BLOW-UP FOR THE b-FAMILY EQUATIONS

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Abstract. In this paper we consider the b-family equations on the torus $u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx} = 0$, which for appropriate values of b reduces to well-known models, such as the Camassa-Holm equation or the Degasperis-Procesi equation. We establish a local-in-space blow-up criterion.

1. Introduction

The bi-Hamiltonian structure of certain evolution equations leads to various remarkable features such as infinitely many symmetries and conserved quantities, and in some cases to the exact solvability of these equations [29, 33]. Examples include the KdV equation [7] and the Benjamin-Ono equation [1]. Years later, R. Camassa and D. Holm [5] in their studies of completely integrable dispersive shallow water equation tackled the following equation,

$$u_t + ku_x - u_{xxt} + 3uu_x = uu_{xxx} + 2u_x u_{xx}, \quad x \in \mathbb{R},\ t > 0. \tag{C-H}$$

where u can be interpreted as a horizontal velocity of the water at a certain depth and k as the dispersion parameter. The equation (C-H) also has been derived independently by B. Fuchssteiner and A. Fokas in [21]. When $k = 0$ (dispersionless case), the equation (C-H) possess soliton solutions peaked at their crest (often named peakons) [2, 5, 6]. Equation (C-H) is obtained by using an asymptotic expansion directly in the Hamiltonian for Euler's equation in the shallow water regime. Like the KdV equation, the Camassa-Holm equation (C-H) describes the unidirectional propagation of waves at the surface of shallow water under the influence of gravity [5, 7]. The equation (C-H) is physically relevant as it also describes the nonlinear dispersive waves in compressible hyperelastic rods [2, 3, 13]. It is convenient to rewrite the Cauchy problem associated with the dispersionless case of (C-H) in the following weak form:

$$\begin{cases} u_t + uu_x + \partial_x p * \left(u^2 + \frac{u_x^2}{2}\right) = 0, & x \in \mathbb{A}, \quad t > 0, \\ u(x,0) = u_0(x) & x \in \mathbb{A}, \end{cases} \tag{1.1}$$

where $p(x)$ is the fundamental solution of the operator $1 - \partial_x^2$ in $\mathbb{A}$. If $\mathbb{A} = \mathbb{R}$, we refer (1.1) as the non-periodic Camassa-Holm equation and $p = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$ in this case. If otherwise that $\mathbb{A} = \mathbb{S} = \mathbb{R}/\mathbb{Z}$ is unit circle, we refer (1.1) as the periodic Camassa-Holm equation, and $p = \frac{\cosh(x-[x]-\frac{1}{2})}{2\sinh(\frac{1}{2})}$ in this case. It is know that both the non-periodic and periodic Camassa-Holm equations are locally well-posed (in the sense Harmard) in the Sobolev space H^s, with $s > \frac{3}{2}$. See [14, 25, 34]. There is an abundance of the literature about the issue of the finite time blowup (see [2–4, 8, 9, 22, 26, 30]) and the related issue of the global existence of strong solutions ([8, 10, 22]).

On the other hand, Degasperis and Procesi [15], in their search of new integrability properties inside a wide class of equations, were led to consider the following integrable equation:

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{D-P}$$

As before, it is convenient to rewrite the Cauchy problem, using the same notations

$$\begin{cases} u_t + uu_x + \partial_x p * \left(\frac{3}{2}u^2\right) = 0, & x \in \mathbb{A}, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{A}. \end{cases} \tag{1.2}$$

A few years later, equation (1.2) as been proved to be relevant in shallow water dynamics, see [11, 12, 16]. Both the Camassa–Holm equation and the the Degasperis-Procesi equation (D-P) possess a bi-Hamiltonian structure (see[15]). The local well-posedness in H^s, with $s > \frac{3}{2}$ for the Cauchy non periodic problem was elaborated in [36], and [37] for the Cauchy periodic problem. With respect to blow-up criteria on the line we refer to [15, 27, 38, 39] and, for the unit tours, to [37, 38]. For the existence globally of the solution, see [27, 36, 38]. Despite sharing some properties with the Camassa-Holm equation, the Degasperis-Procesi has its own peculiarities. A specific feature is that (D-P) admits, beside peakons (i.e., soliton solutions of the form $u(t,x) = ce^{-|x-ct|}$, $c > 0$) also shock peakon solitons (i.e., solutions at the form $u = \frac{1}{t+k}\operatorname{sign}(x)e^{-|x-ct|}$, $k > 0$). For more details see [18, 23, 28]. After these premises, we will now focus on the Cauchy problem for the spatially periodic b-family equations:

$$\begin{cases} u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, & x \in \mathbb{S}, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{S}, \end{cases} \tag{1.3}$$

where $\mathbb{S}$ is the unit tours. Here b is a real parameter, and $u(x,t)$ stands for a horizontal velocity. The b-family equations can be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic-order accuracy for any $b \neq 1$ by an appropriate Kodama transformation [15, 16]. Again, when $b = 2$ and $b = 3$ (1.3) became (C-H) and (D-P) respectively. These values are the only values for which (1.3) is completely integrable. The Cauchy problem for the b-equation is locally well posed in the Sobolev space H^s for any $s > \frac{3}{2}$, [19, 27, 32, 35]. In [31] it is proved that the solution map of the b-family equations is Holder continuous as a map from bounded sets of $H^s(\mathbb{R})$, $s > \frac{3}{2}$ with the $H^r(\mathbb{R})$ $(0 \le r < s)$ topology, to $C([0,T], H^r(\mathbb{R}))$. J. Escher and J. Seiler [19] showed that the periodic b-family equation can be realized as Euler equation on the Lie group $\mathrm{Diff}^\infty(\mathbb{S})$ of all smooth and orientation-preserving diffeomorphisms on the unit tours, if $b = 2$ (C-H equation). The global existence theory of the solution of (1.3) is discussed in [19, 27, 35, 37]. In this paper we rather focus on blow-up criteria as well in estimates about the lifespan of the solutions. The blowup problem for the b-family equations has been already treated, e.g. in [19, 26, 32, 35, 38]: in these references the condition on the initial datum u_0 leading to the blowup typically involves the computation of some global quantities (the Sobolev norm $\|u_0\|_{H^1}$, or some other integral expressions of u_0). Motivated by the recent paper [2] (where earlier blowup results for the Camassa–Holm equations were unified in a single theorem) we address the more subtle problem of finding a *local-in-space* blowup criterion for the b-family equation, i.e., a blowup condition involving only the properties of u_0 in a neighborhood of a single point $x_0 \in \mathbb{S}$.

