q-Hausdorff Summability

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Abstract

We define a q-analog of Cesàro summability and we then construct a class of q-Hausdorff matrices. We define a type of q-difference for sequences and a q-analog of Bernstein polynomials. Using these concepts we define a q-moment problem and relate this moment problem to q-Hausdorff summability.

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1 Introduction

If (z_n) is a sequence of complex numbers then the Cesàro mean (σ_n) is defined by

$$\sigma_n = \frac{z_0 + z_1 + ... + z_n}{n+1}, n = 0, 1, 2, ... \tag{1}$$

If $\lim_{n\to\infty} \sigma_n = \sigma$ then the sequence (z_n) is said to be Cesàro summable to the limit σ. It is also said that (z_n) is summable by the Cesàro means of first order, or is summable $(C, 1)$. This is because the Cesàro mean as defined in (1) belongs to a family of summability methods (C, α) where $\alpha \geqslant 0$. We will speak of these more general Cesàro means subsequently. The first order means (1) have played an important role in analysis. Arguably the most famous application of $(C, 1)$ summability is the classic result of L. Fejér in which he proved that the Cesàro means of the Fourier series of a continuous function converge uniformly. This beautiful theorem may be found in most books on Fourier series. The subject of summability methods was a major research topic in the first half of the twentieth century, an excellent reference to this work is provided by G.H. Hardy's classic book Divergent Series [6].

The last thirty years has seen a remarkable production of research involving q-series and q-differences (cf. [5]). This q-analysis has deep roots going back to Euler. The development of the theory of Askey-Wilson polynomials was a primary catalyst in the current interest in the subject. One of the thrusts in this research has been aimed at finding suitable q -analogs of functions and processes belonging to classical function theory. For example in [1] and [3] first steps were taken in the development of a Fourier theory involving certain q-analogs of trigonometric functions. A complete development of a q-Fourier theory must include a suitable summability theory. In this paper we will take a preliminary step by introducing a q-analog of Cesàro summability and linking it to a q-version of Hausdorff summability.

For the sake of completeness we will make some definitions and fix some notation used in the q-calculus. The standard reference on such things is the book by G. Gasper and M. Rahman [5]. We will always assume that $0 < q < 1$. First, we define the q-coefficient $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$. The infinite version of this product is defined by $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$. The q-binomial coefficient is defined by $\begin{bmatrix} p \\ s \end{bmatrix} = \frac{(q;q)_p}{(q;q)_s(q;q)_{p-s}}$. We will use the notation $[x-a]_q^n = (x-a)(x-aq)...(x-aq^{n-1})$ and throughout the paper we will make frequent use of the finite q-binomial theorem (cf.[5]) which states that

$$[x-a]_q^n = \sum_{j=0}^{n}(-1)^j q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix} a^j x^{n-j}. \tag{2}$$

Lastly, we record the definition of the Jackson q-integral which plays an important role in the q-calculus. If f is a suitably defined function then

$$\int_0^a f(t)d_qt = (1-q)a\sum_{k=0}^{\infty} f(aq^k)q^k. \tag{3}$$

We note that the q-integral (3) is a Riemann-Stieltjes integral with respect to a step function having infinitely many points of increase at the points aq^k, $k = 0, 1, \ldots$. The jump at aq^k is $a(1-q)q^k$.

2 q-Cesàro Summability

Let $A = (a_{nk}), n, k = 0, 1, 2, \ldots$ be an infinite matrix of real numbers. We will define the A-transform of a given sequence $z = (z_n)$ to be the sequence $t = (t_n)$ defined by

$$t_n = \sum_{k=0}^{\infty} a_{nk}z_k, \quad n = 0, 1, \ldots \tag{4}$$

Naturally we presume that the infinite series in (4) converge. The relation (4) can be written in matrix form as $t = Az$. The matrix A is said to be a regular summability method if the convergence of the sequence (z_n) implies the convergence of the transform sequence (t_n) to the same limit. That is, $z_n \to a$ implies that $t_n \to a$. The matrix corresponding to the first order Cesàro means (1) is

$$a_{nk} = \begin{cases} \frac{1}{n+1} & \text{if } k \le n \\ 0 & \text{if } k > n \end{cases} \tag{5}$$

The Silverman-Toeplitz theorem ([6],[8],[9]) provides necessary and sufficient conditions that the matrix A in (4) be regular.

Theorem 1 *(Silverman-Toeplitz): The matrix A is a regular summability method if and only if*

(1) $lim_{n\to\infty}a_{nk} = 0,\ k = 0, 1, \ldots,$

(2) $lim_{n\to\infty}\sum_{k=0}^{\infty} a_{nk} = 1$,
(3) $\sum_{k=0}^{\infty} |a_{nk}| < M, \ n = 0, 1, \ldots$.

