Liouville theorems for a stationary and non-stationary coupled system of liquid crystal flows

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Abstract

We consider here the simplified Ericksen-Leslie system on the whole space $\mathbb{R}^3$. This system deals with the incompressible Navier-Stokes equations strongly coupled with a harmonic map flow, which makes it a challenging study from the mathematical point of view. Using the fairly general framework of a kind of local Morrey spaces, we obtain some Liouville-type theorems for the stationary and moreover for the non-stationary case. Our theorems also improve some well-known results for the particular case of classical Navier-Stokes equations.

Keywords: Simplified Ericksen-Leslie system; local Morrey spaces; Liouville problem; weak suitable solutions
AMS classification : 35Q35, 35B45, 35B53.

1 Introduction

In this paper, we place ourselves on the whole space $\mathbb{R}^3$ and we consider a coupled system of the incompressible Navier-Stokes equations with a harmonic map flow. This system, also known as the *simplified Ericksen-Leslie system*, was proposed by H.F. Lin in [21] as a simplification of the general *Ericksen-Leslie system* which models the hydrodynamic flow of nematic liquid crystal material [8], [24]. The simplified Ericksen-Leslie system, has been successful to model various dynamical behavior for nematic liquid crystals. More precisely, it provides a well macroscopic description of the evolution of the material under the influence of fluid velocity field; and the macroscopic description of the microscopic orientation of fluid velocity of rod-like liquid crystals. See the book [13] for more details. On the other hand, from the mathematical point of view, the simplified Ericksen-Leslie system has recently attired a lot of interest in the research community, see, *e.g.*, the articles [15, 17, 22, 23, 30] and the references therein, where the major challenge is due to the *strong* coupled structure of this system and the presence of a super-critical non-linear term.

In the *stationary* setting, the simplified Ericksen-Leslie system is given as follows:

$$\begin{cases} -\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \operatorname{div}(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v}) + \nabla p = 0, \\ -\Delta \mathbf{v} + (\mathbf{u}\cdot\nabla)\,\mathbf{v} - |\nabla\otimes\mathbf{v}|^2\,\mathbf{v} = 0, \\ \operatorname{div}(\mathbf{u}) = 0. \end{cases} \tag{1}$$

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Here, the fluid velocity $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$, and the pressure $p : \mathbb{R}^3 \to \mathbb{R}$ are the classical unknowns of the fluid mechanics. This system also considers a third unknown $\mathbf{v} : \mathbb{R}^3 \to \mathbb{S}^2$, which is a *unit vector field* representing the macroscopic orientation of the nematic liquid crystal molecules. For the vector field $\mathbf{v} = (v_i)_{1\leq i\leq 3}$, we denote $\nabla \otimes \mathbf{v} = (\partial_i v_j)_{1\leq i,j\leq 3}$. In the first equation of this system, the *super-critical* non-linear term: $\text{div}(\nabla \otimes \mathbf{v} \odot \nabla \otimes \mathbf{v})$, is given as the divergence of a symmetric tensor $\nabla \otimes \mathbf{v} \odot \nabla \otimes \mathbf{v}$, where, for $1 \leq i, j \leq 3$, its components are defined by the expression $(\nabla \otimes \mathbf{v} \odot \nabla \otimes \mathbf{v})_{i,j} = \sum_{k=1}^{3} \partial_i v_k \partial_j v_k$, and then, each component of the vector field $\text{div}(\nabla \otimes \mathbf{v} \odot \nabla \otimes \mathbf{v})$ is explicitly given by the following expression $[\text{div}(\nabla \otimes \mathbf{v} \odot \nabla \otimes \mathbf{v})]_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \partial_j(\partial_i v_k\, \partial_j v_k)$. We may observe that due to the double derivatives in this expression, this super-critical non-linear term is actually more *delicate* to treat than the classical non-linear transport term: $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$, and this fact makes challenging the mathematical study of (1).

We define a weak solution of the coupled system (1) as the triplet $(\mathbf{u}, p, \mathbf{v})$ where: $\mathbf{u} \in L^2_{loc}(\mathbb{R}^3)$, $p \in \mathcal{D}'(\mathbb{R}^3)$ and $\mathbf{v} \in L^\infty(\mathbb{R}^3)$, since by the *physical model* we assume $|\mathbf{v}| = 1$, and moreover it verifies $\nabla \otimes \mathbf{v} \in L^2_{loc}(\mathbb{R}^3)$. Under these hypothesis all the terms in (1) are well-defined in the distributional sense. Remark that the triplet $\mathbf{u} = 0$, $p = 0$ and $\nabla \otimes \mathbf{v} = 0$ (hence $\mathbf{v}$ is a constant unitary vector) is a trivial weak solution of the system (1) and it is natural to ask if this solution is the unique one (modulo constants). The answer to this question is in general *negative* and we are able to exhibit an explicit counterexample. Consider the velocity field $\mathbf{u}$ and the pressure term p defined as $\mathbf{u}(x_1, x_2, x_3) = (2x_1, 2x_2, -4x_3)$ and $p(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 8x_2^3$ respectively, and moreover, the vector field $\mathbf{v}$ defined by $\mathbf{v}(x_1, x_2, x_3) = (x_1, x_2, 0)$ if $x_1^2 + x_2^2 = 1$, and $\mathbf{v}(x_1, x_2, x_3) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ if $x_1^2 + x_2^2 \neq 1$. We have $|\mathbf{v}| = 1$ and using some basic rules of the vector calculus we easily get that the triple $(\mathbf{u}, p, \mathbf{v})$ defined as above is also a solution of the system (1).

Due to the non-uniqueness of the trivial solution, we are interesting to find additional some *a priori* conditions in order to assure the vanishing of stationary solutions. This problem is commonly known as the *Liouville-type problem*. To the best of our knowledge, the first Liouville-type result for the coupled system (1) was recently obtained by Y. Hao, X. Liu & X. Zhang in [15]. In this work, the authors consider a solution $(\mathbf{u}, p, \mathbf{v})$ which verifies $\nabla \otimes \mathbf{u} \in L^2(\mathbb{R}^3)$ and $\nabla \otimes \mathbf{v} \in L^2(\mathbb{R}^3)$ and moreover, under the important assumption: $\mathbf{u} \in L^{9/2}(\mathbb{R}^3)$ and $\nabla \otimes \mathbf{v} \in L^{9/2}(\mathbb{R}^3)$, they obtained the identities $\mathbf{u} = 0$, $p = 0$ and $\nabla \otimes \mathbf{v} = 0$. These *a priori* conditions, which actually are *decaying properties* on $\mathbf{u}$ and $\nabla \otimes \mathbf{v}$ given by the $L^{9/2}-$ norm, are interesting if we compare this result with a well-known result on the Liouville problem for the the classical stationary and incompressible Navier-Stokes equations:

$$-\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = 0, \quad \text{div}(\mathbf{u}) = 0. \tag{2}$$

For these equations, a celebrated result obtained in [14] by G. Galdi shows that if $\mathbf{u} \in L^{9/2}(\mathbb{R}^3)$ then we have $\mathbf{u} = 0$ and $p = 0$, and then, the recent result obtained in [15] can be regarded as a generalizations of Galdi's result to the more delicate setting of the coupled system (1).

Let us recall that the Liouville problem for the stationary Navier-Stokes equations (2) was extensive studied in different functional settings. Galdi's result [14] was extended to setting of the Lorentz spaces by H. Kozono *et. al.* in [19]. Thereafter, this work was improved to a kind of local Lorentz-type spaces by G. Seregin & W. Wang in [28]. Moreover, the Liouville problem for the equations (2) has also largely studied in the more general setting of the Morrey spaces by D. Chamorro *et. al.* in [7] and G. Seregin in [26] and [27]. For more interesting works on the Liouville problem for the stationary Navier-Stokes equations (2) see the articles [3, 4, 5, 18] and the references therein. With all this information in mind, it is quite natural to improve the Galdi's-type result for the system (1) obtained in [15] to different functional settings. Thus, the first aim of this paper is to study the Liouville problem for the coupled system (1) in a fairly general functional setting.

Let us mention that a kind of *local Morrey spaces* (see the expression (7) below for a definition) which, roughly speaking, characterize the averaged decaying properties of functions, have recently attired the attention in the study of the well-posedness issues for the classical the Navier-Stokes, see [1] and [11], and also for the coupled system of the Magneto-hydrodynamic equations, see [10]. In this paper we show that the *local Morrey spaces* also give us an interesting setting to solve the Liouville problem for the coupled system (1) as these spaces contain the classic Lebesgue spaces and the more technical Lorentz and Morrey spaces. As a bi-product, since the equations (2) are a particular case of the system (1) (when we set $\mathbf{v}=0$) we also improve some well-known and recent results on the Liouville problem for (2).

Our methods are essentially based in L^p- local estimates of the functions $\mathbf{u}$ and $\nabla\otimes\mathbf{v}$, and this approach allow us to study the Liouville problem for *non-stationary* case of the coupled system (1). Thus, in the second part of this paper, we will focus on the following Cauchy problem for the simplified Ericksen-Leslie system:

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \text{div}(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v}) + \nabla p = 0, \\ \partial_t \mathbf{v} - \Delta \mathbf{v} + (\mathbf{u}\cdot\nabla)\,\mathbf{v} - |\nabla\otimes\mathbf{v}|^2\,\mathbf{v} = 0, \\ \text{div}(\mathbf{u}) = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \quad \mathbf{v}(0,\cdot) = \mathbf{v}_0 \quad \text{div}(\mathbf{u}_0) = 0. \end{cases} \tag{3}$$

Let us recall that in the particular case of the Cauchy problem for the classical incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = 0, \quad \text{div}(\mathbf{u}) = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \quad \text{div}(\mathbf{u}_0) = 0, \end{cases} \tag{4}$$

J. Serrin proved in [29] that if a *weak solution* $\mathbf{u}$ satisfies the condition $\mathbf{u}\in L^p(0,T,L^r(\mathbb{R}^3))$, for $p>2$ and $r>3$ such that $2/p+3/r\leq 1$, then $\mathbf{u}$ verifies the *global energy equality*. Thereafter, this result was generalized by H. Kozono *et. al.* in [19] as follows. Recall first that the notion of *weak suitable solutions* for the equations (4) were introduced in the celebrated Cafarelli, Konh and Niremberg theory [2]. Then, in [19] it is introduced the notion of *generalized weak suitable solution* (see Definition 3.1, page 5 of [19]). This notion of generalized weak suitable solution is a generalization of the well-known weak suitable solutions and the main difference is that it assumes neither finite energy: $\displaystyle\sup_{0\leq t\leq T}\|\mathbf{u}(t,\cdot)\|^2_{L^2}<+\infty$, nor finite dissipation $\displaystyle\int_0^T \|\mathbf{u}(t,\cdot)\|^2_{\dot{H}^1}dt<+\infty$. In the setting of the *generalized weak suitable solution*, H. Kozono *et. al.* gave a *new a priori condition* which ensures that the well-know *global energy inequality* holds. More precisely, within the more general framework of the Lorentz spaces and for the parameters $3\leq p_1,r_1,p_2,r_2\leq+\infty$ satisfying some technical conditions related to the well-known scaling properties of the equations (4), the condition $\mathbf{u}\in L^3(0,T,L^{p_1,r_1}(\mathbb{R}^3))\cap L^2(0,T,L^{p_2,r_2}(\mathbb{R}^3))$ ensures that $\mathbf{u}\in L^\infty_t L^2_x\cap L^2_t\dot{H}^1_x(]0,T[\times\mathbb{R}^3)$ and moreover it verifies the global energy inequality, *i.e.*, $\mathbf{u}$ becomes a Leray weak solution.