Loosely, the contribution of this paper can be stated as follows: if the parameter b belongs to a suitable range (including the physically relevant cases $b = 2$ and $b = 3$), then then there exists a constant $\beta_b > 0$ such that if

$$\left|u_0'(x_0)\right| \leq -\beta_b \left|u_0(x_0)\right|,$$

in at least one point $x_0 \in \mathbb{S}$, then the solution arising $u_0 \in H^s(\mathbb{S})$ ($s > \frac{3}{2}$) must blow-up in finite time.

This paper is organized as follows. In the next section we start by introducing the relevant notations and function spaces, recalling a few basic results. Then we precisely state and prove our main theorem. An important part of our work will be devoted to the computations of sharp bounds for the constant β_b and the lifespan of the solution. The smallest $b > 0$ to which our main theorem applies is computed numerically in the last part of the paper.

2. Blow-up for the periodic b-family equations

It is convenient to rewrite the periodic Cauchy problem (1.3) in the following weak form (see [35]):

$$\begin{cases} u_t + uu_x + \partial_x p * \left[\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right)u_x^2\right] = 0, & x \in \mathbb{S}, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{S} \\ u(t,x) = u(t,x+1) \quad t \geq 0, \end{cases} \tag{2.1}$$

where

$$p(x) = \frac{\cosh(x - [x] - \frac{1}{2})}{2\sinh\left(\frac{1}{2}\right)}, \tag{2.2}$$

is the fundamental solution of the operator $1 - \partial_x^2$ and $[\cdot]$ stands for the integer part of $x \in \mathbb{R}$. If $u \in C([0,T), H^s(\mathbb{S})) \cap C^1((0,T^*), H^{s-1}(\mathbb{S}))$, with $s > \frac{3}{2}$ satisfies (2.1) then we call u a strong solution to (2.1). If u is a strong solution on $[0,T)$ for every $T > 0$, then is called global strong solution of (2.1).

If $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, an application of Kato's method [24] leads to the following local well-posedness result:

Theorem 2.1 (See [35])**.** *For any constant b, given $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, then there exists a maximal time $T^* = T^*(\|u_0\|_{H^s}) > 0$ and a unique strong solution u to (2.1), such that*

$$u = u(\cdot, u_0) \in C([0,T^*), H^s(\mathbb{S})) \cap C^1([0,T^*), H^{s-1}(\mathbb{S})). \tag{2.3}$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0) : H^s(\mathbb{S}) \to C([0,T^*); H^s(\mathbb{S})) \cap C^1([0,T^*); H^{s-1}(\mathbb{S}))$ is continuous.

Remark 2.2. The maximal lifespan of the solution in Theorem 2.1 may be chosen independently of s in the following sense: If $u = u(\cdot, u_0) \in C([0,T^*), H^s(\mathbb{S})) \cap C^1([0,T^*), H^{s-1}(\mathbb{S}))$

to (2.1) and $u_0 \in H^{s'}(\mathbb{S})$ for some $s' \neq s$, $s' > \frac{3}{2}$, then $u = u(\cdot, u_0) \in C([0,T^*), H^{s'}(\mathbb{S})) \cap C^1([0,T^*), H^{s'-1}(\mathbb{S}))$ and with same T^*. In particular, if $u_0 \in \cap_{s\geq 0} H^s$, then $u \in C([0,T^*), H^{\infty}(\mathbb{S}))$. See [32, 35].

Moreover, by using the Theorem 2.1 and energy estimates, the following precise blow-up scenario of the solution to (2.1) can be obtained.

Theorem 2.3 (See [32, 35])**.** *Assume $b \in \mathbb{R}$ and $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$. Then blow up of the strong solution $u = u(\cdot, u_0)$ in finite time occurs if only if*

$$\lim_{t\to T^*} \inf\{(2b-1) \inf_{x\in\mathbb{R}}[u_x(t,x)]\} = -\infty \tag{2.4}$$

Before presenting our contribution, we will review a few known blow-up theorems with respect to (2.1).

Theorem 2.4 (See [35])**.** *Let $\frac{5}{3} < b \leq 3$ and $\int_{\mathbb{S}} (u_0')^3(x)\,dx < 0$. Assume that $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, $u_0 \not\equiv 0$, and the corresponding solution $u(t)$ (2.1) has a zero for any time $t \geq 0$. Then, the solution $u(t)$ of the (2.1) blows-up finite time.*

The next blow-up theorem uses the fact that if $u(t,x)$ is a solution to (2.1) with initial datum u_0, then $-u(t,-x)$ is also a solution to (2.1) with initial datum $-u_0(-x)$. Hence due to the uniqueness of the solutions, the solution to (2.1) is odd as soon as the initial datum $u_0(x)$ is odd.

Theorem 2.5 (See [32])**.** *Let $1 < b \leq 3$ and $u_0 \in H^s(\mathbb{S})$ $s > \frac{3}{2}$ be odd and nonzero. If $u_0'(0) \leq 0$, then the corresponding solution of (2.1) blow-up in finite time.*

Notations. For any real β, let us consider the 1-periodic function

$$w(x) = p(x) + \beta\partial_x p(x) \tag{2.5}$$

where p is the kernel introduced in (2.1) and $\partial_x p$ denotes the distributional derivative on $\mathbb{R}$, that agrees in this case with the classical i.e pointwise derivative on $\mathbb{R}\setminus\mathbb{Z}$. Notice that the non-negativity condition $w \geq 0$ is equivalent to the inequality $\cosh(1/2) \geq \pm\beta\sinh(1/2)$, i.e., to the condition

$$-\frac{e+1}{e-1} \leq \beta \leq \frac{e+1}{e-1}.$$

Throughout this section, we will work under the above condition on β. Let us now introduce the following weighted Sobolev space:

$$E_\beta \;=\; \{u \in L^1_{loc}(0,1) : \|u\|^2_{E_\beta} = \int_0^1 w(x)(u^2+u_x^2)(x)\,dx < \infty\}, \tag{2.6}$$

where the derivative is understood in the distributional sense. Notice that E_β agrees with the classical Sobolev space $H^1(0,1)$ when $|\beta| < \frac{e+1}{e-1}$, as in this case w is bounded and bounded away from 0, and the two norms $\|\cdot\|_{E_\beta}$ and $\|\cdot\|_{H^1}$ are equivalent. The situation is different for $\beta = \pm\frac{e+1}{e-1}$ as E_β is strictly larger that $H^1(0,1)$ in this case. Indeed, we have

$$w(x) = \frac{2e}{(e-1)^2}\sinh(x), \qquad x \in (0,1), \qquad \left(\text{if } \beta = \tfrac{e+1}{e-1}\right); \tag{2.7}$$