It is obvious that the Cesàro matrix in (5) satisfies the three conditions of Theorem 1. There are many ways to define a q-analog of $(C, 1)$ summability. We will give our suggested analog and then explain why it seems suitable. Define $C_1(q) = (a_{nk}(q))$ where

$$a_{nk}(q) = \begin{cases} \frac{1-q}{1-q^{n+1}} q^{n-k} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases} \tag{6}$$

We will then say that (z_n) is q-Cesàro summable to the limit a if

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{nk}(q) z_k = a. \tag{7}$$

The first reason that this definition is appropriate is that $\lim_{q\to 1} a_{nk}(q) = \frac{1}{n+1}$. Thus the q-Cesàro matrix $C_1(q)$ converges to the Cesàro matrix for (C,1) summability as $q \to 1$. Another reason the definition seems appropriate involves the relation between the binomial theorem and the q-binomial theorem. We will explain this now. The Cesàro means of order α satisfy a power series identity that may be taken as their defining relation. Given an infinite series $\sum_{k=0}^{\infty} u_k$, we define the (C, α) mean of the series to be the sequence $(U_n^{(\alpha)})$ in the power series identity

$$(1 - z)^{-\alpha-1} \sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty} b_n^{(\alpha+1)} U_n^{(\alpha)} z^n, \tag{8}$$

where the numbers $b_n^{(\alpha+1)}$ are the binomial power series coefficients:

$$(1 - z)^{-\alpha-1} = \sum_{n=0}^{\infty} b_n^{(\alpha+1)} z^n. \tag{9}$$

If we denote the partial sums of $\sum_{k=0}^{\infty} u_k$ by s_n then the identity (8) is equivalent to

$$(1 - z)^{-\alpha} \sum_{n=0}^{\infty} s_n z^n = \sum_{n=0}^{\infty} b_n^{(\alpha+1)} U_n^{(\alpha)} z^n. \tag{10}$$

If we set $\alpha = 1$ in (10) we obtain the $(C, 1)$ mean defined in (1). It seems reasonable to write a q-analog of (9) by using the q-binomial series (cf.[5]).

$$\frac{(q^{\alpha+1}z; q)_\infty}{(z; q)_\infty} = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} z^n. \tag{11}$$

If $q \to 1$ in (11) then (9) is obtained. We would then define the q-Cesàro mean of order α of a sequence (u_n) to be the sequence $(U_n^{(\alpha)}(q))$ given by

$$\frac{(q^{\alpha+1}z; q)_\infty}{(z; q)_\infty} \sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} U_n^{(\alpha)}(q) z^n. \tag{12}$$

When $\alpha = 1$ in (12) we get the first order q-Cesàro mean as defined in (1) and as defined by the matrix $C_1(q)$. We will denote the summability matrix that corresponds to $\alpha > 0$ in (12) by $C_\alpha(q)$. Simple calculations establish that the $q-$Cesàro matrix $C_\alpha(q)$ of order α satisfies the conditions of Theorem 1. We thus have

Theorem 2 *The q-Cesàro matrix* $C_\alpha(q)$ *is a regular summability method if* $\alpha > 0$.

If A and B are summability matrices we say that A is stronger than B if every sequence that is summed by B is also summed by A (to the same limit). If conversely every A summable sequence is also B summable then we say that A and B are equivalent. It is natural to ask how the strength of the first order q-Cesàro means varies with q. The answer is provided in the next theorem.

Theorem 3 $C_1(q_1)$ *and* $C_1(q_2)$ *are equivalent for* $0 < q_1, q_2 < 1$

Proof. Set $\alpha = 1$ in equation (12) to get

$$\frac{1}{(1-z)(1-qz)}\sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q} U_n^{(1)}(q) z^n. \tag{13}$$

If we set $q = q_1$ and $q = q_2$ in (13) we easily find that

$$\frac{1-q_2 z}{1-q_1 z}\sum_{n=0}^{\infty} \frac{1-q_2^{n+1}}{1-q_2} U_n^{(1)}(q_2) z^n = \sum_{n=0}^{\infty} \frac{1-q_1^{n+1}}{1-q_1} U_n^{(1)}(q_1) z^n. \tag{14}$$

Expanding $\frac{1-q_2 z}{1-q_1 z}$ in a power series, multiplying the series on the left of (14), and equating power series coefficients yields

$$U_n^{(1)}(q_1) = \sum_{j=0}^{n} a_{nj} U_j^{(1)}(q_2), \tag{15}$$

where the terms a_{nj} have the form

$$a_{nj} = \begin{cases} (q_1-q_2)\frac{1-q_2^{j+1}}{1-q_1^{n+1}}\frac{1-q_1}{1-q_2} q_1^{n-j-1} & \text{if } j = 0,1,...,n-1 \\ \frac{1-q_2^{n+1}}{1-q_1^{n+1}}\frac{1-q_1}{1-q_2} & \text{if } j = n \end{cases} \tag{16}$$

Equation (16) expresses the sequence $(U_n^{(1)}(q_1))$ as a matrix transform of the sequence $(U_n^{(1)}(q_2))$. A routine calculation shows that the matrix (a_{nk}) satisfies the conditions of Theorem 2. Thus every sequence summable $C_1(q_2)$ is also summable $C_1(q_1)$. To complete the proof, we only need to switch q_1 and q_2 in the calculations above. ■

This theorem does not address the comparison of $C_1(q)$ with the usual Cesàro mean $(C, 1)$. The next theorem deals with this.

