathbb{N}}$ the sequence of eigenvalues of T_n. We shall assume that, for each $n \in \mathbb{N}$, $B_n = \{\eta_i^{(n)} \,/\, i \in \mathbb{N}\} \in \mathscr{B}^p_{T_n}$ verifies $\langle\!\langle T_n \rangle\!\rangle_{p,B_n} < +\infty$.

To prove Theorem 1.1 we have some technical steps.

Lemma 4.2. *Assume* (4.1)*. Then*

i) $(\|T_n\|_1)_{n\in\mathbb{N}}$ *is bounded and*

$$\sup_{n\in\mathbb{N}} \sum_{i\in\mathbb{N}} (\nu_i^{(n)})^m < +\infty,$$

where $m = \gamma/(\gamma+1) \in\,]N/(N+p), 1[$*, with* $\gamma > N/p$.

ii) *Up to a subsequence, for each* $i \in \mathbb{N}$*,* $\nu_i^{(n)} \to \overline{\nu}_i \in \mathbb{R}^+ \cup \{0\}$*, as* $n \to +\infty$.

Proof. Let $n \in \mathbb{N}$. By Proposition 2.10 we get $\|T_n\|_1 = \sum_{i\in\mathbb{N}} \nu_i^{(n)} \leq K_p^p C_p K_\infty < +\infty$, which, by the arbitrariness of n, immediately provides point ii). Let's consider the functions β_m and F_γ as in Example 1. By Theorem 3.4, we have, for $V \equiv 0$, that $\mathscr{F}_{\beta_m,p}(T_n) \geq \operatorname{Tr}\left(F_\gamma((-\hat{C}^{-1}\Delta)^{p/2})\right)$ so that, by (4.1),

$$(1-m)^{m-1} m^{-m} \sum_{i\in\mathbb{N}} (\nu_i^{(n)})^m \leq \sum_{i\in\mathbb{N}} F_\gamma\left(\left(\hat{C}^{-1}\lambda_{0,i}\right)^{p/2}\right) + \mathscr{K}_p\,(T_n) < +\infty.$$

Since n was arbitrary, this implies point i). □

Remark 4.3. To facilitate several computations, from now on, we shall assume that $\overline{\nu}_i \neq 0$, for every $i \in \mathbb{N}$.

Lemma 4.4. *Let us assume* (4.1)*. Then, there exists* $\overline{B} = \{\overline{\eta}_i \,/\, i \in \mathbb{N}\} \subseteq \mathrm{W}_0^{1,p}(\Omega)$*, a Hilbert basis of* H*, such that for every* $i \in \mathbb{N}$ *and* $t \in [1,p]$*,*

$$\left\|\eta_i^{(n)} - \overline{\eta}_i\right\|_{\mathrm{L}^t(\Omega)} \longrightarrow 0, \quad \text{as } n \to +\infty, \tag{4.2}$$

up to a subsequence.

Proof. Let $i, n \in \mathbb{N}$. Let $\mu > 0$. By (2.14) we can assume that B_n verifies $\langle\!\langle T_n \rangle\!\rangle_{p,B_n} \leq \langle\!\langle T_n \rangle\!\rangle_p + \mu$. Then, by (4.1), we have that

$$\begin{aligned} \left\|\eta_i^{(n)}\right\|^p_{\mathrm{W}_0^{1,p}(\Omega)} &\leq \|\eta_i^{(n)}\|^p_{V,p} \leq \frac{1}{\nu_i^{(n)}} \sum_{j\in\mathbb{N}} \nu_j^{(n)} \|\eta_j^{(n)}\|^p_{V,p} \\ &= \frac{1}{\nu_i^{(n)}} \langle\!\langle T_n \rangle\!\rangle_{p,B_n} \leq \frac{\langle\!\langle T_n \rangle\!\rangle_p + \mu}{\nu_i^{(n)}}, \end{aligned}$$

which implies that $(\eta_i^{(n)})_{n\in\mathbb{N}}$ is bounded in $\mathrm{W}_0^{1,p}(\Omega)$. Therefore, up to a subsequence, $(\eta_i^{(n)})_{n\in\mathbb{N}}$ converges in $\mathrm{L}^p(\Omega)$, and (4.2) holds.