The elements of $E_{(e+1)/(e-1)}$, after modification on a set of measure zero, are continuous on $(0,1]$, but may be unbounded for $x \to 0^+$ (for instance, $|\log(x/2)|^{1/3} \in E_{(e+1)/(e-1)}$). In the same way,

$$w(x) = \frac{2e}{(e-1)^2}\sinh(1-x), \qquad x \in (0,1), \quad (\text{if } \beta = -\tfrac{e+1}{e-1}); \tag{2.8}$$

after modification on a set of measure zero, the elements of $E_{-(e+1)/(e-1)}$ are continuous on $[0,1)$, but may be unbounded for $x \to 1^-$.
Let us now introduce the closed subspace $E_{\beta,0}$ of E_β defined as the closure of $C_c^\infty(0,1)$ in E_β. The elements of $E_{\beta,0}$ satisfy the weighted Poincaré inequality below:

Lemma 2.6. *For all $|\beta| \le \frac{e+1}{e-1}$, there exists a constant $C>0$ such that*

$$\forall v \in E_{\beta,0}, \qquad \int_0^1 w(x)\; v^2(x)\,dx \le C \int_0^1 w(x)\; v_x^2(x)\,dx. \tag{2.9}$$

Proof. This demonstration is found in [4]. □

We need some notations.

Definition 2.7. For any real constant $b \neq 1$ and β, let $J(b,\beta) \ge -\infty$, be defined by
(2.10)
$$J(b,\beta) = \inf\left\{ \int_0^1 (p+\beta\partial_x p)\left(\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right)u_x^2\right)dx;\; u \in H^1(0,1),\; u(0)=u(1)=1 \right\}$$

and

$$\beta_b = \inf\left\{ \beta > 0 : \quad \beta^2 + \frac{2}{|b-1|}\left(J(b,\beta) - \frac{b}{2}\right) \ge 0 \right\}. \tag{2.11}$$

Notice that a priori $0 \le \beta_b \le +\infty$, as the set on the right-hand side could be empty.

Main results. Let us now formalize the goal of this paper.

Theorem 2.8. *Let $b \in]1,3]$ be such that β_b is finite. Let $u_0 \in H^s(\mathbb{S})$ be with $s > \frac{3}{2}$ and assume that there exists $x_0 \in \mathbb{S}$, such that*

$$u_0'(x_0) < -\beta_b\,|u_0(x_0)|\,. \tag{2.12}$$

then the corresponding solution u of (2.1) in $C([0,T^*),H^s(\mathbb{S})) \cap C^1([0,T^*),H^{s-1}(\mathbb{S}))$ arising from u_0 blows up in finite time. Moreover, the maximal time T^* verifies

$$T^* \le \frac{2}{(b-1)\sqrt{(u_0'(x_0))^2 - \beta_b^2 u_0^2(x_0)}}. \tag{2.13}$$

Remark 2.9. Notice that the Theorem 2.8 relies on the condition that β_b is finite. In section 2, we will prove that one indeed has $\beta_b < +\infty$, as soon as b is outside a very small neighborhood of 1. On the other hand, as we will see later on, for $1 < b < 1.0012\ldots$, $\beta_b = +\infty$ and Theorem 2.8 does not apply in such range.

For the proof of Theorem 2.8, we need the following propositions.

Proposition 2.1. *We have*

$$J(b,\beta) > -\infty \Leftrightarrow \begin{cases} |\beta| \le \frac{e+1}{e-1}, \\ b \le 3, \\ \frac{b}{3-b} > -\frac{1}{C_\beta}, \end{cases} \tag{2.14}$$

where $C_\beta > 0$ is the best Poincaré constant in inequality (2.9).

Proof. Putting $u = v + 1$ and observing that $\int_0^1 w(x)\,dx = 1$, we see that

$$J(b,\beta) = \frac{b}{2} + \inf\{T(v) : v \in H_0^1(0,1)\}, \tag{2.15}$$

where

$$T(v) = \int_0^1 w(x)\left(\frac{b}{2}(v^2 + 2v) + \left(\frac{3-b}{2}\right) v_x^2\right)(x)\,dx. \tag{2.16}$$

Assume that $J(b,\beta) > -\infty$. In order to show $|\beta| \le \frac{e+1}{e-1}$, we refer to the proof of proposition 3.3. in [4]. In order to prove $b \le 3$, we consider $|\beta| \le \frac{e+1}{e-1}$ and

$$u_n(x) = 1 + \frac{1}{2}\sin(n2\pi x) \quad \Rightarrow \quad u_n'(x) = n\pi\cos(n2\pi x). \tag{2.17}$$

For each $n \in \mathbb{N}$ $u_n \in H^1(0,1)$, $u_n(1) = u_n(0) = 1$. Thus there is a constant $c_1 > 0$ independent of n, such that

$$\forall n \in \mathbb{N} \quad 0 \le \frac{b}{2}\int_0^1 w(x)u_n^2(x)\,dx \le c_1,$$

and

$$\frac{3-b}{2}\int_0^1 w(x)(u_n')^2(x)\,dx \;\to\; -\infty,$$

because $b > 3$ and then $J(b,\beta) = -\infty$. In order to prove the third inequality, we only have to treat the case $b < 0$. Applying the inequality

$$\int_0^1 w(x)\left(\frac{b}{2}(n^2v^2 + 2nv) + \left(\frac{3-b}{2}\right) n^2 v_x^2\right)(x)\,dx \ge J(b,\beta) - \frac{b}{2}, \tag{2.18}$$

valid for all $v \in H_0^1(0,1)$ and all $n \in \mathbb{N}$ and letting $n \to \infty$, we get

$$\int_0^1 w(x)\left(\frac{b}{2}v^2 + \left(\frac{3-b}{2}\right) v_x^2\right)(x)\,dx \ge 0.$$

We deduce:

$$\int_0^1 w(x)v^2(x)\,dx \le -\frac{3-b}{b}\int_0^1 w(x)v_x^2(x)\,dx.$$

Then we get $\frac{b}{3-b} \ge -\frac{1}{C_\beta}$. But the equality case $\frac{b}{3-b} = -\frac{1}{C_\beta}$ can be excluded, as otherwise we could find a sequence v_n such that $((b/2)\int_0^1 \omega v_n^2)/((3-b)\int_0^1 \omega (v_n)_x^2)$ converges to 1 and such that $\int b\omega v_n \to -\infty$: for such a sequence we have $T(v_n) \sim \int_0^1 b\omega v_n \to -\infty$, contradicting the assumption $J(b,\beta) > -\infty$.