Following these ideas, we introduce first a notion of *generalized weak suitable solutions* (see Definition 2.1 below) for the coupled system (3). Thereafter, using a time-space version of the *local Morrey spaces* (see the expression (11) below for a definition), we give some *a priori* conditions to ensure that, for a time $0<T<+\infty$ arbitrary large, the *generalized weak suitable solutions* of (3) verify a *global energy inequality* (13). As an interesting application, we obtain some *Liouville-type results* for the non-stationary system (3). More precisely, using the global energy inequality we are able to prove the uniqueness of the trivial solutions $\mathbf{u}=0$, $p=0$ and $\nabla\otimes\mathbf{v}=0$ for the initial data $\mathbf{u}_0=0$ and $\nabla\otimes\mathbf{v}_0=0$.

This paper is organized as follows. In Section 2 below we expose all the results obtained. Then, in Section 3 we summarize some previous results on the local Morrey spaces we shall use here. Section 4 is devoted to a characterization of the pressure term in the coupled systems (1) and (3) which will be useful for the next sections. Finally, in Section 5 we study the Liouville problem for the stationary system (1) and in Section 6 we study the Liouville problem for the non-stationary system (3).

2 Framework and statement of the results

2.1 The stationary case

Recall first that in [15], in order to solve the Liouville problem for (1), the authors need the additional hypothesis on the function $\mathbf{v}$: $\nabla\otimes\mathbf{v}\in L^2(\mathbb{R}^3)$. In our results below we will *relax* this hypothesis as follows: for $R\geq 1$ we denote $\mathcal{C}(R/2,R)=\{x\in\mathbb{R}^3 : R/2<|x|<R\}$; and from now on we will assume

$$\sup_{R\geq 1}\int_{\mathcal{C}(R/2,R)}|\nabla\otimes\mathbf{v}|^2dx<+\infty. \tag{5}$$

Before to state our results, we recall the definition of the Morrey and local Morrey spaces. For more references about this spaces see, *e.g.*, the Chapter 8 of the book [20] and the Section 7 of the paper [12] respectively. Let $1<p<r<+\infty$, the homogeneous Morrey space $\dot{M}^{p,r}(\mathbb{R}^3)$ is the set of functions $f\in L^p_{loc}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{M}^{p,r}}=\sup_{R>0,\ x_0\in\mathbb{R}^3}R^{\frac{3}{r}}\left(\frac{1}{R^3}\int_{B(x_0,R)}|f(x)|^pdx\right)^{\frac{1}{p}}<+\infty, \tag{6}$$

where $B(x_0,R)$ denotes the ball centered at x_0 and with radio R. This is a homogeneous space of degree $-\frac{3}{r}$ and moreover we have the following chain of continuous embedding $L^r(\mathbb{R}^3)\subset L^{r,q}(\mathbb{R}^3)\subset\dot{M}^{p,r}(\mathbb{R}^3)$, where, for $r\leq q\leq+\infty$ the space $L^{r,q}(\mathbb{R}^3)$ is a Lorentz space [6].

Observe that in expression (6) we consider the average in terms of the L^p- norm of the function f on the ball $B(x_0,R)$; and the term $R^{\frac{3}{q}}$ describes the decaying of this averaged quantity when R is large.

The local Morrey spaces we shall consider here describes the averaged decaying of functions in a more general setting. For $\gamma\geq 0$ and $1<p<+\infty$, we define the *local Morrey space* $M^p_\gamma(\mathbb{R}^3)$ as the Banach space of functions $f\in L^p_{loc}(\mathbb{R}^3)$ such that

$$\|f\|_{M^p_\gamma}=\sup_{R\geq 1}\left(\frac{1}{R^\gamma}\int_{B(0,R)}|f(x)|^pdx\right)^{1/p}<+\infty. \tag{7}$$

Here the parameter $\gamma\geq 0$ characterizes the behavior of the quantity $\left(\int_{B(0,R)}|f(x)|^pdx\right)^{1/p}$ when R is large. Moreover, for $\gamma_1\leq\gamma_2$ we have the continuous embedding $M^p_{\gamma_1}(\mathbb{R}^3)\subset M^p_{\gamma_2}(\mathbb{R}^3)$. Remark also that for $1<p<r<+\infty$, setting the parameter γ such that $3(1-p/r)<\gamma$, then we have $\dot{M}^{p,r}(\mathbb{R}^3)=M^p_{3(1-p/r)}(\mathbb{R}^3)\subset M^p_\gamma(\mathbb{R}^3)$, and in this sense the local Morrey space $M^p_\gamma(\mathbb{R}^3)$ can be regarded as a generalization of the homogeneous Morrey space $\dot{M}^{p,r}(\mathbb{R}^3)$.

Finally, we define the space $M^p_{\gamma,0}(\mathbb{R}^3)$ as the set of functions $f\in M^p_\gamma(\mathbb{R}^3)$ such that

$$\lim_{R\to+\infty}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|f(x)|^pdx\right)^{1/p}=0. \tag{8}$$

In the setting of the local Morrey spaces $M^p_{\gamma,0}(\mathbb{R}^3)$ and $M^p_\gamma(\mathbb{R}^3)$ defined above we set the parameter $p=3$ (which as we will comment below in the most interesting value) and we will consider two cases for the parameter γ which characterizes the decaying properties at infinity of the functions $\mathbf{u}$ and $\nabla\otimes\mathbf{v}$ in the coupled system (1). Our first result for the stationary Ericksen-Leslie system (1) reads as follows.

Theorem 1 *Let $(\boldsymbol{u},p,\boldsymbol{v})$ be a smooth solution of stationary coupled system (1) such that $\boldsymbol{v}$ verifies (5).*

1) *Let $\gamma=1$. If $\boldsymbol{u}\in M^3_{1,0}(\mathbb{R}^3)$ and $\nabla\otimes\boldsymbol{v}\in M^3_1(\mathbb{R}^3)$ then we have $\boldsymbol{u}=0$, $p=0$ and $\nabla\otimes\boldsymbol{v}=0$.*

2) *Let $1<\gamma<3/2$. If $\boldsymbol{u}\in M^3_{\gamma,0}(\mathbb{R}^3)$ and $\nabla\otimes\boldsymbol{v}\in M^3_\gamma(\mathbb{R}^3)$, and moreover if the velocity $\boldsymbol{u}$ verifies:*

$$\lim_{R\to+\infty} R^{\gamma-1}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\boldsymbol{u}(x)|^3dx\right)^{1/3}=0, \tag{9}$$

then we have $\boldsymbol{u}=0$, $p=0$ and $\nabla\otimes\boldsymbol{v}=0$.

Let us make the following comments. First, we emphasize on the fact that these regularity needed on the solutions is a technical requirement to reach out our computations and the *key-hypothesis* to study the Liouville problem is the decaying of solutions at infinity characterized through the general setting of the local Morrey spaces.

It is interesting to remark that the spaces $M^3_{1,0}(\mathbb{R}^3)$ and $M^3_1(\mathbb{R}^3)$, considered in point 1) above, contain some well-known functional spaces and then we solve the Liouville problem for the coupled system (1) in a fairly general functional framework. Indeed, for the Lebesgue, Lorentz and Morrey spaces, for $3<r<9/2$ and for $3(1-3/r)<\delta<1$ we have the large chain of strict embedding $L^r(\mathbb{R}^3)\subset L^{r,\infty}(\mathbb{R}^3)\subset \dot{M}^{3,r}(\mathbb{R}^3)\subset M^3_\delta(\mathbb{R}^3)\subset M^3_{1,0}(\mathbb{R}^3)$, where the last embedding is due to point 1 of Lemma 3.1 below. Moreover, for $r=9/2$ and $9/2<q<+\infty$ we also have the embedding $L^{9/2}(\mathbb{R}^3)\subset L^{9/2,q}(\mathbb{R}^3)\subset M^3_{1,0}(\mathbb{R}^3)$. Indeed, let us verify the last inclusion. If $f\in L^{9/2,q}(\mathbb{R}^3)$ then we have $f\in \dot{M}^{3,9/2}(\mathbb{R}^3)$, but due to the identity $\dot{M}^{3,9/2}(\mathbb{R}^3)=M^3_1(\mathbb{R}^3)$, we get $f\in M^3_1(\mathbb{R}^3)$. Moreover, by the following estimate:

$$\int_{\mathbb{C}(R/2,R)}|f|^3dx=\int_{B(0,R)}\left|\mathbb{1}_{\mathbb{C}(R/2,R)}f\right|^3dx\le c\,R\left\|\mathbb{1}_{\mathbb{C}(R/2,R)}f\right\|^3_{L^{9/2,\infty}}\le c\,R\left\|\mathbb{1}_{\mathbb{C}(R/2,R)}f\right\|^3_{L^{9/2,q}}$$

and using the dominated convergence theorem (which is valid in the space $L^{9/2,q}(\mathbb{R}^3)$ for the values $9/2\le q<+\infty$) we obtain: $\displaystyle\lim_{R\to+\infty}\frac{1}{R}\int_{\mathbb{C}(R/2,R)}|f|^3\,dx=0$, hence we have $f\in M^3_{1,0}(\mathbb{R}^3)$. Thus, due to these embedding, the recent result obtained in [15] for the coupled system (1) follows from this theorem.

We may observe that if in the statement of this theorem we set $\mathbf{v}$ a constant unitary vector, then this results holds true for the stationary Navier-Stokes equations (2). In this particular setting, by the last embedding above we get that this theorem improves the classical Galdi's result [14] in the framework of the Lebesgue space $L^{9/2}(\mathbb{R}^3)$ and some recent results [16] in the framework of the Lorentz spaces $L^{9/2,q}(\mathbb{R}^3)$. Moreover, due to the embedding $M^{3,r}(\mathbb{R}^3)\subset M^3_{1,0}(\mathbb{R}^3)$ (with $3<r<9/2$) this theorem also improves some previous results obtained in the setting of the Morrey spaces in [7] and [16]. On the other hand, for $0<\delta\le 1$ we define $w_\delta(x)=\dfrac{1}{(1+|x|)^\delta}$, and we consider the weighted Lebesgue space $L^3_{w_\delta}(\mathbb{R}^3)=L^3(w_\delta\,dx)$. Thus, due to the embedding $L^3_{w_\delta}(\mathbb{R}^3)\subset M^3_{1,0}(\mathbb{R}^3)$ (see the point 1) of Lemma 3.1 below) we obtain that this theorem also contains some results proven in [25] (see Remark 4.9, page 10).

Getting back to the coupled system (1), for the values $1<\gamma<3/2$ considered in point 2) above, we may observe that the general setting of the space $M^3_{\gamma,0}(\mathbb{R}^3)$ seems not to be sufficient to solve the Liouville problem since and we still need some supplementary decaying properties on the velocity $\mathbf{u}$ given in (9). More precisely, we can see that for $\gamma > 1$ the expression $R^{\gamma-1}$ improves the decaying properties at infinity. In this setting, we would emphasize the fact that the value of the parameter $\gamma = 1$ seems to be the critical to solve the Liouville problem in the sense that for $1 < \gamma < 3/2$, if we only consider the information $\mathbf{u} \in M^3_\gamma(\mathbb{R}^3)$, the velocity $\mathbf{u}$ does not decay at infinity fast enough and we need to improve its decaying properties. This interesting phenom was also exhibit in [7] and [16] for the case of the stationary Navier-Stokes (2). Moreover, remark that this improvement on the decay properties (when $\gamma > 1$) are only needed for the velocity $\mathbf{u}$ and not for the function $\nabla \otimes \mathbf{v}$.