Theorem 4 *Any sequence that is summable* $C_1(q)$ *is also summable* $(C, 1)$*. The converse statement does not hold.*

Proof. The proof follows the same lines as the proof of Theorem 3. Let (σ_n) denote the $(C, 1)$ mean of a given sequence and let $(U_n(q))$ denote the $C_1(q)$ mean of the same sequence. Then we have $\sigma_n = \sum_{j=0}^{n} \alpha_{nj} U_j(q)$, where

$$\alpha_{nj} = \begin{cases} \frac{1-q^{j+1}}{n+1} & \text{if } j = 0,1,...,n-1 \\ \frac{1-q^{n+1}}{(n+1)(1-q)} & \text{if } j = n \end{cases} \tag{17}$$

The matrix (α_{nj}) satisfies the conditions of Theorem 1, hence if $(U_n(q))$ converges then so does (σ_n). To prove the second part of the theorem we write $U_n(q) = \sum_{j=0}^{n} \beta_{nj}\sigma_j$, where

$$\beta_{nj} = \begin{cases} \frac{1-q}{1-q^{n+1}}(j+1)(1-q^{-1})q^n & \text{if } j = 0, 1, ..., n-1 \\ \frac{1-q}{1-q^{n+1}}(n+1) & \text{if } j = n \end{cases} . \tag{18}$$

A calculation shows that $\lim_{n\to\infty} \sum_{j=0}^{n} \beta_{nj} \neq 0$. ■

Consider, for example, the sequence (u_n) defined by $u_n = \frac{1}{2} + cos(x) + cos(2x) + ... + cos(nx)$. It is well known that (u_n) is $(C, 1)$ summable to 0 provided $x \neq 2k\pi$. However, it is not $C_1(q)$ summable.

Remark: The q-Cesàro matrix $C_1(q)$ appears in the Pólya-Szegő problem book [7], and in [4]. However neither of these references have placed $C_1(q)$ in the context of Hausdorff summability as will be done here.

3 Hausdorff Summability

The Cesàro means (C, α) belong to an important class of summability methods called Hausdorff Methods. We will give a very brief outline of the subject here. We will follow the development in [8], other presentations may be found in [6] and [9]. Let C denote the matrix that corresponds to $(C, 1)$ summability. We seek a matrix H with the property that $HC = DH$ where D is diagonal. Solving the matrix equation we find that $H = (h_{pq})$ with

$$h_{pq} = (-1)^{p-q}\binom{p}{q}h_{pp}. \tag{19}$$

The numbers h_{pp} are arbitrary as long as they are non-zero. We choose $h_{pp} = (-1)^p$ and then the matrix H has elements given by

$$h_{pq} = (-1)^q\binom{p}{q}. \tag{20}$$

The matrix H is self-inverse, that is, $H^{-1} = H$. The diagonal matrix D has diagonal elements $d_p = \frac{1}{p+1}$. With these matrices we have $C = H^{-1}DH$. Now we define a Hausdorff matrix to be of the form $A = H^{-1}DH$ where H is the matrix with elements as in (20) and D is any diagonal matrix. Thus Hausdorff matrices can be viewed as generalizations of $(C, 1)$ summability. We need three fundamental theorems pertaining to Hausdorff matrices.

Theorem 5 *A triangular matrix A commutes with C (the $(C, 1)$ matrix) if and only if A is a Hausdorff matrix.*

Theorem 6 *A Hausdorff matrix $H^{-1}DH$ is regular if and only if $D = (d_p\delta_{pq})$ with*

$$d_p = \int_0^1 t^p d\phi(t), \ p = 0, 1, \ldots \tag{21}$$

where the function $\phi(t)$ is of bounded variation on $[0, 1]$, $\phi(1) - \phi(0) = 1$, and $\phi(0^+) = \phi(0)$.

A sequence that has the integral form above is called a Hausdorff moment sequence. It is important to record a formula for the elements of a Hausdorff matrix. Given a sequence (d_p) we define the k^{th} forward difference by

$$\Delta^k d_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} d_{n+m}. \tag{22}$$

We define the k^{th} backward difference by

$$\nabla^k d_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} d_{n+k-m}. \tag{23}$$

The backward and forward differences clearly satisfy the identity $\Delta^k d_n = (-1)^k \nabla^k d_n$. Now if $\Lambda = (\lambda_{km})$ is a Hausdorff matrix $\Lambda = H^{-1}DH$ with $D = (d_p \delta_{pq})$ then

$$\lambda_{km} = \binom{k}{m} \Delta^{k-m} d_m. \tag{24}$$

Theorem 7 *The sequence (d_p) has the form*

$$d_p = \int_0^1 t^p d\phi(t), \;\; p = 0, 1, \ldots \tag{25}$$

if and only if

$$(-1)^k \Delta^k d_n \geq 0, n, k = 0, 1, \ldots \tag{26}$$

4 q-Hausdorff Summability

In this section we will parallel the connections between $(C, 1)$ and Hausdorff means for the case of q-Cesàro and a q-analog of Hausdorff matrices. We begin by finding a matrix H_q that plays the role of the self-inverse matrix H given by (19).