Since for every $n\in\mathbb{N}$ and $\phi\in H$, $\phi=\sum_{j\in\mathbb{N}}(\phi,\eta_j^{(n)})\eta_j^{(n)}$, it's quite clear that $\overline{B}$ is a Hilbert basis of H. That $\overline{B}$ is contained in $\mathrm{W}_0^{1,p}(\Omega)$ follows from a standard application of [3, Prop. 9.18], i.e., fixed $i\in\mathbb{N}$ we find $C>0$ such that

$$\forall\psi\in\mathrm{C}_0^1(\mathbb{R}^N),\forall j\in\{1,...,N\}:\quad \left|\int_\Omega \overline{\eta}_i(x)\frac{\partial\psi(x)}{\partial x_j}dx\right|\le C\,\|\psi\|_{\mathrm{L}^{p'}(\Omega)}. \tag{4.3}$$

For this we use (4.2) and the fact that, again by [3, Prop. 9.18], for each $n\in\mathbb{N}$ there exists $C_n>0$ such that (4.3) is verified with $\overline{\eta}_i$ and C replaced by $\overline{\eta}_i^{(n)}$ and C_n, respectively. □

Remark 4.5. Let's recall a *reverse Hölder inequality.* Let $p\in]0,1[$ and $q\in]-\infty,0[$ such that $1/p+1/q=1$. Then, for any $(x_i)_{i\in\mathbb{N}}\in\ell^p(\mathbb{R}^+)$ and any $(y_i)_{i\in\mathbb{N}}\in\ell^q(\mathbb{R}^+)$,

$$\sum_{i\in\mathbb{N}}x_iy_i\ge\left(\sum_{i\in\mathbb{N}}x_i^p\right)^{1/p}\left(\sum_{i\in\mathbb{N}}y_i^q\right)^{1/q}. \tag{4.4}$$

Lemma 4.6. *Let us assume* (4.1)*. Then, for every $\epsilon>0$ there exists $M_0\in\mathbb{N}$ such that*

$$\sup_{n\in\mathbb{N}}\sum_{i=M_0}^{+\infty}(\nu_i^{(n)})^m\le\epsilon, \tag{4.5}$$

where $m=\gamma/(\gamma+1)\in]N/(N+p),1[$ with $\gamma>N/p$. Moreover, up to a subsequence,

$$\lim_{n\to+\infty}\sum_{i\in\mathbb{N}}|\nu_i^{(n)}|^m=\sum_{i\in\mathbb{N}}|\overline{\nu}_i|^m. \tag{4.6}$$

Proof. Let $\epsilon>0$. By Example 2, we know that

$$\forall\epsilon_0>0,\exists N'\in\mathbb{N}:\quad\sum_{\ell=N'}^{+\infty}(\lambda_{0,\ell})^{-\gamma p/2}<\epsilon_0. \tag{4.7}$$

For each $i,n\in\mathbb{N}$ let's consider the expansion

$$\eta_i^{(n)}=\sum_{k\in\mathbb{N}}\mu_{ik}\phi_{0,k},\quad\mu_{ik}=\left(\eta_i^{(n)},\phi_{0,k}\right),\quad\sum_{k\in\mathbb{N}}|\mu_{ik}|^2=1. \tag{4.8}$$