Finally, let us mention that the restriction of the parameter γ: $\gamma < 3/2$, is essentially technical and we think that, with further technical computations, the statement in point 2) above could be improved for the range $3/2 \leq \gamma < 3$. The values $3 \leq \gamma < +\infty$ seems to be more delicate to treat since in this case some useful tools on the local Morrey spaces $M^p_\gamma(\mathbb{R}^3)$ are not valid, see the Section 3 and in particular Lemma 3.2 for more details.

2.2 The non-stationary case

From now on let us fix a time $0 < T < +\infty$. We start introducing the notion of *generalized weak suitable solution* for the non-stationary Ericksen-Leslie system (3).

Definition 2.1 *Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$ such that $div(\boldsymbol{u}_0) = 0$ and let $\boldsymbol{v}_0 \in \dot{H}^1(\mathbb{R}^3)$. We say that the triplet $(\boldsymbol{u}, p, \boldsymbol{v})$ is a generalized weak suitable solution of the coupled system (3) if:*

1) $\boldsymbol{u} \in L^3_{loc}([0,T[\times\mathbb{R}^3)$, $\nabla \otimes \boldsymbol{u} \in L^3_{loc}([0,T[\times\mathbb{R}^3)$ *and* $p \in L^{3/2}_{loc}([0,T[\times\mathbb{R}^3)$.

2) $\boldsymbol{v} \in L^\infty_{loc}([0,T[, L^\infty(\mathbb{R}^3))$, $\nabla \otimes \boldsymbol{v} \in L^3_{loc}([0,T[\times\mathbb{R}^3)$ *and* $\Delta \boldsymbol{v} \in L^3_{loc}([0,T[\times\mathbb{R}^3)$.

3) *The triplet $(\boldsymbol{u}, p, \boldsymbol{v})$ verifies the first three equations of (3) in $\mathcal{D}'(]0,T[\times\mathbb{R}^3)$.*

4) *For every compact set $K \subset \mathbb{R}^3$, the function $\boldsymbol{u}(t,\cdot)$ is continuous for $t \in]0,T[$ in the weak topology of $L^2(K)$ and strongly continuous at $t = 0$. Moreover, the function $\boldsymbol{v}(t,\cdot)$ is continuous for $t \in]0,T[$ in the weak topology of $\dot{H}^1(K)$ and strongly continuous at $t = 0$.*

5) *The triplet $(\boldsymbol{u}, p, \boldsymbol{v})$ verifies the following* local energy inequality*: there exist a* non-negative*, locally finite measure μ on $]0,T[\times\mathbb{R}^3$ such that:*

$$\partial_t \left(\frac{|\boldsymbol{u}|^2 + |\nabla \otimes \boldsymbol{v}|^2}{2} \right) + |\nabla \otimes \boldsymbol{u}|^2 = -|\Delta \boldsymbol{v}|^2 + \Delta \left(\frac{|\boldsymbol{u}|^2 + |\nabla \otimes \boldsymbol{v}|^2}{2} \right)$$
$$-div \left(\left[\frac{|\boldsymbol{u}|^2 + |\nabla \otimes \boldsymbol{v}|^2}{2} + p \right] \boldsymbol{u} \right) - \sum_{k=1}^{3} \partial_k ([\boldsymbol{u} \cdot \nabla) \, \boldsymbol{v}] \cdot \partial_k \boldsymbol{v}) - |\nabla \otimes \boldsymbol{v}|^2 \boldsymbol{v} \cdot \Delta \boldsymbol{v} - \mu. \tag{10}$$

Observe that by the hypothesis given in points 1) and 2) we have that μ is well-defined in the distributional sense. However, the most important fact in this definition is the positivity assumed on μ which is the whole point in the notion of *suitable* solutions.

This notion of generalized weak suitable solution is close to the definition of a weak suitable solution for the coupled system (3) given in [22] (for the case of a bounded and smooth domain $\Omega \subset \mathbb{R}^3$). In comparison with [22], it is worth to remark that here we suppose neither $\mathbf{u} \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$, nor $\mathbf{v} \in L^\infty_t \dot{H}^1_x$ and we consider here only locally integrable properties. Moreover, we assume on the pressure term a local $L^{3/2}-$integrability, while the authors in [22] assume a $L^{5/3}-$ integrability.

We introduce now a time-space version of the *local Morrey spaces*, for more references on these spaces see always the Section 7 of [12]. For $1 < p < +\infty$ and $\gamma \geq 0$, we define the space $M^p_\gamma L^p(0,T)$ as the Banach space of functions $f \in L^p_{loc}([0,T] \times \mathbb{R}^3)$ such that

$$\|f\|_{M^p_\gamma L^p(0,T)} = \sup_{R\geq 1} \left(\frac{1}{R^\gamma} \int_0^T \int_{B(0,R)} |f(t,x)|^p dx\, dt \right)^{1/p} < +\infty. \tag{11}$$

Moreover, we define the space $M^p_{\gamma,0} L^p(0,T)$ as the set of functions $f \in M^p_\gamma L^p(0,T)$ which verifies

$$\lim_{R\to+\infty} \left(\frac{1}{R^\gamma} \int_0^T \int_{B(0,R)} |f(t,x)|^p dx\, dt \right)^{1/p} = 0. \tag{12}$$

As mentioned in the introduction, in the general framework of the time-space local Morrey spaces, we give some *a priori* conditions on the generalized weak suitable solutions defined above to ensure that these solutions verify a global energy inequality.

Theorem 2 *Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$, with $div(\boldsymbol{u}_0) = 0$, and let $\boldsymbol{v}_0 \in \dot{H}^1(\mathbb{R}^3)$ be the initial data. Let a time $0 < T < +\infty$, and let $(\boldsymbol{u}, p, \boldsymbol{v})$ be a* generalized weak suitable solution *of the non-stationary coupled system (3) given in Definition 2.1. If $\boldsymbol{u} \in M^3_{1,0} L^3(0,T)$ and $\nabla \otimes \boldsymbol{v} \in M^3_1 L^3(0,T)$, then we have $\boldsymbol{u} \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x([0,T] \times \mathbb{R}^3)$, $\boldsymbol{v} \in L^\infty_t \dot{H}^1_x([0,T] \times \mathbb{R}^3)$, and moreover, for all $t \in [0,T]$ the global energy inequality is verified:*

$$\|\boldsymbol{u}(t,\cdot)\|^2_{L^2} + 2\int_0^t \|\boldsymbol{u}(s,\cdot)\|^2_{\dot{H}^1} ds + \|\boldsymbol{v}(t,\cdot)\|^2_{\dot{H}^1} \leq \|\boldsymbol{u}_0\|^2_{L^2} + \|\boldsymbol{v}_0\|^2_{\dot{H}^1}. \tag{13}$$

As a direct application of the global energy inequality above we have the following Liouville-type result.

Corollary 1 *Let the initial data $\boldsymbol{u}_0 = 0$ and $\boldsymbol{v}_0 = 0$. let $(\boldsymbol{u}, p, \boldsymbol{v})$ be a* generalized weak suitable solution *of the non-stationary coupled system (3) given by Definition 2.1. If $\boldsymbol{u} \in M^3_{1,0} L^3(0,T)$ and $\nabla \otimes \boldsymbol{v} \in M^3_1 L^3(0,T)$, then we have the identities $\boldsymbol{u} = 0$, $p = 0$ and $\nabla \otimes \boldsymbol{v} = 0$ on $[0,T] \times \mathbb{R}^3$.*

To close this section, let us make the following comments. As for the stationary case, we may observe that if we set $\mathbf{v}_0$ and $\mathbf{v}$ two constant unitary vectors, then Theorem 2 and Corollary 1 hold true for the classical Navier-Stokes equations (4) provided that $(\mathbf{u}, p)$ is a generalized weak suitable solution in the sense of Definition 2.1 (with $\nabla \otimes \mathbf{v} = 0$) and $\mathbf{u} \in M^3_{1,0} L^3(0,T)$. In this setting, it is interesting to observe that the space $M^3_{1,0} L^3(0,T)$ contains some well-known spaces of the Navier-Stokes literature [20]. We observe first that for $3 < p, r < +\infty$ such that $2/p + 3/r \leq 1$ and $r \leq q < +\infty$, we have the following chain of embedding $L^p(0,T,L^r(\mathbb{R}^3)) \subset L^p(0,T,L^{r,q}(\mathbb{R}^3)) \subset M^3_{1,0} L^3(0,T)$. Indeed, it is sufficient to verify the last inclusion which is a direct consequence of the following local estimate

$$\int_0^T \int_{\mathbb{C}(R/2,R)} |f(t,x)|^3 dx\, dt \leq c\, R \int_0^T \|\mathbb{1}_{\mathbb{C}(R/2,R)} f(t,\cdot)\|^3_{L^{r,+\infty}} dt \leq c\, R \int_0^T \|\mathbb{1}_{\mathbb{C}(R/2,R)} f(t,\cdot)\|^3_{L^{r,q}} dt,$$

and the convergence dominated theorem. On the other hand, we also have the inclusion $\mathcal{C}([0,T], L^3(\mathbb{R}^3)) \subset M^3_{1,0} L^3(0,T)$, which follows directly from point 2) of Lemma 3.2.

3 The local Morrey spaces

In this section, for the completeness of the paper, we summarize some previous results on the local Morrey spaces $M^p_\gamma(\mathbb{R}^3)$ and $M^p_{\gamma,0}(\mathbb{R}^3)$ given in (7) and (8) respectively, and its time-space version $M^p_\gamma L^p(0,T)$ and $M^p_{\gamma,0} L^p(0,T)$ defined in (11) and (12) respectively.

These kind of local Morrey spaces are strongly lied with the weighted Lebesgue spaces $L^p_{w_\gamma}(\mathbb{R}^3)$ which are defined as follows: for $\gamma \geq 0$ we consider the weight

$$w_\gamma(x) = \frac{1}{(1+|x|)^\gamma} \tag{14}$$

and then for $1 < p < +\infty$ we define the space $L^p_{w_\gamma}(\mathbb{R}^3) = L^p(w_\gamma\, dx)$. Thus, we have the following useful result.

Lemma 3.1 (Lemma 2.1 of [10]) *Let $0 \leq \gamma < \delta$ and $1 < p < +\infty$.*

1) *We have the continuous embedding: $L^p_{w_\gamma}(\mathbb{R}^3) \subset M^p_{\gamma,0}(\mathbb{R}^3) \subset M^p_\gamma(\mathbb{R}^3) \subset L^p_{w_\delta}(\mathbb{R}^3)$.*

2) *Moreover, for $0 < T < +\infty$ we also have the continuous embedding:*

$$L^p\left([0,T], L^p_{w_\gamma}(\mathbb{R}^3)\right) \subset M^p_{\gamma,0}L^p(0,T) \subset M^p_\gamma L^p(0,T) \subset L^p\left([0,T], L^p_{w_\delta}(\mathbb{R}^3)\right).$$

Thereafter, a second useful result is the following one.