Theorem 8 *If D is a diagonal matrix then the matrix equation $H_q C_1(q) = DH_q$ has solution $H_q = (h_{ps})$ with*

$$h_{ps} = (-1)^{p-s} \begin{bmatrix} p \\ s \end{bmatrix} h_{pp} q^{(s^2-s-p^2+p)/2}, \;\; s = 0, 1, \ldots p \tag{27}$$

The diagonal matrix D is given by $D = (d_p \delta_{ps})$ with

$$d_p = \frac{1-q}{1-q^{p+1}}. \tag{28}$$

Proof. The proof is a standard matrix calculation. ■
The terms h_{pp} in (27) are arbitrary as long as they are non-zero. Accordingly, taking $h_{pp} = (-1)^p$, the matrix H_q is found to be given by

$$h_{ps} = (-1)^s \begin{bmatrix} p \\ s \end{bmatrix} q^{(s^2-s-p^2+p)/2}, \;\; s = 0, 1, \ldots, p. \tag{29}$$

The matrix H_q is not self-inverse as is the case with the matrix H that was defined in (20). It is easy however to compute the inverse and we find $H_q^{-1} = (h^*_{ps})$ where

$$h^*_{ps} = h_{ps} q^{(p-s)(p-s-1)/2}. \tag{30}$$

It should be noted that the sequence defined in (28) is a Hausdorff moment sequence and hence the q-Cesàro matrix is a Hausdorff matrix. This is seen by writing

$$d_p = (1-q)\sum_{k=0}^{\infty} q^{kp} q^k = \int_0^1 t^p d_q t, \tag{31}$$

and recalling that the q-integral is a Riemann-Stieltjes integral. The more general q-Cesàro matrix of order α defined by (2.8) also involves a moment sequence. To see this we denote the matrix by $C_\alpha(q) = (a_{n,k})$ and note that $a_{n,n} = \frac{(q;q)_n}{(q^{\alpha+1};q)_n}$. Now we appeal to Lemma 2.1 in [3] which states:

Lemma 1 *If 0<b<a<1 then*

$$\frac{(a;q)_k}{(b;q)_k} = \int_0^1 t^k d\Psi(t) \tag{32}$$

where $\Psi(t)$ is a monotone increasing step function.

We can thus conclude that if $\alpha > 0$ then the general q-Cesàro matrix is a Hausdorff matrix. We now define a q-Hausdorff matrix to be a lower triangular matrix of the form $H_q^{-1} D H_q$ where D is a diagonal matrix. Thus as $q \to 1$ a q-Hausdorff matrix $H_q^{-1} D H_q$ approaches a Hausdorff matrix HDH.
Next, the form of the matrix elements in a q-Hausdorff matrix will be determined.

Definition 1 *For a given sequence (d_p) we define the $k^{\rm th}$ forward q-difference of (d_p) by*

$$\Delta_q^{(k)} d_p = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{(k-j)(k-j-1)}{2}} d_{j+p}, k = 0, 1, \ldots \tag{33}$$

We define the $k^{\rm th}$ backward q-difference by

$$\nabla_q^{(k)} d_p = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{j(j-1)}{2}} d_{k+p-j}. \tag{34}$$

Note that as $q \to 1$ the forward q-difference approaches the standard forward difference defined in (22) and the backward q-difference approaches the backward difference in (23). Also, we have the identity $\Delta_q^{(k)} d_s = (-1)^k \nabla_q^{(k)} d_s$. A matrix calculation shows that we have:

$$\begin{gathered} H_q^{-1} D H_q = (\lambda_{ps}),\ \lambda_{ps} = (-1)^s h_{ps} \Delta_q^{(p-s)} d_p = (-1)^p h_{ps} \nabla_q^{(p-s)} d_p, \\ s = 0, 1, ..., p;\ p = 0, 1, \ldots \end{gathered} \tag{35}$$

The forward difference defined by (22) satisfies the identity

$$\Delta^n d_p = \Delta^{n-1} d_p - \Delta^{n-1} d_{p+1} \tag{36}$$

The forward q-difference defined by (33) satisfies a similar identity as we prove next.

Theorem 9 *The forward q-difference defined in (33) satisfies the identity*

$$\Delta_q^{(n)} d_s = q^{n-1}\Delta_q^{(n-1)} d_s - \Delta_q^{(n-1)} d_{s+1}. \tag{37}$$

Proof. Use the identity $\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} n-1 \\ j \end{bmatrix}$ to write

$$\Delta_q^{(n)} d_s = \sum_{j=0}^{n-1} (-1)^j q^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{\frac{(n-j)(n-j-1)}{2}} d_{j+s} -$$
$$- \sum_{j=0}^{n-1} (-1)^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{\frac{(n-j-1)(n-j-2)}{2}} d_{j+s+1}.$$

A simple rearrangement of the sums gives (37). ■
The identity (37) written in terms of the backward difference becomes

$$\nabla_q^{(n)} d_p = \nabla_q^{(n-1)} d_p - q^{n-1}\nabla_q^{(n-1)} d_{p+1}. \tag{38}$$

5 A Class of q-Hausdorff Matrices

The q-Cesàro matrix $C_1(q) = H_q^{-1} D H_q$ is generated by the moment sequence $d_p = \int_0^1 t^p d_q t$. In this section, a class of q-Hausdorff matrices that generalize $C_1(q)$ will be introduced. Given a sequence of positive numbers a_k with $a_0 = 1$, $a_{k+1} < a_k, k = 0, 1, \ldots,$ and $a_k \to 0$. Define a function $\Psi_q(t)$ by $\Psi_q(t) = a_k - a_{k+1}, q^k \le t < q^{k-1}, k = 1, 2, \ldots, \Psi_q(0) = 0, \Psi_q(t) = 1,\ t \ge 1$. For each such sequence and each such resulting weight function $\Psi(t)$ we have a q-Hausdorff matrix where the diagonal matrix D has entries given by

$$d_p = \int_0^1 t^p d\Psi_q(t). \tag{39}$$

In particular when $a_k = q^k$ then $d\Psi_q(t) = d_q t$ and the q-Hausdorff matrix is $C_1(q)$.