Conversely, assume that $|\beta| \le \frac{e+1}{e-1}$. By the weighted Poincairé inequality (2.9), we can consider an equivalent norm in $E_{\beta,0}$:

$$\|v\|_{E_{\beta,0}} = \int_0^1 w(x) v_x(x)\, dx. \tag{2.19}$$

Since $\frac{b}{3-b} > -\frac{1}{C_\beta}$, the symmetric bilinear form

$$B(u,v) = \int_0^1 w(x) \left(\frac{b}{2} uv + \left(\frac{3-b}{2}\right) u_x v_x\right)(x)\, dx, \tag{2.20}$$

is coercive on the Hilbert space $E_{\beta,0}$. Applying the Lax-Milgram theorem yields the existence and uniqueness of a minimizer $\hat{v} \in E_{\beta,0}$ for the functional T. But $H_0^1(0,1) \subset E_{\beta,0}$, so in particular, we get $J(b,\beta) > -\infty$. Moreover, if $|\beta| < \frac{e+1}{e-1}$, then recalling $E_{\beta,0} = H_0^1(0,1)$ we see that $J(b,\beta)$ is in fact a minimun, achieved at $\hat{u} = 1 + \hat{v} \in H^1(0,1)$. □

The next lemma provides some useful information about $J(b,\beta)$.

Lemma 2.10. *The function $(b,\beta) \mapsto J(b,\beta) \in \mathbb{R} \cup \{-\infty\}$ defined for all $(b,\beta) \in \mathbb{R}^2$ is concave with respect to each one of its variables, and is even with respect to the variable β. Also for all $b \in \mathbb{R}$ and $|\beta| \le \frac{e+1}{e-1}$, $\quad -\infty \le J(b, \frac{e+1}{e-1}) \le J(b,\beta) \le J(b,0) \le \frac{b}{2}$.*

Proof. The proof is similar to that of the proposition 3.4. in [4] □

The next lemma motivates the introduction of quantity the $J(b,\beta)$ in relation with the b-family equations.

Proposition 2.2. *Let $(\alpha,\beta) \in \mathbb{R}^2$ and $u \in H^1(\mathbb{S})$, we get*

$$\forall x \in \mathbb{S}, \quad (p + \beta\partial_x p) * \left(\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right) u_x^2)\right)(x) \ge J(b,\beta)\ u^2(x).$$

Proof. Let $\alpha = \alpha(b,\beta)$ be some constant. Because of the invariance under translation, we get that the inequality

$$(p + \beta\partial_x p) * \left(\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right) u_x^2)\right)(x) \ge \alpha\ u^2(x), \tag{2.21}$$

holds true for all $u \in H^1(\mathbb{S})$ and all $x \in \mathbb{S}$ if and only if the inequality

$$(p + \beta\partial_x p) * \left(\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right) u_x^2)\right)(1) \ge \alpha\ u^2(1), \tag{2.22}$$

holds true for all $u \in H^1(\mathbb{S})$. But on the interval $]0,1[$, $(p+\beta\partial_x p)(1-x) = (p - \beta\partial_x p)(x)$. Then we get

$$(p + \beta\partial_x p) * \left(\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right) u_x^2)\right)(1) = \int_0^1 (p - \beta\partial_x p) \left(\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right) u_x^2\right)(x)\, dx. \tag{2.23}$$

Normalizing to obtain $u(1) = 1$, we get that the best constant α in inequality (2.21) satisfies $\alpha = J(b,-\beta) = J(b,\beta)$. □

The next proposition provides a first lower bound estimate of $J(b,\beta)$, when $b \in [-1,3]$.

Proposition 2.3. *Let $-1 \leq b \leq 3$ and $|\beta| \leq \frac{e+1}{e-1}$. Then, if $u \in H^1(0,1)$ such that $u(1)=u(0)$, we get*

$$(p \pm \beta\partial_x p) * \left(\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right)u_x^2\right) \geq \begin{cases} \delta_b\ u^2, \ \text{if} \ \ |\beta| \leq 1 \\ \\ \frac{\delta_b}{2}[(e+1) - |\beta|\,(e-1)]u^2, \ \text{if} \ \ 1 \leq |\beta| \leq \frac{e+1}{e-1}, \end{cases}$$

where

$$\delta_b = \frac{\sqrt{3-b}}{4}\left(\sqrt{3(1+b)} - \sqrt{3-b}\right). \tag{2.24}$$

Remark 2.11. Notice that $\delta_b \geq 0$ if and only if for $0 \leq b \leq 3$.

Proof. It is sufficient to consider the case $0 \leq \beta \leq \frac{e+1}{e-1}$. We make the convolution estimates for $(p+\beta\partial_x p)$, the convolution estimates for $(p-\beta\partial_x p)$ being similar. First observe that:

$$\forall x \in \mathbb{R} \ \ p(x) = \frac{e^{x-\frac{1}{2}-[x]}}{4\sinh\frac{1}{2}} + \frac{e^{-x+\frac{1}{2}+[x]}}{4\sinh\frac{1}{2}} = p_1(x) + p_2(x). \tag{2.25}$$

We start with the estimate of $p_1 * (a^2u^2 + u_x^2)(1)$, with $a \in \mathbb{R}$ to be determined later. We get

$$\begin{aligned} p_1 * (a^2u^2 + u_x^2)(1) &= \frac{1}{4\sinh(\frac{1}{2})}\int_0^1 e^{\frac{1}{2}-\xi}(a^2u^2 + u_x^2)(\xi)\,d\xi \\ &\geq \frac{-a}{4\sinh(\frac{1}{2})}\int_0^1 e^{\frac{1}{2}-\xi}(2uu_x)(\xi)\,d\xi \\ &= \frac{-a}{4\sinh(\frac{1}{2})}(e^{\frac{-1}{2}} - e^{\frac{1}{2}})u^2(1) - \frac{1}{4\sinh(\frac{1}{2})}\int_0^1 e^{\frac{1}{2}-\xi}au^2\,d\xi \\ &= \frac{a}{2}u^2(1) - p_1 * (au^2)(1). \end{aligned}$$