By (4.4), we have, for an arbitrary $N_0\in\mathbb{N}$, that

$$\sum_{i=N_0}^{+\infty}\nu_i^{(n)}\left\|\eta_i^{(n)}\right\|^p_{\mathrm{H}_0^1(\Omega)}\ge\left(\sum_{i=N_0}^{+\infty}(\nu_i^{(n)})^m\right)^{1/m}\left(\sum_{i=N_0}^{+\infty}\left\|\eta_i^{(n)}\right\|^{-\gamma p}_{\mathrm{H}_0^1(\Omega)}\right)^{-1/\gamma},$$

$$\left(\sum_{i=N_0}^{\infty}(\nu_i^{(n)})^m\right)^{1/m}\le\hat{C}_p^p\left(\sum_{i=N_0}^{+\infty}\nu_i^{(n)}\left\|\eta_i^{(n)}\right\|^p_{V,p}\right)\left(\sum_{i=N_0}^{+\infty}\left\|\eta_i^{(n)}\right\|^{-\gamma p}_{\mathrm{H}_0^1(\Omega)}\right)^{1/\gamma}. \tag{4.9}$$

Now using the equality $\left\|\eta_i^{(n)}\right\|^2_{\mathrm{H}^1_0(\Omega)} = \sum_{\ell\in\mathbb{N}} |\mu_{i\ell}|^2\lambda_{0,\ell}$, we get by the convexity of $]0,+\infty[\ni s\mapsto s^{-\gamma p/2}\in\mathbb{R}$ and (4.8) that

$$\left\|\eta_i^{(n)}\right\|^{-\gamma p}_{\mathrm{H}^1_0(\Omega)} \le \sum_{\ell\in\mathbb{N}} |\mu_{i\ell}|^2\lambda_{0,\ell}^{-\gamma p/2}. \tag{4.10}$$

Then, since $(\lambda_{0,\ell})_{\ell\in\mathbb{N}}$ is non-decreasing we get, for a fixed $M\in\mathbb{N}\smallsetminus\{1\}$,

$$\begin{aligned}\sum_{i=N_0}^{+\infty}\left\|\eta_i^{(n)}\right\|^{-\gamma p}_{\mathrm{H}^1_0(\Omega)} &\le \sum_{i=N_0}^{+\infty}\sum_{\ell\in\mathbb{N}}|\mu_{i\ell}|^2\lambda_{0,\ell}^{-\gamma p/2} = \sum_{\ell=1}^{M-1}\sum_{i=N_0}^{+\infty}\cdots+\sum_{\ell=M}^{+\infty}\sum_{i=N_0}^{+\infty}\cdots\\ &\le \frac{M-1}{(\hat{\lambda}_{0,1})^{\gamma p/2}}\sum_{i=N_0}^{+\infty}|\mu_{i\ell}|^2+\sum_{\ell=M}^{+\infty}\sum_{i=N_0}^{+\infty}|\mu_{i\ell}|^2\lambda_{0,\ell}^{-\gamma p/2}.\end{aligned} \tag{4.11}$$

Since $\overline{B}$ is a Hilbert basis of H, for each $\ell\in\mathbb{N}$, $\sum_{i=1}^{+\infty}|(\overline{\eta}_i,\phi_{0,\ell})|^2 = \|\phi_{0,\ell}\|_{\mathrm{L}^2(\Omega)} = 1$. Therefore given any $\epsilon_1>0$, there exists $N_1\in\mathbb{N}$ such that for some $n_0\in\mathbb{N}$ and $\ell=1,...,N_1-1$,

$$\sum_{i=N_1}^{+\infty}|\mu_{i\ell}|^2<\epsilon_1,\qquad n\ge n_0.$$

Then, by taking $M=N'$, $N_0=N_1$ in (4.11), and choosing $\epsilon_0,\epsilon_1>0$ such that $\epsilon_1\left((M-1)\lambda_{0,1}^{-\gamma p/2}+\epsilon_0\right)<\epsilon$, we get

$$\sum_{i=N_1}^{+\infty}\left\|\eta_i^{(n)}\right\|^{-\gamma p}_{\mathrm{H}^1_0(\Omega)} \le \frac{M-1}{(\lambda_{0,1})^{\gamma p/2}}\epsilon_1+\epsilon_0\epsilon_1<\epsilon,$$

which together with (4.9) provides (4.5) as well as (4.6), up to a subsequence. □