Theorem 10 *The matrices $H_q^{-1} D H_q$ where the elements of D are given by (39) are regular.*

Proof. We must show that if d_p is given by (39) then the matrix elements λ_{ps} given by (34) satisfy the three conditions of Theorem 2. We will consider the three conditions in order.

(i) To prove that $\lambda_{ps} \to 0$ as $p \to \infty$ for each $s = 0, 1, \ldots$ we must first compute the difference $\nabla_q^{(p-s)} d_s$. We have

$$\begin{gathered} \nabla_q^{(p-s)} d_s = \textstyle\sum_{j=0}^{p-s} (-1)^j \begin{bmatrix} p-s \\ j \end{bmatrix} q^{\frac{j(j-1)}{2}} d_{p-j} \\ = \int_0^1 \sum_{j=0}^{p-s} (-1)^j \begin{bmatrix} p-s \\ j \end{bmatrix} q^{\frac{j(j-1)}{2}} t^{p-j} d\Psi_q(t) = \int_0^1 t^s [t-1]_q^{p-s} d\Psi_q(t). \end{gathered} \tag{40}$$

Note that $[t-1]_q^{p-s} = (t-1)(t-q)\ldots(t-q^{p-s-1}) = 0$ when $t = q^m$, $m = 0, 1, \ldots p-s-1$. Thus

$$\nabla_q^{(p-s)} d_s = \int_0^{q^{p-s}} t^s [t-1]_q^{p-s} d\Psi_q(t). \tag{41}$$

After some calculations, it is found that

$$\left|\nabla_q^{(p-s)} d_s\right| \leq q^{\frac{(p-s)(p-s-1)}{2}}(q;q)_{p-s} q^{(p-s)s} q^{p-s}\left[\Psi_q(q^{p-s})-\Psi_q(0)\right]. \tag{42}$$

Thus we have $|\lambda_{ps}| \leq \frac{(q;q)_p}{(q;q)_s} q^{p-s}$. This proves that $\lambda_{ps} \to 0$ as $p \to \infty$ for fixed s.

(ii) Here, it will be proven that $\lim_{p\to\infty} \sum_{s=0}^{p} \lambda_{ps} = 1$. From (34) and from (39) we get

$$\sum_{s=0}^{p} \lambda_{ps} = (-1)^p q^{-\frac{p(p-1)}{2}} \int_0^1 \sum_{s=0}^{p} (-1)^s \begin{bmatrix} p \\ s \end{bmatrix} q^{\frac{s(s-1)}{2}} t^s [t-1]_q^{p-s} d\Psi_q(t). \tag{43}$$

In the right side of (43) use the expansion $[t-1]_q^{p-s} = \sum_{j=0}^{p-s}(-1)^j \begin{bmatrix} p-s \\ j \end{bmatrix} q^{\frac{j(j-1)}{2}} t^{p-s-j}$, and use the identity $\begin{bmatrix} p \\ s \end{bmatrix}\begin{bmatrix} p-s \\ j \end{bmatrix} = \begin{bmatrix} p-j \\ s \end{bmatrix}\begin{bmatrix} p \\ j \end{bmatrix}$, and interchange the sums to get

$$\begin{aligned} &\int_0^1 \sum_{s=0}^{p}(-1)^s \begin{bmatrix} p \\ s \end{bmatrix} q^{\frac{s(s-1)}{2}} t^s [t-1]_q^{p-s} d\Psi_q(t) = \\ &\int_0^1 \sum_{j=0}^{p} \begin{bmatrix} p \\ j \end{bmatrix} (-1)^j q^{\frac{j(j-1)}{2}} \sum_{s=0}^{p-j}(-1)^s \begin{bmatrix} p-j \\ s \end{bmatrix} q^{\frac{s(s-1)}{2}} t^{p-j} d\Psi_q(t). \end{aligned} \tag{44}$$

Note that $\sum_{s=0}^{p-j}(-1)^s \begin{bmatrix} p-j \\ s \end{bmatrix} q^{\frac{s(s-1)}{2}} t^{p-j} = \delta_{pj}$, and thus the right side of (44) reduces to $(-1)^p q^{\frac{p(p-1)}{2}} \int_0^1 d\Psi_q(t)$. Thus we have

$$\sum_{s=0}^{p} \lambda_{ps} = \int_0^1 d\Psi_q(t) = 1. \tag{45}$$

(iii) Here we must prove that $\sum_{s=0}^{p} |\lambda_{ps}|$ is uniformly bounded. But it is easy to use an argument like that in (i) to see that $\lambda_{ps} \geq 0$, the bound then follows from (ii). ■

As a further example of such a q-Hausdorff matrix we discuss a q-analog of Euler summability (cf.[6]). Here we will take the q-Hausdorff matrix to have elements

$$\lambda_{ps} = \frac{\begin{bmatrix} p \\ s \end{bmatrix} q^{(p-s)(p-s-1)/2} a^{p-s} x^s}{[x+a]_q^p}, 0 < a < x. \tag{46}$$