Hence

$$p_1 * ((a^2+a)u^2 + u_x^2)(1) \geq \frac{a}{2}u^2(1),$$

and because of the invariance under translations, we get

$$p_1 * ((a^2+a)u^2 + u_x^2) \geq \frac{a}{2}u^2. \tag{2.26}$$

Similarily:

$$\begin{aligned} p_2 * (a^2u^2 + u_x^2)(1) &= \frac{1}{4\sinh(\frac{1}{2})}\int_0^1 e^{\xi-\frac{1}{2}}(a^2u^2 + u_x^2)(\xi)\,d\xi \\ &\geq \frac{a}{4\sinh(\frac{1}{2})}\int_0^1 e^{\xi-\frac{1}{2}}(2uu_x)(\xi)\,d\xi \\ &= \frac{a}{4\sinh(\frac{1}{2})}(e^{\frac{1}{2}} - e^{\frac{-1}{2}})u^2(1) - \frac{1}{4\sinh(\frac{1}{2})}\int_0^1 e^{\xi-\frac{1}{2}}au^2\,d\xi \\ &= \frac{a}{2}u^2(1) - p_2 * (au^2)(1). \end{aligned}$$

Hence, again using the invariance under translations, we get

$$p_2 * ((a^2+a)u^2 + u_x^2) \geq \frac{a}{2}u^2. \tag{2.27}$$

Choose a such that $a^2 + a = \frac{b}{3-b}$. This is indeed possible if $-1 \leq b < 3$ (if $b = 3$, the proposition is trivial and there is nothing to prove). We get:

$$p_1 * \left(\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right)u_x^2\right) \quad \geq \quad \frac{\delta_b}{2}u^2, \tag{2.28}$$

$$p_2 * \left(\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right)u_x^2\right) \quad \geq \quad \frac{\delta_b}{2}u^2. \tag{2.29}$$

Now, from the identity $p = p_1 + p_2$ and $\partial_x p = p_1 - p_2$, that holds both in the distributional and in the a.e. pointwise sense, we get

$$p + \beta\partial_x p = (1+\beta)p_1 + (1-\beta)p_2. \tag{2.30}$$

If $0 \leq \beta \leq 1$, then from (2.28) and (2.30), we deduce

$$(p + \beta\partial_x p) * \left(\frac{b}{2}u^2 + \left(\frac{3-b}{2}u_x^2\right)\right) \geq [(1+\beta) + (1-\beta)]\frac{\delta_b}{2}u^2 = \delta_b u^2. \tag{2.31}$$

We proved as follows. From

$$p_2(x) \leq e\ p_1(x), \quad \forall x \in (0,1), \tag{2.32}$$

we get, for $1 \leq \beta \leq \frac{e+1}{e-1}$

$$\begin{aligned} p + \beta\partial_x p &= (1+\beta)p_1 - (\beta-1)p_2, \\ &\geq [(e+1) - \beta(e-1)]p_1. \end{aligned} \tag{2.33}$$

We deduce, using (2.28):

$$\forall\ 1 \leq \beta \leq \tfrac{e+1}{e-1}, \quad (p + \beta\partial_x p)\left(\tfrac{b}{2}u^2 + \left(\tfrac{3-b}{2}u_x^2\right)\right) \geq [(e+1) - \beta(e-1)]\tfrac{\delta_b}{2}u^2. \tag{2.34}$$

□

Remark 2.12. If $-1 \leq b \leq 3$, then it follows by the preceding proposition that $|\beta| \leq 1$, then $J(b,\beta) \geq \delta_b$, and if $1 \leq |\beta| \leq \frac{e+1}{e-1}$ then $J(b,\beta) \geq \frac{\delta_b}{2}[(e+1) - |\beta|(e-1)]$.

Proof of Theorem 2.8. By the well-posedness result in $H^s(\mathbb{S})$, with $s > 3/2$, the density of $H^3(\mathbb{S})$ in $H^s(\mathbb{S})$ and a simple approximation argument, we only need to prove Theorem 2.8 assuming $u_0 \in H^3(\mathbb{S})$. We thus obtain a unique solution of (2.1), defined in some nontrivial interval $[0,T[$, and such that $u \in C([0,T[, H^3(\mathbb{S})) \cap C^1([0,T[, H^2(\mathbb{S}))$. The starting point is the analysis of the flow map $q(t,x)$ of (2.1)

$$\begin{cases} q_t(t,x) = u(t,q(t,x)) & x \in \mathbb{S}, \quad t \in [0,T^*), \\ q(0,x) = x, & x \in \mathbb{S}. \end{cases} \tag{2.35}$$

As $u \in C^1([0,T[, H^2(\mathbb{S}))$, we can see that u and u_x are continuous on $[0,T[\times\mathbb{S}$ and $x \mapsto u(t,x)$ is Lipschitz, uniformly with respect to t in any compact time interval in $[0,T[$. Then the flow map $q(t,x)$ is well defined by (2.35) in the time interval $[0,T[$ and $q \in$

$C^1([0,T[\times\mathbb{R},\mathbb{R})$. Differentiating (2.1) with respect to the x variable and applying the identity $\partial_x^2 p * f = p * f - f$, we get:

$$u_{tx} + uu_{xx} = \frac{b}{2}u^2 - \left(\frac{b-1}{2}\right)u_x^2 - p * \Big[\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right)u_x^2\Big].$$

Let us introduce the two C^1 functions of the time variable depending on β. The constant β, will be chosen later on

$$f(t) = (-u_x + \beta u)(t, q(t,x_0)) \quad \text{and} \quad g(t) = -(u_x + \beta u)(t, q(t,x_0)).$$

Using (2.35) and differentiating with respect to t, we get

$$\begin{aligned}\frac{df}{dt}(t) &= [(-u_{tx} - uu_{xx}) + \beta(u_t + uu_x)](t, q(t,x_0)) \\ &= -\frac{b}{2}u^2 + \left(\frac{b-1}{2}\right)u_x^2 + (p - \beta\partial_x p) * \Big[\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right)u_x^2\Big](t, q(t,x_0)),\end{aligned}$$

and

$$\begin{aligned}\frac{dg}{dt}(t) &= [(-u_{tx} - uu_{xx}) - \beta(u_t + uu_x)](t, q(t,x_0)) \\ &= -\frac{b}{2}u^2 + \left(\frac{b-1}{2}\right)u_x^2 + (p + \beta\partial_x p) * \Big[\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right)u_x^2\Big](t, q(t,x_0)).\end{aligned}$$

Let us first consider $b \in]1,3]$. Recall that we work under the condition $\beta_b < \infty$. By the definition of β_b (2.11) we deduce that there exists $\beta \geq 0$ such that

$$\beta^2 \geq \frac{2}{b-1}\left(\frac{b}{2} - J(b,\beta)\right). \tag{2.36}$$

Applying the convolution estimate of (2.2) and the fact that $J(b,\beta) = J(b,-\beta)$, we get

$$\begin{aligned}\frac{df}{dt}(t) &\geq \left(\frac{b-1}{2}\right)u_x^2 + \left(J(b,-\beta) - \frac{b}{2}\right)u^2(t, q(t,x_0)) \\ &\geq \frac{b-1}{2}\left(u_x^2 - \beta^2 u^2\right)(t, q(t,x_0)) \\ &= \frac{b-1}{2}[f(t)g(t)]\end{aligned}$$