Lemma 3.2 (Lemma 2.1 of [9] and Corollary 2.1 of [10]) *Let $0 < \gamma < 3$ and $1 < p < +\infty$.*

1) *The Riesz transform $\mathcal{R}_i = \dfrac{\partial_i}{\sqrt{-\Delta}}$ is bounded on $M^p_\gamma(\mathbb{R}^3)$ and we have $\|\mathcal{R}_i f\|_{M^p_\gamma} \leq c_{p,\gamma}\|f\|_{M^p_\gamma}$.*

2) *The Hardy-Littlewood maximal function operator $\mathcal{M}$ is also bounded on the space $M^p_\gamma(\mathbb{R}^3)$ and we have $\|\mathcal{M}_f\|_{M^p_\gamma} \leq c_{p,\gamma}\|f\|_{M^p_\gamma}$.*

3) *The points 1) and 2) also hold for the weighted Lebesgue spaces $L^p_{w_\gamma}(\mathbb{R}^3)$.*

4 Characterization of the pressure term

With the technical results stated in the previous section, and following some of the ideas of the proof of Proposition 2.1 in [9], we are able to give a characterization of the pressure term in the first equation of the coupled systems (1) and (3). We may observe that every solution of the *stationary* system (1) is also a solution of the *non-stationary* system (3) since the term involving the derivative in time is equals to zero. Thus, we will state the following result in the more general setting of the *non-stationary* coupled system (3).

Proposition 4.1 *Let $(\boldsymbol{u}, p, \boldsymbol{v})$ be a solution of the coupled system (3) such that, for $0 < \gamma < 3/2$, $2 < p < +\infty$ and $0 < T < +\infty$, it verifies $\boldsymbol{u} \in M^p_\gamma L^p(0,T)$, $p \in \mathcal{D}'([0,T]\times\mathbb{R}^3)$ and $\nabla \otimes \boldsymbol{v} \in M^p_\gamma L^p(0,T)$. Then, the term ∇p is necessary related to $\boldsymbol{u}$ and $\nabla \otimes \boldsymbol{v}$ through the Riesz transforms $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula*

$$\nabla p = \nabla \left(\sum_{i,j=1}^{3} \mathcal{R}_i\mathcal{R}_j(u_i\, u_j) + \sum_{i,j,k=1}^{3} \mathcal{R}_i\mathcal{R}_j\left(\partial_i v_k\, \partial_j v_k\right) \right).$$

Proof. First, we define q given by the expression

$$q = \sum_{i,j=1}^{3} \mathcal{R}_i\mathcal{R}_j(u_i\, u_j) + \sum_{i,j,k=1}^{3} \mathcal{R}_i\mathcal{R}_j\left(\partial_i v_k\, \partial_j v_k\right), \tag{15}$$

where, for $9/4 < \delta < 3$ we have $q \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$. Indeed, since we have assumed $\mathbf{u} \in M^p_\gamma L^p(0,T)$ and $\nabla \otimes \mathbf{v} \in M^p_\gamma L^p(0,T)$, with $0 < \gamma < 3/2$, then by point 2) of Lemma 3.1 we get $\mathbf{u} \in L^p([0,T], L^p_{w_\delta}(\mathbb{R}^3))$

and $\nabla \otimes \mathbf{v} \in L^p([0,T], L^p_{w_\delta}(\mathbb{R}^3))$. With this information we are able to write $\mathbf{u}\otimes\mathbf{u} \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$ and $\nabla \otimes \mathbf{v} \odot \nabla \otimes \mathbf{v} \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$, and moreover, as by point 3) of Lemma 3.2 the operator $\mathcal{R}_i\mathcal{R}_i$ is bounded in $L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$, then we obtain $q \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$.

Now, we will prove the identity $\nabla p = \nabla q$. For this, let $\varepsilon > 0$ (small enough) and let $\alpha \in \mathcal{C}^\infty_0(\mathbb{R})$ be a function such that $\alpha(t) = 0$ for $|t| > \varepsilon$. Moreover, let $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^3)$. We may observe that we have $(\alpha\varphi) * \nabla p \in \mathcal{D}'(]\varepsilon, T-\varepsilon[\times\mathbb{R}^3)$ and $(\alpha\varphi) * \nabla q \in \mathcal{D}'(]\varepsilon, T-\varepsilon[\times\mathbb{R}^3)$ and then, for $t \in]\varepsilon, T-\varepsilon[$ fix we define $A_\varepsilon(t) = (\alpha\varphi) * \nabla p(t,\cdot) - (\alpha\varphi) * \nabla q(t,\cdot) \in \mathcal{D}'(\mathbb{R}^3)$, where we must verify that we have $A_\varepsilon(t) \in \mathcal{S}'(\mathbb{R}^3)$. We write $A_\varepsilon(t) = (\alpha\varphi) * \nabla p(t,\cdot) - (\alpha\nabla\varphi) * q(t,\cdot)$, and moreover, since $(\mathbf{u}, p, v)$ verify the coupled system (3) then we have

$$\nabla p = -\partial_t \mathbf{u} + \Delta \mathbf{u} - div(\mathbf{u}\otimes\mathbf{u}) - div(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v}),$$

and thus we obtain

$$\begin{aligned} A_\varepsilon(t) = & \left[(-(\partial_t\alpha)\varphi + \alpha\Delta\varphi) * \mathbf{u}\right](t,\cdot) - \left[(\alpha * \nabla\varphi) * (\mathbf{u}\otimes\mathbf{u})\right](t,\cdot) \\ & - \left[(\alpha * \nabla\varphi) * (\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})\right](t,\cdot) - \left[(\alpha\nabla\varphi) * q\right](t,\cdot). \end{aligned} \tag{16}$$

In this identity we will prove that each term in the right side belong to the space $L^{p/2}_{w_\delta}(\mathbb{R}^3)$ (where $9/4 < \delta < 3$). For the first term to the right in (16), recall that we have $\mathbf{u} \in L^p([0,T], L^p_{w_\delta}(\mathbb{R}^3))$. Moreover, since for $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^3)$ and a function f we have the pointwise estimate $|(\varphi * f)(x)| \leq c_\varphi \mathcal{M}_f(x)$ (where $\mathcal{M}$ always denote the Hardy-Littlewood maximal function operator) then by point 3) of Lemma 3.2 we obtain that convolution with test functions is a bounded operator on $L^p_{w_\delta}(\mathbb{R}^3)$. Thus, we have $\left[(-(\partial_t\alpha)\varphi + \alpha\Delta\varphi) * \mathbf{u}\right](t,\cdot) \in L^p_{w_\delta}(\mathbb{R}^3)$. On the other hand, for $9/4 < \delta < 3$ we have the continuous embedding $L^p_{w_\delta}(\mathbb{R}^3) \subset L^{p/2}_{w_\delta}(\mathbb{R}^3)$. Indeed, by definition of the weight $w_\delta(x)$ given by (14) and using the Cauchy-Schwarz inequalities we write

$$\int_{\mathbb{R}^3} |f|^{p/2} w_\delta dx = \int_{\mathbb{R}^3} |f|^{p/2} w_{3/4} w_{\delta-3/4} \leq \left(\int_{\mathbb{R}^3} |f|^p w_{3/2} dx\right)^{1/2} \left(\int_{\mathbb{R}^3} w_{2\delta-3/2} dx\right)^{1/2},$$

where, as we have $9/4 < \delta < 3$ then the last integral in the right side convergences. Thus we obtain $\left[(-(\partial_t\alpha)\varphi + \alpha\Delta\varphi) * \mathbf{u}\right](t,\cdot) \in L^{p/2}_{w_\delta}(\mathbb{R}^3)$.

For the second and third terms to the right in (16), recall that we have $\mathbf{u}\otimes\mathbf{u} \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$ and $\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v} \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$, hence, always by the fact that convolution with test functions is a bounded operator on the space $L^{p/2}_{w_\delta}(\mathbb{R}^3)$, we obtain $\left[(\alpha * \nabla\varphi) * (\mathbf{u}\otimes\mathbf{u})\right](t,\cdot) \in L^{p/2}_{w_\delta}(\mathbb{R}^3)$ and $\left[(\alpha * \nabla\varphi) * (\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})\right](t,\cdot) \in L^{p/2}_{w_\delta}(\mathbb{R}^3)$.

Finally, for the fourth term to the right in identity (16), as we have $q \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$ then we obtain $\left[(\alpha\nabla\varphi) * p\right](t,\cdot) \in L^{p/2}_{w_\delta}(\mathbb{R}^3)$.

Once we have this information, getting back to identity (16) we get $A_\varepsilon(t) \in L^{p/2}_{w_\delta}(\mathbb{R}^3)$ and then we have $A_\varepsilon(t) \in \mathcal{S}'(\mathbb{R}^3)$. On the other hand, since we have $div(\mathbf{u}) = 0$ then taking the divergence operator in the first equation of (3) we obtain $\Delta(p-q) = 0$. Then we have $\Delta A_\varepsilon(t) = 0$ and since $A_\varepsilon(t) \in \mathcal{S}'(\mathbb{R}^3)$ we get that $A_\varepsilon(t)$ is a polynomial. But, as we also have $A_\varepsilon(t) \in L^{p/2}_{w_\delta}(\mathbb{R}^3)$ then we necessary have $A_\varepsilon(t) = 0$. Finally, we use the approximation of the identity $\dfrac{1}{\varepsilon^4}\alpha\left(\dfrac{t}{\varepsilon}\right)\varphi\left(\dfrac{x}{\varepsilon}\right)$ to write $\nabla(p-q)(t,\cdot) = \lim\limits_{\varepsilon\to 0} A_\varepsilon(t) = 0$. ■

5 The stationary case: proof of Theorem 1

All our results stated in this theorem deeply base on the following local estimate, also know as a Caccioppoli-type estimate.

Proposition 5.1 *Let* $(\boldsymbol{u}, p, \boldsymbol{v})$ *be a smooth solution of the coupled system* (1). *Let* $3 \leq p < +\infty$. *If* $(\boldsymbol{u}, p) \in L^p_{loc}(\mathbb{R}^3)$ *and moreover, if* $\nabla \otimes \boldsymbol{v} \in L^p_{loc}(\mathbb{R}^3)$, *then there exists a constant* $c > 0$ *such that for all* $R \geq 1$ *we have:*

$$\begin{aligned}\int_{B_{R/2}} |\nabla \otimes \boldsymbol{u}|^2 dx \leq c \Bigg[\left(\int_{\mathcal{C}(R/2,R)} |\boldsymbol{u}|^p dx \right)^{2/p} + \left(\int_{\mathcal{C}(R/2,R)} |\nabla \otimes \boldsymbol{v}|^3 dx \right)^{2/p} \\ + \left(\int_{\mathcal{C}(R/2,R)} |p|^{p/2} dx \right)^{2/p} \Bigg] \times R^{2-9/p} \left(\int_{\mathcal{C}(R/2,R)} |\boldsymbol{u}|^p dx \right)^{1/p} + \frac{c}{R^2} \int_{\mathcal{C}(R/2,R)} |\boldsymbol{u}|^2 dx.\end{aligned} \tag{17}$$

Proof. We start by introducing the following cut-off function. Let $\theta \in \mathcal{C}^\infty_0(\mathbb{R}^3)$ be a positive and radial function such that $\theta(x) = 1$ for $|x| < 1/2$ and $\theta(x) = 0$ for $|x| \geq 1$. Then, for $R \geq 1$ we define the function

$$\theta_R(x) = \theta(x/R). \tag{18}$$

Remark that this function verifies the following properties: we have $\theta_R(x) = 1$ for $|x| < R/2$, $\theta_R(x) = 0$ for $|x| > R$, and moreover we have $\|\nabla \theta_R\|_{L^\infty} \leq \frac{c}{R}$ and $\|\Delta \theta_R\|_{L^\infty} \leq \frac{c}{R^2}$.