A calculation shows that the associated diagonal matrix has elements given by

$$d_p = \frac{1}{(-\frac{a}{x};q)_p}. \tag{47}$$

Write $\alpha = \frac{a}{x}$, we have $0 < \alpha < 1$. We can then write

$$d_p = \frac{(-\alpha q^p;q)_\infty}{(-\alpha;q)_\infty} = \frac{1}{(-\alpha;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} \alpha^n q^{np}}{(q;q)_n}. \tag{48}$$

The right side of (48) is a Riemann-Stieltjes integral of the form (39) in which the weight function $\Psi(t)$ has jumps at the points q^n and the jump $j(q^n)$ at q^n has value

$$j(q^n) = \frac{q^{\binom{n}{2}} \alpha^n}{(q;q)_n(-\alpha;q)_\infty}. \tag{49}$$

We note that when $q \to 1$ the matrix elements in (46) approach the matrix elements for Euler summability.

The examples of q-Hausdorff summability shown here all have weight functions that are purely discrete and have jumps at the points q^j, the resulting Riemann-Stieltjes integrals thus are all very similar to the Jackson q-integral. In the next section it will be shown that this is not accidental.

6 Relation to the Hausdorff Moment Problem

It is known that a Hausdorff matrix HDH is regular if and only if the sequence that forms the main diagonal in D is a Hausdorff moment sequence ([6], [8], [9]). We will now form a similar connection for a q-Hausdorff matrix. We will say that a sequence (d_p) is totally q-monotone if $\Delta_q^{(n)} d_p \geq 0, n, p = 0, 1, \ldots$. We define a class of weight functions $\mathcal{F}$ as follows.

Definition 2 *$\alpha(t)$ belongs to the class $\mathcal{F}$ if $\alpha(t)$ is bounded and monotone increasing with jumps at the points q^j, $j = 0, 1, \ldots$, $\alpha(0) = 0$, and if $\alpha(t)$ has no other point of increase.*

Theorem 11 *(d_p) is totally q-monotone if and only if $d_p = \int_0^1 t^p d\Psi(t)$, where $\Psi(t) \in \mathcal{F}$.*

Proof. First, suppose that d_p is of the form stated with $\Psi(t) \in \mathcal{F}$. We compute the q-difference and find that if $a_j > 0$ is the jump at q^j then

$$\begin{aligned} \Delta_q^{(k)} d_s &= \int_0^1 (1-t)(q-t)(q^2-t)\ldots(q^{k-1}-t)t^s d\Psi(t) \\ &= \sum_{j=k}^{\infty} (1-q^j)(q-q^j)\ldots(q^{k-1}-q^j)a_j q^{js} > 0 \end{aligned}$$

In the other direction the proof follows the lines of the presentation given by Wall [8], the original idea of the proof is due to Schoenberg . We begin with the observation that if $\Delta_q^{(n)} d_s \geq 0, n, s = 0, 1, \ldots$ then for any integer p we have

$$\begin{array}{cc} d_n \geq 0, & n = 0, 1, ..., p \\ \Delta_q^{(1)} d_n \geq 0, & n = 0, 1, ..., p-1 \\ \cdots & \\ \Delta_q^{(p-1)} d_n \geq 0, & n = 0, 1 \\ \Delta_q^{(p)} d_n \geq 0, & n = 0 \end{array} \tag{50}$$

From (37) it follows that the above equations are equivalent to the inequalities

$$\begin{array}{c} \Delta_q^{(p)} d_0 \geq 0 \\ \Delta_q^{(p-1)} d_1 \geq 0 \\ \cdots \\ \Delta_q^{(1)} d_{p-1} \geq 0 \\ \Delta_q^{(0)} d_n \geq 0 \end{array} \tag{51}$$

If we define $r_{p,n} = \Delta_q^{(p-n)} d_n$ the system (51) can be written using (33) as

$$r_{p,n} = \sum_{m=0}^{p} (-1)^{m-n} \begin{bmatrix} p-n \\ m-n \end{bmatrix} d_m q^{\frac{(p-m)(p-m-1)}{2}}, \; n = 0, 1, \ldots, p. \tag{52}$$

Note that the terms in the sum in (52) vanish if $m \leq n-1$. The system of equations (52) can be solved for d_m, the result is

$$d_m = \sum_{k=0}^{p} \begin{bmatrix} p-m \\ p-k \end{bmatrix} q^{m(p-k)} r_{p,k} q^{\frac{k(k-1)-p(p-1)}{2}}. \tag{53}$$

Again, the terms in the above sum vanish if $k \leq m-1$. Define $L_{p,k} = \left[\begin{smallmatrix} p \\ k \end{smallmatrix}\right] r_{p,k} q^{\frac{k(k-1)-p(p-1)}{2}}$, and use this definition in (53) to get

$$d_m = \sum_{k=0}^{p} \frac{\left[\begin{smallmatrix} p-m \\ p-k \end{smallmatrix}\right]}{\left[\begin{smallmatrix} p \\ k \end{smallmatrix}\right]} q^{m(p-k)} L_{p,k}. \tag{54}$$