Remark 4.7. Under the conditions of Lemma 4.6 and working as in the proof of [5, Th. 3.4-ii] it's proved that, for every $m'\in]m,1]$, up to a subsequence,

$$\lim_{n\to+\infty}\sum_{i\in\mathbb{N}}|\nu_i^{(n)}|^{m'} = \sum_{i\in\mathbb{N}}|\overline{\nu}_i|^{m'}.$$

Proof of Theorem 1.1. By point ii) of Lemma 4.2 and Remark 4.7 with $m'=1$, we have that $\sum_{i\in\mathbb{N}}\overline{\nu}_i<+\infty$. Let's write, for $\eta\in\mathrm{L}^2(\Omega)$,

$$\overline{T}\eta = \sum_{i\in\mathbb{N}}(\eta,\overline{\eta}_i)_{\mathrm{L}^2(\Omega)}\,\overline{\nu}_i\overline{\eta}_i$$

and observe that, by Cauchy-Schwarz, $\left\|\overline{T}\eta\right\|_{\mathrm{L}^2(\Omega)}\le\|\eta\|_{\mathrm{L}^2(\Omega)}\sum_{i\in\mathbb{N}}\overline{\nu}_i<+\infty$, so that the operator $\overline{T}:\mathrm{L}^2(\Omega)\to\mathrm{L}^2(\Omega)$ is well defined. Moreover, for each $i\in\mathbb{N}$, $\overline{T}\overline{\eta}_i=\overline{\nu}_i\overline{\eta}_i$, so that, by Lemma 4.4, $\overline{B}=\{\overline{\eta}_i\,/\,i\in\mathbb{N}\}\in\mathscr{B}^p_{\overline{T}}$. It's quite easy to show that $\overline{T}$ is self-adjoint and positive.

Let's prove that $\langle\!\langle\overline{T}\rangle\!\rangle_p<+\infty$. For the moment let's fix $N_0\in\mathbb{N}$. For $n\in\mathbb{N}$ and $B_n\in\mathscr{B}^p_{T_n}$ we write

$$f_n(x)=\sum_{i=1}^{N_0}\nu_i^{(n)}(|\nabla\eta_i^{(n)}(x)|^p+V(x)|\eta_i^{(n)}(x)|^p),\quad \text{for a.e. } x\in\Omega.$$

It's clear that $\sup_{n\in\mathbb{N}}\int_\Omega f_n(x)dx \geq 0$. Hence, by Fatou's lemma (see e.g.[15]), Lemma 4.2 and Lemma 4.4, we get

$$\int_\Omega \sum_{i=1}^{N_0} \left[\overline{\nu}_i(|\nabla\overline{\eta}_i|^p + V(x)|\overline{\eta}_i|^p)\right] dx \leq \liminf_{n\to+\infty}\int_\Omega f_n(x)dx. \tag{4.12}$$

Given $\epsilon > 0$, we can choose $B_n \in \mathscr{B}^p_{T_n}$ such that

$$\begin{aligned}\liminf_{n\to+\infty}\inf_{B_n\in\mathscr{B}^p_{T_n}}\int_\Omega f_n(x)dx &\leq \liminf_{n\to+\infty}\int_\Omega f_n(x)dx \\ &\leq \liminf_{n\to+\infty}\inf_{B_n\in\mathscr{B}^p_{T_n}}\int_\Omega f_n(x)dx + \epsilon \end{aligned}\tag{4.13}$$