For $R \geq 1$, we multiply the first equation of the system (1) by $\theta_R \mathbf{u}$ and integrating on the ball B_R (since we have $supp(\theta_R) \subset B_R$) we obtain:

$$\begin{aligned}-\int_{B_R} \Delta \mathbf{u} \cdot \theta_R \mathbf{u} dx + \int_{B_R} \text{div}(\mathbf{u} \otimes \mathbf{u}) \cdot \theta_R \mathbf{u} dx + \int_{B_R} \text{div}(\nabla \otimes \mathbf{v} \odot \nabla \otimes \mathbf{v}) \cdot \theta_R \mathbf{u} dx \\ + \int_{B_R} \nabla p \cdot \theta_R \mathbf{u} dx = 0.\end{aligned} \tag{19}$$

Moreover, we multiply the second equation of the system (1) by $-\theta_R \Delta \mathbf{v}$, then we integrate on the ball B_R to get:

$$\int_{B_R} \Delta \mathbf{v} \cdot \theta_R \Delta \mathbf{v} dx - \int_{B_R} div(\mathbf{v} \otimes \mathbf{u}) \cdot \theta_R \Delta \mathbf{v} dx + \int_{B_R} |\nabla \otimes \mathbf{v}|^2 \mathbf{v} \cdot \theta_R \Delta \mathbf{v} dx = 0. \tag{20}$$

At this point remark that as $\mathbf{u}, P$ and $\mathbf{v}$ are smooth functions then all the terms in equations (19) and (20) are well-defined.

Now, we need to study each term in these equations. We start by equation (19). For the first term in the left-hand side, by integration by parts we have

$$\begin{aligned}-\int_{B_R} \Delta \mathbf{u} \cdot \theta_R \mathbf{u} dx &= -\sum_{i,j=1}^{3} \int_{B_R} (\partial_j^2 u_i)(\theta_R u_i) d = \sum_{i,j=1}^{3} \int_{B_R} \partial_j u_i \partial_j (\theta_R u_i) dx \\ &= \sum_{i,j=1}^{3} \int_{B_R} (\partial_j u_i)(\partial_j \theta_R) u_i dx + \sum_{i,j=1}^{3} \int_{B_R} (\partial_j u_i) \theta_R (\partial_j u_i) dx \\ &= \frac{1}{2} \sum_{i,j=1}^{3} \int_{B_R} (\partial_j \theta_R) \partial_j (u_i^2) dx + \int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx \\ &= -\frac{1}{2} \int_{B_R} |\mathbf{u}|^2 \Delta \theta_R dx + \int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx.\end{aligned}$$

For the second term in the left-hand side, by integration by parts and moreover, as we have $div(\mathbf{u})=0$, we can write

$$\int_{B_R} div(\mathbf{u}\otimes\mathbf{u})\cdot\theta_R\mathbf{u}dx = \sum_{i,j=1}^{3}\int_{B_R}\partial_j(u_iu_j)\theta_R u_i dx$$
$$= -\sum_{i,j=1}^{3}\int_{B_R} u_iu_j(\partial_j\theta_R)u_i dx - \sum_{i,j=1}^{3}\int_{B_R} u_iu_j\theta_R(\partial_j u_i)dx$$
$$= -\int_{B_R}|\mathbf{u}|^2(\mathbf{u}\cdot\nabla\theta_R)dx - \frac{1}{2}\sum_{i,j=1}^{3}\int_{B_R} u_j\theta_R\partial_j(u_i^2)dx$$
$$= -\int_{B_R}|\mathbf{u}|^2(\mathbf{u}\cdot\nabla\theta_R)dx + \frac{1}{2}\sum_{i,j=1}^{3}\int_{B_R}\partial_j(u_j\theta_R)u_i^2 dx$$
$$= -\int_{B_R}|\mathbf{u}|^2(\mathbf{u}\cdot\nabla\theta_R)dx + \frac{1}{2}\int_{B_R}(\mathbf{u}\cdot\nabla\theta_R)|\mathbf{u}|^2 dx$$
$$= -\frac{1}{2}\int_{B_R}|\mathbf{u}|^2(\mathbf{u}\cdot\nabla\theta_R)dx.$$

In order to study the third term in the left-hand side, we need the following technical identity:

$$div(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v}) = \nabla\left(\frac{1}{2}|\nabla\otimes\mathbf{v}|^2\right) + \Delta\mathbf{v}(\nabla\otimes\mathbf{v}).$$

Indeed, recall that for $i=1,2,3$ each component of the vector field $div(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})$ is given by

$$(div(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v}))_i = \sum_{j,k=1}^{3}\partial_j(\partial_i v_k\,\partial_j v_k) = \sum_{j,k=1}^{3}\partial_j(\partial_i v_k)\partial_j v_k + \sum_{j,k=1}^{3}\partial_i v_k\partial_j^2 v_k$$
$$= \sum_{j,k=1}^{3}\partial_i(\partial_j v_k)\partial_j v_k + \sum_{k=1}^{3}\partial_i v_k\,\Delta v_k = \partial_i\left(\frac{1}{2}\sum_{j,k=1}^{3}(\partial_j v_k)^2\right) + \sum_{k=1}^{3}\Delta v_k\,\partial_i v_k$$
$$= \partial_i\left(\frac{1}{2}|\nabla\otimes\mathbf{v}|^2\right) + (\Delta\mathbf{v}(\nabla\otimes\mathbf{v}))_i.$$

With this identity at hand, we get back to the third term in the left-hand side in (19) and, by integration by parts and the fact that $div(\mathbf{u})=0$, we write

$$\int_{B_R} div(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})\cdot\theta_R\mathbf{u}dx = \sum_{i=1}^{3}\int_{B_R}\partial_i\left(\frac{1}{2}|\nabla\otimes\mathbf{v}|^2\right)\theta_R U_i dx$$
$$+\sum_{i,j=1}^{3}\int_{B_R}\Delta V_j(\partial_i V_j)\theta_R u_i dx = -\frac{1}{2}\int_{B_R}|\nabla\otimes\mathbf{v}|^2(\mathbf{u}\cdot\nabla\theta_R)dx$$
$$+\sum_{i,j=1}^{3}\int_{B_R}\Delta v_j(\partial_i v_j)\theta_R u_i dx.$$

Finally, for the fourth term in the left-hand side in (19), always by integration by parts and since $div(\mathbf{u})=0$ we have

$$\int_{B_R}\nabla p\cdot\theta_R\mathbf{u}dx = \sum_{i=1}^{3}\int_{B_R}(\partial_i p)\theta_R u_i dx = -\int_{B_R} p(\mathbf{u}\cdot\nabla\theta_R)dx.$$

Once we dispose of these identities, we get back to equation (19) and thus we obtain

$$\begin{aligned}\int_{B_R} |\nabla\otimes\mathbf{u}|^2\theta_R dx &= \int_{B_R}\left(\frac{|\mathbf{u}|^2}{2}+\frac{|\nabla\otimes\mathbf{v}|^2}{2}+p\right)(\mathbf{u}\cdot\nabla\theta_R)dx\\ &+\frac{1}{2}\int_{B_R}|\mathbf{u}|^2\Delta\theta_R dx-\sum_{i,j=1}^{3}\int_{B_R}\Delta v_j(\partial_i v_j)\theta_R u_i dx.\end{aligned} \tag{21}$$

We study now the terms in the left-hand side in equation (20). For the first term we write directly

$$\int_{B_R}\Delta\mathbf{v}\cdot\theta_R\Delta\mathbf{v}dx=\int_{B_R}|\Delta\mathbf{v}|^2\theta_R dx.$$

For the second term, integrating by parts and as $div(\mathbf{u})=0$ then we get

$$\begin{aligned}-\int_{B_R} div(\mathbf{v}\otimes\mathbf{u})\cdot\theta_R\Delta\mathbf{v}dx &= -\sum_{i,j=1}^{3}\int_{B_R}\partial_j(v_iu_j)\theta_R\Delta v_i\\ =-\sum_{i,j=1}^{3}\int_{B_R}(\partial_j v_i)u_j\theta_R\Delta v_i &= -\sum_{i,j=1}^{3}\int_{B_R}\Delta v_i(\partial_j v_i)\theta_R u_j dx.\end{aligned}$$

For the third term we write

$$\int_{B_R}|\nabla\otimes\mathbf{v}|^2\mathbf{v}\cdot\theta_R\Delta\mathbf{v}dx=\sum_{i=1}^{3}\int_{B_R}|\nabla\otimes\mathbf{v}|^2v_i\theta_R\Delta v_i dx=\int_{B_R}|\nabla\otimes\mathbf{v}|^2(\mathbf{v}\cdot\Delta\mathbf{v})\theta_R dx.$$

Thus, with these identities at hand, from equation (20) we obtain:

$$\int_{B_R}|\Delta\mathbf{v}|^2\theta_R dx=\sum_{i,j=1}^{3}\int_{B_R}\Delta v_i(\partial_j v_i)\theta_R u_j dx-\int_{B_R}|\nabla\otimes\mathbf{v}|^2(\mathbf{v}\cdot\Delta\mathbf{v})\theta_R dx. \tag{22}$$

Now, adding the equations (21) and (22) we get

$$\begin{aligned}&\int_{B_R}|\nabla\otimes\mathbf{u}|^2\theta_R dx+\int_{B_R}|\Delta\mathbf{v}|^2\theta_R dx=\int_{B_R}\left(\frac{|\mathbf{u}|^2}{2}+\frac{|\nabla\otimes\mathbf{v}|^2}{2}+p\right)(\mathbf{u}\cdot\nabla\theta_R)dx\\ &+\int_{B_R}\frac{|\mathbf{u}|^2}{2}\Delta\theta_R dx\underbrace{-\sum_{i,j=1}^{3}\int_{B_R}\Delta v_j(\partial_i v_j)\theta_R u_i dx+\sum_{i,j=1}^{3}\int_{B_R}\Delta v_i(\partial_j v_i)\theta_R u_j dx}_{(a)}\\ &-\int_{B_R}|\nabla\otimes\mathbf{v}|^2(\mathbf{v}\cdot\Delta\mathbf{v})\theta_R dx,\end{aligned}$$

but, we may observe that we have $(a)=0$ and then we write

$$\begin{aligned}&\int_{B_R}|\nabla\otimes\mathbf{u}|^2\theta_R dx+\int_{B_R}|\Delta\mathbf{v}|^2\theta_R dx=\int_{B_R}\left(\frac{|\mathbf{u}|^2}{2}+\frac{|\nabla\otimes\mathbf{v}|^2}{2}+p\right)(\mathbf{u}\cdot\nabla\theta_R)dx\\ &+\int_{B_R}\frac{|\mathbf{u}|^2}{2}\Delta\theta_R dx-\int_{B_R}|\nabla\otimes\mathbf{v}|^2(\mathbf{v}\cdot\Delta\mathbf{v})\theta_R dx.\end{aligned}$$