In the same way,

$$\begin{aligned}\frac{dg}{dt}(t) &\geq \left(\frac{b-1}{2}\right)u_x^2 + \left(J(b,\beta) - \frac{b}{2}\right)u^2(t, q(t,x_0)) \\ &\geq \frac{b-1}{2}\left(u_x^2 - \beta^2 u^2\right)(t, q(t,x_0)) \\ &= \frac{b-1}{2}[f(t)g(t)].\end{aligned}$$

The assumption $u_0'(x_0) < -\beta_b |u_0(x_0)|$ guarantees that we may choose β satisfying (2.36) with $\beta - \beta_b > 0$ small enough so that

$$u_0'(x_0) < -\beta |u_0(x_0)|.$$

For such a choice of β we have $f(0) > 0$ and $g(0) > 0$.

We now make use of the following result:

Lemma 2.13 (See [4]). *Let $0 < T^* \leq \infty$ and $f, g \in C^1([0, T^*[, \mathbb{R})$ be such that, for some constant $c > 0$ and all $t \in [0, T^*[$,*

$$\begin{aligned} \frac{df}{dt}(t) &\geq cf(t)g(t) \\ \frac{dg}{dt}(t) &\geq cf(t)g(t). \end{aligned}$$

If $f(0) > 0$ and $g(0) > 0$, then

$$T^* \leq \frac{1}{c\sqrt{f(0)g(0)}}.$$

The blow-up of u then follows immediately from our previous estimates applying the above lemma. □

3. ESTIMATES OF β_b

Theorem 2.8 is meaningful only if $b \in (1, 3]$ is such that $\beta_b < \infty$. We recall here that β_b is defined by Eq. (2.11):

$$\beta_b = \inf \left\{ \beta > 0 : \quad \beta^2 + \frac{2}{|b-1|} \left(J(b, \beta) - \frac{b}{2} \right) \geq 0 \right\}.$$

Next, we propose three lower bound estimates for the convolution term

$$(p \pm \beta \partial_x p) * \left(\frac{b}{2} u^2 + \left(\frac{3-b}{2} \right) u_x^2 \right),$$

or —what is equivalent, owing to Proposition 2.2— three lower bound estimates for $J(b, \beta)$). Such estimates will allow us to determinate sufficient conditions on $b \in (1, 3]$ in order to β_b to be finite and will provide upper bounds for β_b.

Estimate 1 and Estimate 2 below are presented mainly for pedagogical purposes, as they are self-contained. But these two estimates will be later on improved by Estimate 3, which is more technical and deeply relies on a few involved computations made in [4]. We point out however that Estimate 1 suffices to claim that Theorem 2.8 is not vacuous.

3.1. **Estimate 1.** Let $0 \leq \beta \leq \frac{e+1}{e-1}$ and $1 < b \leq 3$. We start considering the obvious estimate

$$(p \pm \beta \partial_x p) * \left(\frac{b}{2} u^2 + \left(\frac{3-b}{2} \right) u_x^2 \right) \geq 0.$$

Thanks to definition (2.11), we see that a sufficient condition on b which entails $\beta_b < \infty$, is the existence of a constant β satisfying

$$\sqrt{\frac{b}{b-1}} \leq \beta \leq \frac{e+1}{e-1}. \tag{3.1}$$

This holds when $b \geq \frac{(e+1)^2}{4e} \equiv \alpha$. In this case, the corresponding bound for β_b is

$$\beta_b \leq \sqrt{\frac{b}{b-1}} < +\infty, \qquad \text{for } \frac{(e+1)^2}{4e} \leq b \leq 3. \tag{3.2}$$

(See Figure 1a).

3.2. **Estimate 2.** Proposition 2.3 provides a better sufficient condition ensuring that $\beta_b < +\infty$. Namely:

$$\exists\ 0 \le \beta \le 1 \quad \text{such that} \quad \beta^2 + \frac{2}{b-1}\left(\delta_b - \frac{b}{2}\right) \ge 0, \tag{3.3}$$

or

$$\exists\ 1 \le \beta \le \frac{e+1}{e-1} \quad \text{such that} \quad \beta^2 + \frac{2}{b-1}\left([(e+1) - \beta(e-1)]\frac{\delta_b}{2} - \frac{b}{2}\right) \ge 0, \tag{3.4}$$

where δ_b is as (2.24). The study of the function $b \mapsto \sqrt{\frac{2}{b-1}\left(\frac{b}{2} - \delta_b\right)}$ in the interval $(1,3]$ however reveals that condition (3.3) is satisfied only for $b = 2$. We have $\delta_2 = \frac{1}{2}$ and so $\beta = 1$. The corresponding estimate for β_2 is then $\beta_2 \le 1$. This situation corresponds to the Camassa–Holm equation. We thus recover the result in [2]. (See Figure 1b.) On the other hand, solving (3.4) is possible if and only if the largest real zero $\phi(b)$ of the quadratic polynomial $\beta \mapsto P_b(\beta) = \beta^2 + \beta\,\delta_b\left(\frac{e+1}{b-1}\right) + \left(\delta_b\left(\frac{e+1}{b-1}\right) - \frac{b}{b-1}\right)$ is inside the interval $[1, \frac{e+1}{e-1}]$.