Thus, by (4.12) and (4.13), we have that

$$\begin{aligned}\int_\Omega \sum_{i=1}^{N_0} \left[\overline{\nu}_i(|\nabla\overline{\eta}_i(x)|^p + V(x)|\overline{\eta}_i(x)|^p)\right] dx &\leq \liminf_{n\to+\infty}\inf_{B_n\in\mathscr{B}^p_{T_n}}\int_\Omega f_n(x)dx \\ &\leq \liminf_{n\to+\infty}\langle\!\langle T_n\rangle\!\rangle_p,\end{aligned}$$

whence, by doing $N_0 \to +\infty$, we get

$$\langle\!\langle \overline{T}\rangle\!\rangle_p \leq \liminf_{n\to+\infty}\langle\!\langle T_n\rangle\!\rangle_p \leq K_\infty < +\infty, \tag{4.14}$$

so that $\overline{T} \in \mathscr{W}^{1,p}_+$. Finally, to prove that $\|T - T_n\|_1 \to 0$, as $n \to +\infty$, we work as in the proof of [5, Th. 3.4-iii]. □

5. Ground states for free energy functionals

Let's use our main result, Theorem 1.1, to minimize free energy functionals.

Theorem 5.1. *Let $p \geq 2$. Assume that V is bounded away from 0 and let $\lambda \leq \gamma_V^{(p)}$ and that β is an entropy seed generated by $F \in \mathscr{C}^{(\epsilon/2)}_{p,0}$, for some $\epsilon \in]0,1]$. Then there exists a unique $T_\infty \in \mathscr{W}^{1,p}_+$ such that*

$$\mathscr{F}^{(\lambda)}_{\beta,p}(T_\infty) = \inf_{T\in\mathscr{W}^{1,p}_+}\mathscr{F}^{(\lambda)}_{\beta,p}(T) \tag{5.1}$$

Proof. By Proposition 3.7, $\mathscr{F}^{(\lambda)}_{\beta,p}$ is bounded from below on $\mathscr{W}^{1,p}$. Let $(T_n)_{n\in\mathbb{N}} \subseteq \mathscr{W}^{1,p}_+$ be a minimizing sequence for $\mathscr{F}^{(\lambda)}_{\beta,p}(\cdot)$. Since $(\mathscr{F}^{(\lambda)}_{\beta,p}(T_n))_{n\in\mathbb{N}}$ is bounded, Corollary 3.8 implies that $(\langle\!\langle T_n\rangle\!\rangle_p)_{n\in\mathbb{N}}$, $(\mathcal{S}_\beta(T_n))_{n\in\mathbb{N}}$ and $(\|T_n\|_{n\in\mathbb{N}})_{n\in\mathbb{N}}$ are bounded and $K_\infty = \sup_{n\in\mathbb{N}}\langle\!\langle T_n\rangle\!\rangle_p < +\infty$. Then we can apply Theorem 1.1 and extract a subsequence, still denoted $(T_n)_{n\in\mathbb{N}}$, such that $\|T_n - \overline{T}\|_1 \to 0$, as $n \to +\infty$, for some $\overline{T} \in \mathscr{W}^{1,p}_+$. This implies that $\|T_n\|_1 \to \|\overline{T}\|_1$, as $n \to +\infty$ and, moreover, by (4.14), we have that

$$\langle\!\langle \overline{T}\rangle\!\rangle_p \leq \liminf_{n\to+\infty}\langle\!\langle T_n\rangle\!\rangle_p.$$

Let's prove that

$$\mathcal{S}_\beta(\overline{T}) \leq \liminf_{n\to+\infty}\mathcal{S}_\beta(T_n). \tag{5.2}$$

The convexity of β implies the convexity of the set

$$\Lambda_+ = \left\{ \theta = (\theta_i)_{i\in\mathbb{N}} \in \ell^1(\mathbb{R}^+) \,/\, \sum_{i\in\mathbb{N}} \beta(\theta_i) < \sup_{n\in\mathbb{N}} \mathcal{S}_\beta(T_n) \right\}$$

and of the mapping $J : \Lambda_+ \to \mathbb{R}$ given by $J(\theta) = \sum_{i\in\mathbb{N}} \beta(\theta_i)$. Since J is lower semi-continuous we have that

$$\mathcal{S}_\beta(\overline{T}) = J((\overline{\nu}_i)_{i\in\mathbb{N}}) \le \liminf_{n\to+\infty} J((\nu_i^{(n)})_{i\in\mathbb{N}}) = \liminf_{n\to+\infty} \mathcal{S}_\beta(T_n).$$