Moreover, the last term is estimated as follows:

$$-\int_{B_R}|\nabla\otimes\mathbf{v}|^2(\mathbf{v}\cdot\Delta\mathbf{v})\theta_R dx\leq\int_{B_R}|\Delta\mathbf{v}|^2\theta_R dx.$$

Indeed, recall that by hypothesis we have $|\mathbf{v}|^2 = 1$ and then we get $\frac{1}{2}\Delta|\mathbf{v}|^2 = 0$. Thus, we can write

$$
\begin{aligned}
&-|\nabla \otimes \mathbf{v}|^2 = -\sum_{i,j=1}^{3} (\partial_i v_j)^2 = -\sum_{i,j=1}^{3} (\partial_i v_i)^2 = -\sum_{i,j=1}^{3} (\partial_i v_j)^2 + \frac{1}{2}\Delta|\mathbf{v}|^2 \\
&= -\sum_{i,j=1}^{3} (\partial_i v_j)^2 + \frac{1}{2}\sum_{i,j=1}^{2} \partial_i^2(v_i^2) - \sum_{i,j=1}^{3} (\partial_i v_j)^2 + \sum_{i,j=1}^{3} \partial_j \left(\frac{1}{2}\partial_i v_i^2\right) \\
&= -\sum_{i,j=1}^{3} (\partial_i v_j)^2 + \sum_{i,j=1}^{3} \partial_j (v_i \partial_j v_i) - \sum_{i,j=1}^{3} (\partial_i v_j)^2 + \sum_{i,j=1}^{3} (\partial_j v_i)^2 + \sum_{i,j=1}^{3} v_i \partial_j^2 v_i \\
&= \sum_{i,j=1}^{3} v_i \partial_j^2 v_i = \mathbf{v} \cdot \Delta \mathbf{v}.
\end{aligned}
\tag{23}
$$

With the identity $-|\nabla \otimes \mathbf{v}|^2 = \mathbf{v} \cdot \Delta \mathbf{v}$ at hand, and moreover, as we have $\theta_R \geq 0$ and as we have $|\mathbf{v}|^2 = 1$, we obtain

$$
-\int_{B_R} |\nabla \otimes \mathbf{v}|^2 (\mathbf{v} \cdot \Delta \mathbf{v}) \theta_R dx = \int_{B_R} |\mathbf{v} \cdot \Delta \mathbf{v}|^2 \theta_R dx \leq \int_{B_R} |\mathbf{v}|^2 |\Delta \mathbf{v}|^2 \theta_R dx \leq \int_{B_R} |\Delta \mathbf{v}|^2 \theta_R dx.
$$

Once we have this estimate then we can write

$$
\begin{aligned}
&\int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx + \int_{B_R} |\Delta \mathbf{v}|^2 \theta_R dx \leq \int_{B_R} \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\nabla \otimes \mathbf{v}|^2}{2} + p \right) (\mathbf{u} \cdot \nabla \theta_R) dx \\
&+ \int_{B_R} \frac{|\mathbf{u}|^2}{2} \Delta \theta_R dx + \int_{B_R} |\Delta \mathbf{v}|^2 \theta_R dx,
\end{aligned}
$$

hence we get

$$
\int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx \leq \int_{B_R} \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\nabla \otimes \mathbf{v}|^2}{2} + p \right) (\mathbf{u} \cdot \nabla \theta_R) dx + \int_{B_R} \frac{|\mathbf{u}|^2}{2} \Delta \theta_R dx.
$$

Recalling that we have $\theta_R(x) = 1$ for $|x| < R/2$, then we obtain

$$
\int_{B_{R/2}} |\nabla \otimes \mathbf{u}|^2 dx \leq \int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx,
$$

and from the previous inequality we are able to write

$$
\int_{B_{R/2}} |\nabla \otimes \mathbf{u}|^2 dx \leq \int_{B_R} \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\nabla \otimes \mathbf{v}|^2}{2} + p \right) (\mathbf{u} \cdot \nabla \theta_R) dx + \int_{B_R} \frac{|\mathbf{u}|^2}{2} \Delta \theta_R dx.
$$

Moreover, recalling that we have $supp(\nabla \theta_R) \subset \mathcal{C}(R/2, R)$ and $supp(\Delta \theta_R) \subset \mathcal{C}(R/2, R)$, then we obtain the following estimate

$$
\begin{aligned}
\int_{B_{R/2}} |\nabla \otimes \mathbf{u}|^2 dx &\leq \int_{\mathcal{C}(R/2,R)} \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\nabla \otimes \mathbf{v}|^2}{2} + p \right) (\mathbf{u} \cdot \nabla \theta_R) dx + \int_{\mathcal{C}(R/2,R)} \frac{|\mathbf{u}|^2}{2} \Delta \theta_R dx \\
&\leq \int_{\mathcal{C}(R/2,R)} \frac{|\mathbf{u}|^2}{2} (\mathbf{u} \cdot \nabla \theta_R) dx + \int_{\mathcal{C}(R/2,R)} \frac{|\nabla \otimes \mathbf{v}|^2}{2} (\mathbf{u} \cdot \nabla \theta_R) dx \\
&+ \int_{\mathcal{C}(R/2,R)} p (\mathbf{u} \cdot \nabla \theta_R) dx + \int_{\mathcal{C}(R/2,R)} \frac{|\mathbf{u}|^2}{2} \Delta \theta_R dx = I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{24}
$$

From this estimate we will derive the desired inequality (17) and for this we will study each term I_i for $i=1,\cdots 4$. For the term I_1, by the Hölder inequalities (with $1=2/p+1/q$) and moreover, as we have $\|\nabla \theta_R\|_{L^\infty}\leq c/R$, we get

$$\begin{aligned} I_1 &\leq \int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 |\mathbf{u}\cdot \nabla \theta_R| dx \leq \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{2/p} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}\cdot \nabla \theta_R|^q dx\right)^{1/q} \\ &\leq \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{2/p} \frac{c}{R}\left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^q dx\right)^{1/q}. \end{aligned}$$

But, since we have $3\leq p<+\infty$ and $1=2/p+1/q$ then the parameter q verifies $q\leq 3\leq p$ and thus, for the last expression we can write

$$\frac{c}{R}\left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^q dx\right)^{1/q} \leq \frac{c}{R} R^{3(1/q-1/p)} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{1/p} \leq c\, R^{2-9/p} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{1/p},$$

hence we have

$$I_1 \leq c\left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{2/p} R^{2-9/p} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{1/p}. \tag{25}$$

Following the same computations, the terms I_2 and I_3 are estimated as follows:

$$I_2 \leq c\left(\int_{\mathcal{C}(R/2,R)} |\nabla\otimes \mathbf{v}|^p dx\right)^{2/p} R^{2-9/p} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{1/p}, \tag{26}$$

and

$$I_3 \leq c\left(\int_{\mathcal{C}(R/2,R)} |p|^{p/2} dx\right)^{2/p} R^{2-9/p} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{1/p}. \tag{27}$$

Finally, for the term I_4, always by the Hölder inequalities, with $1=2/p+1/q$, by the fact that $\|\Delta \theta_R\|_{L^\infty}\leq c/R^2$ we obtain

$$I_4 \leq c\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 |\Delta \theta_R| dx \leq \frac{c}{R^2}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 dx.$$

With the estimates, we get back to the inequality (24) to obtain the desired estimate (17). ∎

From now on, let $(\mathbf{u},p,\mathbf{v})$ be a smooth solution of (1) such that $\mathbf{v}$ verifies (5). Recall that in the statement of Theorem 1 we have $\mathbf{u}\in M^3_{\gamma,0}(\mathbb{R}^3)$ and $\nabla\otimes\mathbf{v}\in M^3_\gamma(\mathbb{R}^3)$, with $1\leq\gamma<3/2$. In particular, by definition of the local Morrey spaces $M^3_{\gamma,0}(\mathbb{R}^3)$ and $M^3_\gamma(\mathbb{R}^3)$ given in (8) and (7) respectively, we have $\mathbf{u}\in L^3_{loc}(\mathbb{R}^3)$ and $\nabla\otimes\mathbf{v}\in L^3_{loc}(\mathbb{R}^3)$ and thus, setting $p=3$, by Proposition 5.1 we can write the following local estimate for all $R\geq 1$:

$$\begin{aligned} \int_{B_{R/2}} |\nabla\otimes \mathbf{u}|^2 dx \leq& \frac{c}{R^2}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 dx + c\left[\left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{2/3} + \left(\int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^3 dx\right)^{2/3}\right. \\ &\left. + \left(\int_{\mathcal{C}(R/2,R)} |p|^{3/2} dx\right)^{2/3}\right] \times \frac{1}{R}\left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{1/3}. \end{aligned}$$

Then, for $1 \leq \gamma < 3/2$ we write

$$\begin{aligned}
\int_{B_{R/2}} |\nabla \otimes \mathbf{u}|^2 dx &\leq \frac{c}{R^2}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 dx + \frac{c}{R^{\frac{2}{3}\gamma}}\left[\left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{2/3} + \left(\int_{\mathcal{C}(R/2,R)} |\nabla \otimes \mathbf{v}|^3 dx\right)^{2/3}\right.\\
&\quad \left. + \left(\int_{\mathcal{C}(R/2,R)} |P|^{3/2} dx\right)^{2/3}\right] \times R^{\frac{2}{3}\gamma - 1}\left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{1/3}\\
&= \frac{c}{R^2}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 dx + c\left[\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{2/3} + \left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |\nabla \otimes \mathbf{v}|^3 dx\right)^{2/3}\right.\\
&\quad \left. + \left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |p|^{3/2} dx\right)^{2/3}\right] \times R^{\gamma - 1}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{1/3}.
\end{aligned}$$

At this point, remark that by Proposition 4.1 the pressure term p writes down as in formula (15), and moreover, as we $\mathbf{u} \in M^3_{\gamma,0}(\mathbb{R}^3)$ and $\nabla \otimes \mathbf{v} \in M^3_\gamma(\mathbb{R}^3)$ then, by point 1) of Lemma 3.2 we can write

$$\|p\|_{M^{3/2}_\gamma} \leq c\left(\|\mathbf{u}\|^2_{M^3_\gamma} + \|\nabla \otimes \mathbf{v}\|^2_{M^3_\gamma}\right). \tag{28}$$

Thus, getting back to the previous estimate we get

$$\begin{aligned}
\int_{B_{R/2}} |\nabla \otimes \mathbf{u}|^2 dx \leq \frac{c}{R^2}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 dx + c\left(\|\mathbf{u}\|^2_{M^3_\gamma} + \|\nabla \otimes \mathbf{v}\|^2_{M^3_\gamma}\right)&\\
\times R^{\gamma-1}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{1/3}&,
\end{aligned} \tag{29}$$

where, we shall study study each term in the right-hand side. For the first term, as $1 \leq \gamma < 3/2$ we have

$$\lim_{R \to +\infty} \frac{c}{R^2}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 dx = 0. \tag{30}$$

Indeed, we write

$$\frac{c}{R^2}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 dx \leq c\, R^{-1}\left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3\right)^{2/3} \leq c\, R^{\frac{2}{3}\gamma-1}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p\right)^{2/3} \leq c\, R^{\frac{2}{3}\gamma-1}\|\mathbf{u}\|_{M^3_\gamma}.$$

But, as $1 \leq \gamma < 3/2$ then we get $\frac{2}{3}\gamma - 1 < 0$, hence (30) follows.