Note that

$$\frac{\left[\begin{smallmatrix} p-m \\ p-k \end{smallmatrix}\right]}{\left[\begin{smallmatrix} p \\ k \end{smallmatrix}\right]} = \frac{(q^{k-m+1}; q)_m}{(q^{p-m+1}; q)_m} \tag{55}$$

which yields

$$\begin{aligned} d_m &= \textstyle\sum_{k=0}^{p} \frac{(q^{k-m+1};q)_m}{(q^{p-m+1};q)_m} q^{m(p-k)} L_{p,k} \\ &= \textstyle\sum_{k=0}^{p} \frac{\left[q^{p-k} - q^{p-m+1}\right]_q^m}{(q^{p-m+1};q)_m} L_{p,k} \end{aligned} \tag{56}$$

Now make a change of index $j = p-k$ in (56) and write $B_{p,j} = L_{p,p-j}$ to finally obtain

$$d_m = \frac{1}{(q^{p-m+1}; q)_m} \sum_{j=0}^{p} \left[q^j - q^{p-m+1}\right]_q^m B_{p,j}. \tag{57}$$

The sum on the right side of (57) represents the evaluation of a Riemann-Stieltjes integral with jumps at the points q^j, $j = 0, 1, \ldots, p$, the jump at each such point is $B_{p,j}$. If we define the step function $\Lambda_p(t)$ by

$$\Lambda_p(t) = \begin{cases} 0, & t < q^p \\ B_{p,p}, & q^p \leq t < q^{p-1} \\ B_{p,p} + B_{p,p-1}, & q^{p-1} \leq t < q^{p-2} \\ \cdots & \\ B_{p,0} + B_{p,1} + \ldots + B_{p,p-1} + B_{p,p}, & 1 \leq t \end{cases} \tag{58}$$

then we may write equation (57) in the form

$$d_m = \frac{1}{(q^{p-m+1}; q)_m} \int_0^1 [t - q^{p-m+1}]_q^m \, d\Lambda_p(t). \tag{59}$$

Note that the function $\Lambda_p(t)$ is bounded because it is monotone increasing and $\Lambda_p(1) = d_0$ from (53). Now observe that

$$\frac{1}{(q^{p-m+1}; q)_m} = 1 + q^p O(1) \text{ as } p \to \infty. \tag{60}$$

Also,

$$[t-q^{p-m+1}]_q^m = \sum_{j=0}^{m} \begin{bmatrix} p \\ j \end{bmatrix} (-1)^j q^{\frac{j(j-1)}{2}} q^{(p-m+1)j} t^{m-j} = t^m + q^p O(1), \text{ as } p \to \infty. \tag{61}$$

Equation (59) can thus be written as

$$d_m = \int_0^1 t^m d\Lambda_p(t) + q^p O(1). \tag{62}$$

We can now apply the Helly-Bray Selection Theorem (cf.[9]) to (62) and allowing $p \to \infty$, the existence of a bounded and non-decreasing function $\Lambda(t)$ such that

$$d_m = \int_0^1 t^m d\Lambda(t) \tag{63}$$

is established. Further, since each function $\Lambda_p(t)$ has jumps at $1, q, q^2, \ldots q^p$, and $\Lambda_p(0) = 0$ it follows that the limit function $\Lambda(t)$ has jumps at $q^j, j = 0, 1, 2, \ldots,$ and that $\Lambda(0) = 0$. Thus $\Lambda(t) \in \digamma$. This proves the theorem. ■

We now need some lemmas. The proofs are direct and we only outline one proof.

Lemma 2 $x^n = \sum_{k=0}^{n} \left[{n \atop k}\right] [x-1]_q^k, \ n = 0, 1, \ldots$

Definition 3 *Let $\Lambda_{ps}[x]$ be the polynomial of degree p defined by*

$$\Lambda_{ps}[x] = (-1)^p h_{ps} x^s [x-1]_q^{p-s}. \tag{64}$$

Also, for a given sequence (d_n) define a linear functional M acting on polynomials by $M(x^n) = d_n$.

A calculation shows that $M[\Lambda_{ps}[x]] = \lambda_{ps}$. We will make use of the following identity that has a straightforward induction proof, which is omited.

Lemma 3 *If $0 \le n \le p$ then*

$$x^n = \sum_{s=n}^{p} \frac{\left[{s \atop n}\right]}{\left[{p \atop n}\right]} q^{n(p-s)} \Lambda_{ps}[x]. \tag{65}$$

Next, for a function f defined on the points q^k define the q-Bernstein polynomial associated with f to be

$$B_p[f[x]] = \sum_{s=0}^{p} f(q^{p-s}) \Lambda_{ps}[x]. \tag{66}$$

Lemma 4 *If $0 \le n \le s \le p$, then* $\left\{ \frac{\left[{s \atop n}\right]}{\left[{p \atop n}\right]} - 1 \right\} q^{p-s} = q^p O(1)$ *as $p \to \infty$.*

Proof. The integer n is considered to be fixed. We have

$$\frac{\left[{s \atop n}\right]}{\left[{p \atop n}\right]} = \frac{(q^{s-n+1}; q)_n}{(q^{p-n+1}; q)_n}. \tag{67}$$

Also, $(q^{s-n+1};q)_n = \sum_{j=0}^{n}(-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2} q^{(s-n+1)j} = 1 + q^s O(1)$ as $s \to \infty$. Using the q-binomial theorem we have

$$\frac{1}{(q^{p-n+1};q)_n} = \frac{(q^{p+1};q)_\infty}{(q^{p-n+1};q)_\infty} = \sum_{j=0}^{\infty} \frac{(q^n;q)_j}{(q;q)_j} q^{(p-n+1)j} = 1 + q^p O(1). \tag{68}$$

Using these expressions we get the result. ■

Lemma 5 *If* $\sum_{s=0}^{p} |\lambda_{ps}| < K$ *for* $p = 0, 1, \dots$ *then* $\lim_{p\to\infty} M[B_p[x^n]] = d_n$.