A simple computation shows that this is indeed the case when $\alpha \le b \le 3$. Here $\alpha = \frac{(e+1)^2}{4e}$ is the same as in Estimate 1. For $\alpha \le b \le 3$, now we get the bound

$$\beta_b \le \phi(b) < +\infty, \tag{3.5}$$

that considerably improves our earlier estimate (3.2). See Figure 1b

3.3. **Estimate 3.** This part relies on the properties of $J(b,\beta)$ which are described in Lemma 2.10 and the computations made in [4]

Let $I(\alpha,\beta)$ as in [4, Section 2]. For $b \in (1,3]$, and $|\beta| \le \frac{e+1}{e-1}$, the relation between I and J is the following:

$$J(b,\beta) = \begin{cases} \frac{3-b}{2}\ I\left(\frac{b}{3-b}, \beta\right), & \text{if } b \ne 3 \\ \frac{3}{2}\ \inf\left\{\int_0^1 w(x)\ u^2\,dx;\ u \in H^1(0,1),\ u(0) = u(1) = 1\right\}, & \text{if } b = 3. \end{cases}$$

where $I(\alpha,\beta)$ is as in [4]. If $b \ne 3$, borrowing the computation made in [4], we get

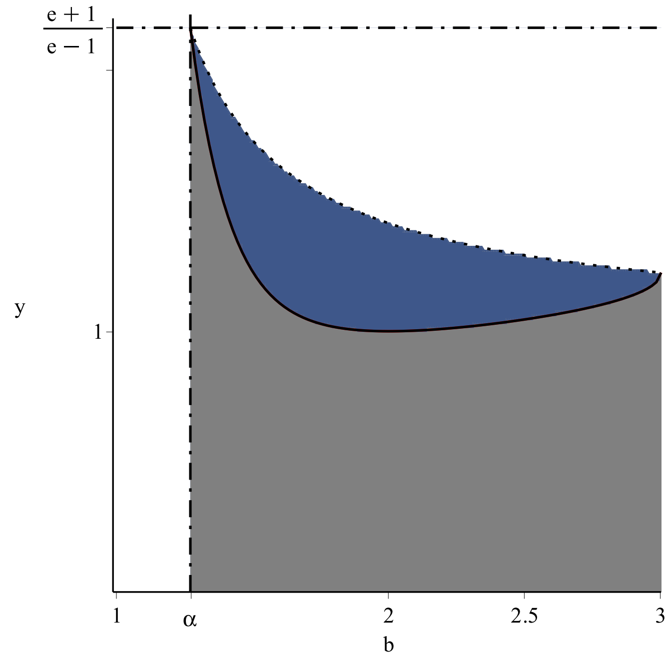
$$\begin{aligned} J\left(b, \frac{e+1}{e-1}\right) &= \frac{3-b}{2}\ I\left(\frac{b}{3-b}, \frac{e+1}{e-1}\right) \\ &= \frac{3-b}{4e}\,(e+1)^2\,\frac{P'_{\upsilon(b)}}{P_{\upsilon(b)}}(\cosh 1) \end{aligned}$$

where

$$\upsilon(b) = -\frac{1}{2} + \frac{1}{2}\cdot\sqrt{1 + 4\cdot\left(\frac{b}{3-b}\right)} \in \{z \in \mathbb{C} : \Im(z) \ge 0\}.$$

and $P_{\upsilon(b)}$ is Legendre function of the first kind, of the degree $\upsilon(b)$, arising when solving the Euler–Lagrange equation associated with the minimization problem of $I(\alpha, \frac{e+1}{e-1})$. The reason for considering here the limit case $\beta = \frac{e+1}{e-1}$ is twofold: on one hand, in this case the weight function has a simpler expression, namely $w(x)$ becomes in this case

$$w(x) = p(x) + \tfrac{e+1}{e-1}\,\partial_x p(x) = \tfrac{2e}{(e-1)^2}\sinh x, \qquad x \in (0,1);$$



(A) The plot of the function $b \mapsto \sqrt{\frac{b}{b-1}}$, providing the bound (3.2). The upper-bound estimate of β_b given by Eq.(3.2), showing that Theorem 2.8 applies for $b \in [\alpha, 3]$, where $\alpha = \frac{(e+1)^2}{4e}$ (blue and gray region).

(B) The function $b \mapsto \phi(b)$, providing the bound (3.5). The upper-bound estimates of β_b given by Eq.(3.5) and the Theorem 2.8 are valid inside the interval $[\alpha, 3]$ (grey region).

FIGURE 1. First and Second estimate of β_b.

this allow to reduce the Euler-Lagrange equation to a linear second order ordinary differential equation of Legendre type. See [4] for more details. On the other hand, by Lemma 2.10, we have $J(b,\beta) \geq J\left(b, \frac{e+1}{e-1}\right)$ for all $0 \leq \beta \leq \frac{e+1}{e-1}$.

Now, for $0 \leq \beta \leq \frac{e+1}{e-1}$, we have

$$\beta^2 + \frac{2}{b-1}\left(J(b,\beta) - \frac{b}{2}\right) \geq \beta^2 + \frac{2}{b-1}\left(\frac{3-b}{4e}(e+1)^2 \frac{P'_{\upsilon(b)}}{P_{\upsilon(b)}}(\cosh 1) - \frac{b}{2}\right). \tag{3.6}$$

Computing the Legendre function shows that the right hand-side of the above expression is nonnegative when $\gamma \leq b \leq 3$, with $\gamma \approx 1.012$. See Figure 2. Therefore, in the range $b \in [\gamma, 3]$ we have $\beta_b < +\infty$

$$\beta_b \leq \sqrt{\frac{2}{b-1}\left(\frac{b}{2} - \frac{3-b}{4e}(e+1)^2 \frac{P'_{\upsilon(b)}}{P_{\upsilon(b)}}(\cosh 1)\right)}, \qquad \text{for } \gamma \leq b \leq 3, \tag{3.7}$$

and Theorem 2.8 applies in such range.

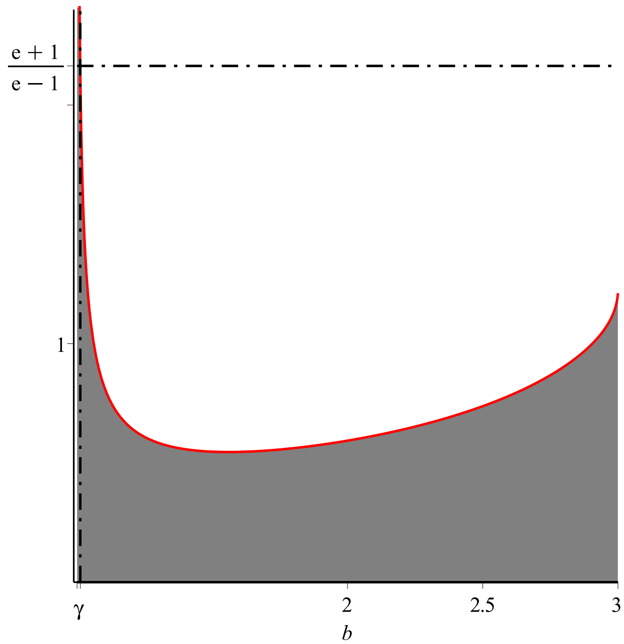


FIGURE 2. The function $b \mapsto \sqrt{\frac{2}{b-1}\left(\frac{b}{2} - \frac{3-b}{4e}(e+1)^2 \frac{P'_{\upsilon(b)}}{P_{\upsilon(b)}}(\cosh 1)\right)}$, providing the bound (3.7). The upper-bound estimates of β_b given by Eq.(3.7) and the Theorem 2.8 are valid inside the interval $[\gamma, 3]$ (grey region)