In order to study the second term in the estimate above we will distinguish two cases for the parameter γ.

1) **The case** $\gamma = 1$. As we have $\mathbf{u} \in M^3_{1,0}(\mathbb{R}^3)$, then we can write $\displaystyle\lim_{R \to +\infty}\left(\frac{1}{R}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{1/3} = 0.$

   Thus, we obtain

$$\lim_{R \to \infty} c\left(\|\mathbf{u}\|^2_{M^3_1} + \|\nabla \otimes \mathbf{v}\|^2_{M^3_1}\right)\left(\frac{1}{R}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{1/3} = 0.$$

   Then, we back to (29) and taking the limit when $R \to +\infty$, we obtain $\displaystyle\int_{\mathbb{R}^3} |\nabla \otimes \mathbf{u}|^2 dx = 0$ and thus $\mathbf{u}$ is a constant vector. But, as $\mathbf{u} \in M^3_{1,0}(\mathbb{R}^3)$ we have $\mathbf{u} = 0$.

2) **The case** $1 < \gamma < 3/2$**.** Recall that by (9) we have $\lim_{R\to+\infty} R^{\gamma-1}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{1/3} = 0$. Then we obtain

$$\lim_{R\to+\infty} c\left(\|\mathbf{u}\|^2_{M^3_\gamma} + \|\nabla\otimes\mathbf{v}\|^2_{M^3_\gamma}\right) R^{\gamma-1}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{1/3} = 0,$$

hence, following the same ideas of previous case we conclude that $\mathbf{u} = 0$.

Until now we have proven that $\mathbf{u} = 0$ and then it remains to prove the identities $\nabla\otimes\mathbf{v} = 0$ and $p = 0$. We start by proving that $\nabla\otimes\mathbf{v} = 0$. As $\mathbf{u} = 0$ then by (1) we have that $\mathbf{v}$ solves the following elliptic equation

$$-\Delta\mathbf{v} - |\nabla\otimes\mathbf{v}|^2\mathbf{v} = 0.$$

In this equation, we multiply by $\theta_R((x\cdot\nabla)\mathbf{v})$, where for $R \geq 1$ the cut-off function $\theta_R(x)$ was defined in (18), and integrating on the ball B_R by [15], page 6, we have the following local estimate:

$$\int_{B_{R/2}} |\nabla\otimes\mathbf{v}|^2 dx \leq c\int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^2 dx. \tag{31}$$

Now, recall that $\mathbf{v}$ verifies (5) and then we have

$$\int_{B_{R/2}} |\nabla\otimes\mathbf{v}|^2 dx \leq c\sup_{R\geq 1}\int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^2 dx < +\infty,$$

hece we obtain $\int_{\mathbb{R}^3} |\nabla\otimes\mathbf{v}|^2 dx < +\infty$. With this information, we get back to (31) and taking the limit when $R\to+\infty$ we get $\int_{\mathbb{R}^3} |\nabla\otimes\mathbf{v}|^2 dx = 0$. Hence we have $\nabla\otimes\mathbf{v}$. Once we have the identities $\mathbf{u} = 0$ and $\nabla\otimes\mathbf{v} = 0$, the identity $p = 0$ follows directly from the estimate (28). Theorem 1 is proven. ■

6 The non-stationary case

6.1 Proof of Theorem 2

We will apply the local energy balance (10) to a suitable test function and for this we will follow some of the ideas of [11]. Let $0 < t_0 < t_1 < T$. For a parameter $\varepsilon > 0$, we will consider a function $\alpha_{\varepsilon,t_0,t_1}(t)$ which converges $a.e.$ to $\mathbb{1}_{[t_0,t_1]}(t)$ and such that $\frac{d}{dt}\alpha_{\varepsilon,t_0,t_1}(t)$ is the difference between two identity approximations: the first one in t_0 and the second one in t_1. For this, let $\alpha\in\mathcal{C}^\infty(\mathbb{R})$ be a function such that $\alpha(t) = 0$ for $-\infty < t < 1/2$ and $\alpha(t) = 1$ for $1 < t < +\infty$. Then, for $\varepsilon < \min(t_0/2, T-t_1)$ we set the function $\alpha_{\varepsilon,t_0,t_1}(t) = \alpha\left(\frac{t-t_0}{\varepsilon}\right) - \alpha\left(\frac{t-t_1}{\varepsilon}\right)$. On the other hand, for $R\geq 1$ let $\theta_R(x)$ be function test given in (18). Thus, we consider the function test $\alpha_{\varepsilon,t_0,t_1}(t)\theta_R(x)$ and by (10) we can write

$$\begin{aligned}
&-\int_{\mathbb{R}}\int_{\mathbb{R}^3} \frac{|\mathbf{u}|^2 + |\nabla\otimes\mathbf{v}|^2}{2}\partial_s\alpha_{\varepsilon,t_0,t_1}\theta_R\, dx\, ds + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |\nabla\otimes\mathbf{u}|^2\alpha_{\varepsilon,t_0,t_1}\theta_R dx\, ds + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |\Delta\mathbf{v}|^2\alpha_{\varepsilon,t_0,t_1}\theta_R dx\, ds\\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}^3}\left(\frac{|\mathbf{u}|^2 + |\nabla\otimes\mathbf{v}|^2}{2}\right)\alpha_{\varepsilon,t_0,t_1}\Delta\theta_R dx\, ds + \int_{\mathbb{R}}\int_{\mathbb{R}^3}\left(\left[\frac{|\mathbf{u}|^2 + |\nabla\otimes\mathbf{v}|^2}{2} + p\right]\mathbf{u}\right)\cdot\alpha_{\varepsilon,t_0,t_1}\nabla\theta_R dx\, ds\\
&\int_{\mathbb{R}}\int_{\mathbb{R}^3}\sum_{k=1}^{3}([\mathbf{u}\cdot\nabla)\,\mathbf{v}]\cdot\partial_k\mathbf{v})\alpha_{\varepsilon,t_0,t_1}\partial_k\theta_R dx\, ds - \int_{\mathbb{R}}\int_{\mathbb{R}^3} |\nabla\otimes\mathbf{v}|^2\mathbf{v}\cdot\Delta\mathbf{v}\,\alpha_{\varepsilon,t_0,t_1}\theta_R dx\, ds.
\end{aligned}$$

Now, taking the limit when $\varepsilon \to 0$, by the dominated convergence theorem we obtain (when the limit in the left side is well-defined)

$$
\begin{aligned}
&-\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\partial_s\alpha_{\varepsilon,t_0,t_1}\theta_R\,dx\,ds+\int_{t_0}^{t_1}\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{u}|^2\theta_R dx\,ds+\int_{t_0}^{t_1}\int_{\mathbb{R}^3}|\Delta\mathbf{v}|^2\theta_R dx\,ds\\
&\leq\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\left(\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\right)\Delta\theta_R dx\,ds+\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\left(\left[\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}+p\right]\mathbf{u}\right)\cdot\nabla\theta_R dx\,ds\\
&\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\sum_{k=1}^{3}([\mathbf{u}\cdot\nabla)\,\mathbf{v}]\cdot\partial_k\mathbf{v})\partial_k\theta_R dx\,ds-\int_{t_0}^{t_1}\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{v}|^2\mathbf{v}\cdot\Delta\mathbf{v}\,\theta_R dx\,ds.
\end{aligned}
$$

At this point, we must study the expression $-\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\partial_s\alpha_{\varepsilon,t_0,t_1}\theta_R\,dx\,ds$. To make the writing more simple, let us define the function $A_R(s)=\int_{\mathbb{R}^3}\frac{|\mathbf{u}(s,x)|^2+|\nabla\otimes\mathbf{v}(s,x)|^2}{2}\theta_R\,dx$. Then, assuming that t_0 and t_1 are Lebesgue points of the function $A_R(s)$, and moreover, since

$$
\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\partial_s\alpha_{\varepsilon,t_0,t_1}\theta_R\,dx\,ds=-\frac{1}{2}\int_{\mathbb{R}}A_R(s)\partial_s\alpha_{\varepsilon,t_0,t_1}ds,
$$

then we have

$$
-\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\partial_s\alpha_{\varepsilon,t_0,t_1}\theta_R\,dx\,ds=\frac{1}{2}(A_R(t_1)-A_R(t_0)).
$$

On the other hand, recall that by point 4) in Definition 2.1 we have that the functions $\mathbf{u}(t,\cdot)$ and $\nabla\otimes\mathbf{v}(t,\cdot)$ are strong continuous at $t=0$ and then we can replace t_0 by 0. Moreover, for $0<t<T$, always by point 4) in Definition 2.1 we have that the functions $\mathbf{u}(t,\cdot)$ and $\nabla\otimes\mathbf{v}(t,\cdot)$ are weak continuous at t and then we obtain $A_R(t)\leq\liminf_{t_1\to t}A_R(t_1)$. Thus, we can also replace t_1 for t.

With this information, for every $0\leq t\leq T$ we can write

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\frac{|\mathbf{u}(t,\cdot)|^2+|\nabla\otimes\mathbf{v}(t,\cdot)|^2}{2}\theta_R\,dx+\int_0^t\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{u}|^2\theta_R dx\,ds+\int_0^t\int_{\mathbb{R}^3}|\Delta\mathbf{v}|^2\theta_R dx\,ds\\
&\leq\int_{\mathbb{R}^3}\frac{|\mathbf{u}_0|^2+|\nabla\otimes\mathbf{v}_0|^2}{2}\theta_R\,dx+\int_0^t\int_{\mathbb{R}^3}\left(\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\right)\Delta\theta_R dx\,ds\\
&+\int_0^t\int_{\mathbb{R}^3}\left(\left[\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}+p\right]\mathbf{u}\right)\cdot\nabla\theta_R dx\,ds+\int_0^t\int_{\mathbb{R}^3}\sum_{k=1}^{3}([\mathbf{u}\cdot\nabla)\,\mathbf{v}]\cdot\partial_k\mathbf{v})\partial_k\theta_R dx\,ds\\
&-\int_0^t\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{v}|^2\mathbf{v}\cdot\Delta\mathbf{v}\,\theta_R dx\,ds.
\end{aligned}
$$

In this inequality we must study now the term $-\int_0^t\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{v}|^2\mathbf{v}\cdot\Delta\mathbf{v}\theta_R dx\,ds$. Recall that by (23) we have the identity (in the distributional sense) $|\nabla\otimes\mathbf{v}|^2=-\mathbf{v}\cdot\Delta\mathbf{v}$, moreover, as we have $|\mathbf{v}|=1$, then we can write

$$
-\int_0^t\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{v}|^2\mathbf{v}\cdot\Delta\mathbf{v}\theta_R dx\,ds=\int_0^t\int_{\mathbb{R}^3}|\mathbf{v}\cdot\Delta\mathbf{v}|^2\theta_R dx\,ds\leq\int_0^t\int_{\mathbb{R}^3}|\mathbf{v}|^2|\Delta\mathbf{v}|^2\theta_R dx\,ds\leq\int_0^t\int_{\mathbb{R}^3}|\Delta\mathbf{v}|^2\theta_R dx\,ds.
$$