Then, by the superadditivity of lim inf, we get

$$\begin{aligned}
\mathscr{F}^{(\lambda)}_{\beta,p}(\overline{T}) &= \mathcal{S}_\beta(\overline{T}) + \langle\!\langle \overline{T} \rangle\!\rangle_p - \lambda \|\overline{T}\|_1 \\
&\le \liminf_{n\to+\infty} \mathcal{S}_\beta(T_n) + \liminf_{n\to+\infty} \langle\!\langle T_n \rangle\!\rangle_p - \lambda \lim_{n\to\infty} \|T_n\|_1 \\
&\le \liminf_{n\to+\infty} \mathscr{F}^{(\lambda)}_{\beta,p}(T_n) = \inf_{T\in\mathscr{W}^{1,p}_+} \mathscr{F}^{(\lambda)}_{\beta,p}(T),
\end{aligned}$$

so that $\overline{T}$ is a minimizer for $\mathscr{F}^{(\lambda)}_{\beta,p}$. As it's worked out in [6], the minimizer of (5.1) at mixed states level is unique, up to a choice of a basis for non-simple eigenvalues. Therefore the operator T_∞ is the unique minimizer of (5.1). □

Remark 5.2. Using the scheme of the proof of [6, Prop.4], for $p = 2$ the minimizer in Theorem 5.1 has the explicit form $T_\infty = (\beta')^{-1}(\Delta - V + \lambda)$, provided β is strictly convex and differentiable on the interior of its support.

Whenever it makes sense, let's write for $T \in \mathscr{W}^{1,p}_+$ and for some function $z : [0, +\infty[\to \mathbb{R}$,

$$\mathscr{Z}(T) = \int_\Omega z(\rho_T(x))\, dx \quad \text{and} \quad \mathscr{F}^{\lambda,z}_{\beta,p}(T) = \mathscr{F}^{(\lambda)}_{\beta,p}(T) + \mathscr{Z}(T).$$

Corollary 5.3. *Assume the conditions of Theorem 5.1 and that for some* $s \in [1, pN/(2N-2-p)]$ *and* $c_1, c_2 \ge 0$,

$$\forall y \in [0, +\infty[: \quad c_1 \le z(y) \le c_2 y^s. \tag{5.3}$$

Then there exists $T_* \in \mathscr{W}^{1,p}_+$ *such that*

$$\mathscr{F}^{\lambda,z}_{\beta,p}(T_*) = \inf_{T\in\mathscr{W}^{1,p}_+} \mathscr{F}^{\lambda,z}_{\beta,p}(T).$$

Proof. The proof is similar to that of Theorem 5.1. Observe that, by (5.3) and Theorem 2.14, $\mathscr{Z}(\cdot)$ is well defined on $\mathscr{W}^{1,p}_+$. By using (5.3) and Fatou's lemma, it's obtained that $\mathscr{Z}(T_*) \le \liminf_{n\to+\infty} \mathscr{Z}(T_n)$, where $(T_n)_{n\in\mathbb{N}} \subseteq \mathscr{W}^{1,p}_+$ is a minimizing sequence for $\mathscr{F}^{\lambda,z}_{\beta,p}(\cdot)$. □

Remark 5.4. Using the scheme of the proof of [6, Prop.4], for $p = 2$, it can be proved that the minimizer in Theorem 5.1 has the explicit form $T_\infty = (\beta')^{-1}(\Delta - V + \lambda)$, provided β is strictly convex and differentiable on the interior of its support. In the same way, T_*, the minimizer in Corollary 5.3, is a fixed point of the map $W : \mathscr{W}^{1,2}_+ \to \mathscr{W}^{1,2}_+$, given by $W(T) = (\beta')^{-1}(\Delta - V + \lambda - g' \circ \rho_T)$, assuming that g is of class C^1.

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