3.4. **Numerical Analysis of β_b.** In this last part we compute numerically β_b. We need first to compute numerically $J(\beta, b)$. Recall that

$$J(b,\beta) = \frac{b}{2} + \inf\{T(v) : v \in H_0^1(0,1)\},$$

where

$$T(v) = \int_0^1 w(x)\left(\frac{b}{2}(v^2 + 2v) + \left(\frac{3-b}{2}\right) v_x^2\right)(x)\,dx. \tag{3.8}$$

The Euler-Lagrange equation associated with the above minimization problem is

$$(3-b)w(x)v_{xx} + (3-b)w_x v_x - bwv - bw = 0. \tag{3.9}$$

Let $\bar{v}$ be the solution such that $\bar{v}(0) = \bar{v}(1) = 0$, i.e $\bar{v}$ is the minimiser :

$$J(b,\beta) = \frac{b}{2} + \int_0^1 w(x)\left(\frac{b}{2}\bar{v}^2 + b\bar{v} + \left(\frac{3-b}{2}\right)\bar{v}_x^2\right)(x)\,dx. \tag{3.10}$$

On the other hand, multiplying (3.9) by $\bar{v}$ and integrating with respect to the spatial variable, we get

$$\int_0^1 (3-b)w\bar{v}_{xx}\bar{v}\,dx + \int_0^1 (3-b)w_x\bar{v}\bar{v}_x\,dx - \int_0^1 bw(\bar{v}^2 + \bar{v})\,dx = 0.$$

Integrating by parts $\int_0^1 (3-b)w_x\bar{v}\bar{v}_x\,dx$, and using that $\bar{v}(0)=\bar{v}(1)=0$, we get

$$\int_0^1 (3-b)w\bar{v}_x^2 + \int_0^1 b(w\bar{v}^2+\bar{v})\,dx = 0$$
$$\int_0^1 bw\bar{v}\,dx = \int_0^1 w\left(b(\bar{v}^2+2\bar{v})+(3-b)\bar{v}_x^2\right)\,dx.$$

Thus, using $\int_0^1 w\,dx = 1$ and $(3-b)(wv_{xx}+w_xv_x) = bw(v+1)$, we get

$$\begin{aligned} J(b,\beta) &= \frac{b}{2}+\int_0^1 \frac{b}{2}w\bar{v}\,dx \\ &= \frac{3-b}{2}\int_0^1 [w\bar{v}_x]_x\,dx \\ &= \frac{3-b}{2}\left[(w\bar{v}_x)(1^-)-(w\bar{v}_x)(0^+)\right]. \end{aligned}$$

The above solution $\bar{v}$ of the minimization problem, depending on the parameters b and β, cannot be computed analytically, but it it can be computed numerically with the standard numerical schemes for linear ODEs, with an arbitrary good precision. This allow to compute numerically the above function $J(b,\beta)$. This being done, a simple algorithm allows to compute numerically the quantity β_b (with an arbirary good precision). Such numerical computations illustrate that in fact $\beta_b < +\infty$ for $1.0012\ldots \le b \le 3$, which is (slightly !) better than the range $1.012 \le b \le 3$ obtained via Estimate 3. The actual value of β_b is actually slightly smaller than its upper bound computed in (3.7). See Figure 3 and 4. We summarize in the last picture all our previous estimates and numerical approximate of β_b.

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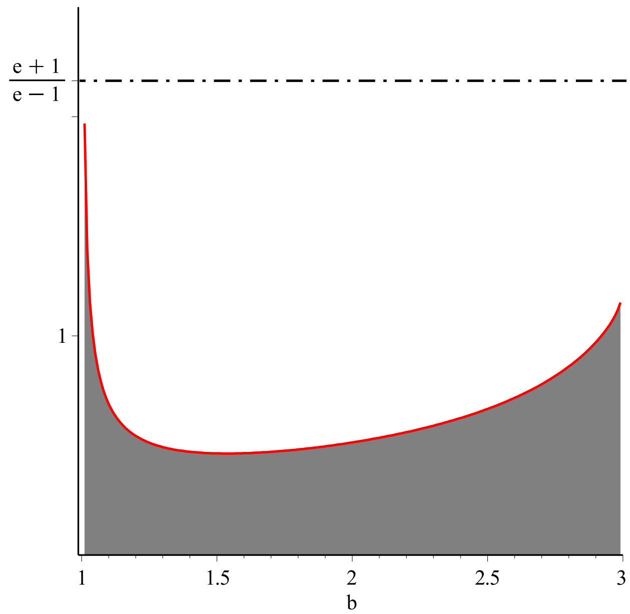

FIGURE 3. The plot of the function $b \mapsto \beta_b$. This numerical approach of β_b, allows us to say: if $3 \geq b \geq \alpha_0 \approx 1.0012$, then the Theorem 2.8 is valid (gray region).

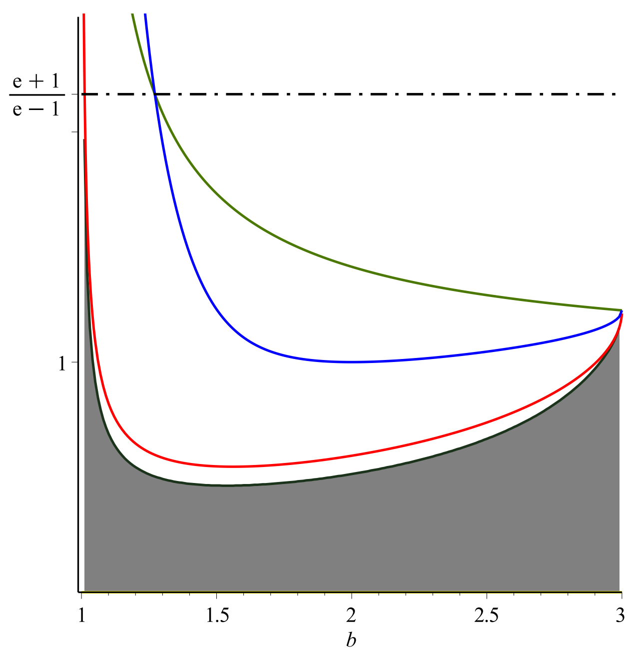

FIGURE 4. In this plot we can see the different estimates that we have worked out (green curve first estimate, blue curve second estimate and red curve third estimates), as well as the numerical approach of β_b.

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