By this estimate and the previous inequality we get

$$\int_{\mathbb{R}^3} \frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\theta_R\,dx + \int_0^t\int_{\mathbb{R}^3} |\nabla\otimes\mathbf{u}|^2\theta_R dx\,ds \leq \int_{\mathbb{R}^3} \frac{|\mathbf{u}_0|^2+|\nabla\otimes\mathbf{v}_0|^2}{2}\theta_R\,dx$$
$$+\int_0^t\int_{\mathbb{R}^3}\left(\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\right)\Delta\theta_R dx\,ds + \int_0^t\int_{\mathbb{R}^3}\left(\left[\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}+p\right]\mathbf{u}\right)\cdot\nabla\theta_R dx\,ds$$
$$-\int_0^s\int_{\mathbb{R}^3}\sum_{k=1}^{3}\partial_k([\mathbf{u}\cdot\nabla)\,\mathbf{v}]\cdot\partial_k\mathbf{v})\theta_R dx\,ds.$$

Now, as we have $u_0\in L^2(\mathbb{R}^3)$ and $\mathbf{v}_0\in\dot{H}^1(\mathbb{R}^3)$, and moreover, recalling that $supp(\theta_R)\subset B_R$, $supp(\nabla\theta_R)\subset \mathcal{C}(R/2,R)$ and $supp(\Delta\theta_R)\subset\mathcal{C}(R/2,R)$, then we write

$$\int_{B_R} \frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\theta_R\,dx + \int_0^t\int_{B_R} |\nabla\otimes\mathbf{u}|^2\theta_R dx\,ds \leq \|u_0\|^2_{L^2}+\|\mathbf{v}_0\|^2_{\dot{H}^1}$$
$$+\int_0^t\int_{\mathcal{C}(R/2,R)}\left(\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\right)\Delta\theta_R dx\,ds + \int_0^t\int_{\mathcal{C}(R/2,R)}\left(\left[\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}+p\right]\mathbf{u}\right)\nabla\theta_R dx\,ds$$
$$+\int_0^t\int_{\mathcal{C}(R/2,R)}\sum_{k=1}^{3}([\mathbf{u}\cdot\nabla)\,\mathbf{v}]\cdot\partial_k\mathbf{v})\partial_k\theta_R dx\,ds$$
$$= \|u_0\|^2_{L^2}+\|\mathbf{v}_0\|^2_{\dot{H}^1}+I_1+I_2+I_3, \tag{32}$$

where we will show that we have $\lim_{R\to+\infty} I_i=0$ for $i=1,2,3$. Indeed, for the term I_1 recall that we have $\|\Delta\theta_R\|_{L^\infty}\leq\frac{c}{R^2}$, and the we get

$$I_1\leq\frac{c}{R^2}\int_0^t\int_{\mathcal{C}(R/2,R)}(|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2)dx\,ds\leq\frac{c}{R}\int_0^t\left(\int_{\mathcal{C}(R/2,R)}(|\mathbf{u}|^3+|\nabla\otimes\mathbf{v}|^3)dx\right)^{2/3}ds,$$

thereafter, by the HÏder inequalities in the temporal variable (with $2/3+1/3=1$) we have

$$\frac{c}{R}\int_0^t\left(\int_{\mathcal{C}(R/2,R)}(|\mathbf{u}|^3+|\nabla\otimes\mathbf{v}|^3)dx\right)^{2/3}ds\leq\frac{c}{R}\left(\int_0^t\int_{\mathcal{C}(R/2,R)}(|\mathbf{u}|^3+|\nabla\otimes\mathbf{v}|^3)dx\,ds\right)^{2/3}t^{1/3}$$
$$\leq c\frac{t^{1/3}}{R^{1/3}}\left(\frac{1}{R}\int_0^t\int_{\mathcal{C}(R/2,R)}(|\mathbf{u}|^3+|\nabla\otimes\mathbf{v}|^3)dx\,ds\right)^{2/3}$$
$$\leq c\frac{T^{1/3}}{R^{1/3}}\left(\frac{1}{R}\int_0^T\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^3dx\,ds\right)^{2/3}+c\frac{T^{1/3}}{R^{1/3}}\left(\frac{1}{R}\int_0^T\int_{\mathcal{C}(R/2,R)}|\nabla\otimes\mathbf{v}|^3dx\,ds\right)^{2/3}$$
$$\leq cT^{1/3}\left(\frac{1}{R}\int_0^T\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^3dx\,ds\right)^{2/3}+c\frac{T^{1/3}}{R^{1/3}}\left(\frac{1}{R}\int_0^T\int_{B_R}|\nabla\otimes\mathbf{v}|^3dx\,ds\right)^{2/3}$$
$$\leq cT^{1/3}\left(\frac{1}{R}\int_0^T\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^3dx\,ds\right)^{2/3}+c\frac{T^{1/3}}{R^{1/3}}\|\nabla\otimes\mathbf{v}\|^2_{M^3_1L^3(0,T)},$$

hence we finally write

$$I_1\leq cT^{1/3}\left(\frac{1}{R}\int_0^T\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^3dx\,dt\right)^{2/3}+c\frac{T^{1/3}}{R^{1/3}}\|\nabla\otimes\mathbf{v}\|^2_{M^3_1L^3(0,T)}.$$

But, as we have the information $\mathbf{u} \in M^3_{1,0}L^3(0,T)$ and $\nabla \otimes \mathbf{v} \in M^3_1 L^3(0,T)$, taking the limit when $R \to +\infty$ we obtain $\displaystyle\lim_{R\to+\infty} I_1 = 0$.

For the term I_2, by estimates (25), (26) and (27) (with $p=3$), we have

$$\begin{aligned} I_2 &\leq \int_0^t \left(\frac{1}{R}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{2/3} \left(\frac{1}{R}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{1/3} ds \\ &\quad + \int_0^t \left(\frac{1}{R}\int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^3 dx\right)^{2/3} \left(\frac{1}{R}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{1/3} ds \\ &\quad + \int_0^t \left(\frac{1}{R}\int_{\mathcal{C}(R/2,R)} |p|^{3/2} dx\right)^{2/3} \left(\frac{1}{R}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\right)^{1/3} ds, \end{aligned}$$

and then, applying the Hölder inequalities in the temporal variable (with $2/3 + 1/3 = 1$) in each term to the right side we obtain

$$\begin{aligned} I_2 &\leq \left(\frac{1}{R}\int_0^t\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\, ds\right)^{2/3} \left(\frac{1}{R}\int_0^t\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\, ds\right)^{1/3} \\ &\quad + \left(\frac{1}{R}\int_0^t\int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^3 dx\, ds\right)^{2/3} \left(\frac{1}{R}\int_0^t\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\, ds\right)^{1/3} \\ &\quad + \left(\frac{1}{R}\int_0^t\int_{\mathcal{C}(R/2,R)} |p|^{3/2} dx\, ds\right)^{2/3} \left(\frac{1}{R}\int_0^t\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\, ds\right)^{1/3}. \end{aligned}$$

At this point, recall that always by Proposition 4.1 the pressure term p writes down as in formula (15), and moreover, as we have $\mathbf{u} \in M^3_{1,0}L^3(0,T)$ and $\nabla \otimes \mathbf{v} \in M^3_1 L^3(0,T)$, then by point 1) of Lemma 3.2 we get

$$\|p\|_{M^{3/2}_1 L^{3/2}(0,T)} \leq c\left(\|\mathbf{u}\|^2_{M^3_1 L^3(0,T)} + \|\nabla\otimes\mathbf{v}\|^2_{M^3_1 L^3(0,T)}\right). \tag{33}$$

Thus, getting back to the previous estimate we can write

$$\begin{aligned} I_2 &\leq \left(\|\mathbf{u}\|^2_{M^3_1 L^3(0,T)} + \|\nabla\otimes\mathbf{v}\|^2_{M^3_1 L^3(0,T)} + \|p\|_{M^{3/2}_1 L^{3/2}(0,T)}\right) \left(\frac{1}{R}\int_0^T\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\, ds\right)^{1/3} \\ &\leq c\left(\|\mathbf{u}\|^2_{M^3_1 L^3(0,T)} + \|\nabla\otimes\mathbf{v}\|^2_{M^3_1 L^3(0,T)}\right) \left(\frac{1}{R}\int_0^T\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^3 dx\, ds\right)^{1/3}, \end{aligned}$$

and then, taking the limit when $R \to +\infty$ we have $\displaystyle\lim_{R\to+\infty} I_2 = 0$.

Finally, for the term I_3, as we have $\|\nabla\theta_R\|_{L^\infty} \leq \dfrac{c}{R}$, and moreover, applying the Hölder inequalities first in the spatial variable ans thereafter in the temporal variable (both with $2/3 + 1/3 = 1$), we write

$$
\begin{aligned}
I_3 &= \sum_{i,j,k=1}^{3} \int_0^t \int_{\mathcal{C}(R/2,R)} (u_j \partial_j v_i)(\partial_k v_i) \partial_k \theta_R dx\, ds \leq c \int_0^t \int_{\mathbb{C}(R/2,R)} |\mathbf{u}| |\nabla \otimes \mathbf{v}|^2 |\nabla \theta_R| dx\, ds \\
&\leq \frac{c}{R} \int_0^t \left(\int_{\mathbb{C}(R/2,R)} |\mathbf{u}|^3 dx \right)^{1/3} \left(\int_{\mathbb{C}(R/2,R)} |\nabla \otimes \mathbf{v}|^3 dx \right)^{2/3} ds \\
&\leq c \int_0^t \left(\frac{1}{R} \int_{\mathbb{C}(R/2,R)} |\mathbf{u}|^3 dx \right)^{1/3} \left(\frac{1}{R} \int_{\mathbb{C}(R/2,R)} |\nabla \otimes \mathbf{v}|^3 dx \right)^{2/3} ds \\
&\leq c \left(\frac{1}{R} \int_0^t \int_{\mathbb{C}(R/2,R)} |\mathbf{u}|^3 dx\, ds \right)^{1/3} \left(\frac{1}{R} \int_0^t \int_{\mathbb{C}(R/2,R)} |\nabla \otimes \mathbf{v}|^3 dx\, ds \right)^{2/3} \\
&\leq c \left(\frac{1}{R} \int_0^T \int_{\mathbb{C}(R/2,R)} |\mathbf{u}|^3 dx\, ds \right)^{1/3} \|\nabla \otimes \mathbf{v}\|^2_{M^3_1 L^3(0,T)}.
\end{aligned}
$$

Hence, taking the limit when $R \to +\infty$ we obtain $\lim\limits_{R\to+\infty} I_3 = 0$.

Once we have proven that $\lim\limits_{R\to+\infty} I_i = 0$ for $i = 1, 2, 3$, we get back to (32) where we take the limit when $R \to +\infty$, and thus for $0 \leq t \leq T$ we get the global energy inequality (13). Theorem 2 is proven. $\blacksquare$

6.2 Proof of Corollary 1

This proof is straightforward. Just observe that by the global energy inequality (13) if the initial datum verify $\mathbf{u}_0 = 0$ and $\nabla \otimes \mathbf{v}_0 = 0$ then for all time $0 < t \leq T$ we have $\|\mathbf{u}(t,\cdot)\|^2_{L^2} = 0$ and $\|\mathbf{v}(t,\cdot)\|^2_{\dot{H}^1} = 0$, hence $\mathbf{u} = 0$ and $\nabla \otimes \mathbf{v} = 0$ on $[0,T] \times \mathbb{R}^3$. Thereafter, by estimate (33) we also have $p = 0$ on $[0,T] \times \mathbb{R}^3$. $\blacksquare$

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