Proof. We have $B_p[x^n] = \sum_{s=0}^{p} q^{n(p-s)} \Lambda_{ps}[x]$ and consequently $M[B_p[x^n]] = \sum_{s=0}^{p} q^{n(p-s)} \lambda_{ps}$. From Lemma 4 recalling that $M[x^n] = d_n$ and applying M on both sides of (65) we get

$$d_n = \sum_{s=n}^{p} \frac{\begin{bmatrix} s \\ n \end{bmatrix}}{\begin{bmatrix} p \\ n \end{bmatrix}} q^{n(p-s)} \lambda_{ps}, \tag{69}$$

thus we may write

$$d_n - M[B_p[x^n]] = \sum_{s=n}^{p} \left\{ \frac{\begin{bmatrix} s \\ n \end{bmatrix}}{\begin{bmatrix} p \\ n \end{bmatrix}} - 1 \right\} q^{n(p-s)} \lambda_{ps} - \sum_{s=0}^{n} q^{n(p-s)} \lambda_{ps}. \tag{70}$$

Note that the right side of the above expression vanishes when $n = 0$ and the lemma then holds trivially. We may then suppose that $n \geq 1$ for the remainder of the proof. The second sum on he right of (70) is of the form $q^p O(1)$ as $p \to \infty$.The first sum also has that form by Lemma (4). This proves the result. ■

Definition 4 $\alpha(t) \in F^*$ *if* $\alpha(t)$ *has points of increase at* $q^k, k = 0, 1, \dots$ *and nowhere else,* $\alpha(0) = 0$, *and if* $\alpha(t)$ *is of bounded variation on* $[0, 1]$.

Theorem 12 *A q-Hausdorff matrix is regular if and only if* d_m *is given by (63) with* $\Lambda(t) \in F^*$.

Proof. If d_m is given by (63) with $\Lambda(t) \in F^*$ then a very slight modification of the proof of Theorem 10 gives the necessary conclusion. So we must prove that d_m is a q-moment sequence with weight function in the class F^* if the q-Hausdorff matrix is regular. Suppose first that

$$\sum_{s=0}^{p} |\lambda_{ps}| < K, p = 0, 1, \dots \tag{71}$$

We rewrite (69) in the form

$$d_n = \frac{1}{(q^{p-n+1};q)_n} \sum_{k=0}^{p-n} [q^k - q^{p-n+1}]_q^n \lambda_{p,p-k} \tag{72}$$

We may write the right side of (72) as a Riemann-Stieltjes integral in the form

$$d_n = \frac{1}{(q^{p-n+1};q)_n} \int_0^1 [t - q^{p-n+1}]_q^n d\Psi_p(t) \tag{73}$$

The weight function $\Psi_p(t)$ is defined by

$$\Psi_p(t) = \begin{cases} 0 & \text{if } t < q^p \\ \lambda_{p0} + \lambda_{p1} & \text{if } q^{p-1} \le t < q^{p-2} \\ \cdots & \\ \lambda_{p0} + \ldots + \lambda_{p,p-1} & \text{if } q \le t < 1 \\ \lambda_{p0} + \ldots \lambda_{pp} & \text{if } 1 \le t \end{cases} \tag{74}$$

The function $\Psi_p(t)$ thus defined is of uniformly bounded variation because $\sum_{s=0}^{p} |\lambda_{ps}| < K, p = 0, 1, \ldots$ We may apply the reasoning that led to equation (62) and then appeal to the Helly-Bray Theorem [9] to conclude that

$$d_n = \int_0^1 t^n d\Psi(t) \tag{75}$$

where $\Psi(t) \in F^*$. Now suppose that $\lim_{p\to\infty} \sum_{s=0}^{p} \lambda_{ps} = 1$. Using (43) we have that

$$\sum_{s=0}^{p} \lambda_{ps} = \int_0^1 d\Lambda(t). \tag{76}$$

We thus have that $\Lambda(1) - \Lambda(0^+) = 1$. Lastly suppose that $\lim_{p\to\infty} \lambda_{ps} = 0$. Then

$$\lim_{p\to\infty} (-1)^s \begin{bmatrix} p \\ s \end{bmatrix} q^{(s^2-s-p^2+p)/2} \int_0^{q^{p-s}} t^s [t-1]_q^{p-s} d\Psi(t) = 0. \tag{77}$$

The above implies that $\lim_{p\to\infty} \int_0^{q^{p-s}} t^s [t-1]_q^{p-s} d\Psi(t) = 0$. It is not difficult to show that this implies $\Psi(0^+) = \Psi(0) = 0$. ■

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