A remark on Besov spaces interpolation over the 2-adic group

Diego Chamorro*

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Abstract

Motivated by a recent result which identifies in the special setting of the 2-adic group the Besov space $\dot{B}^{1,\infty}_1(\mathbb{Z}_2)$ with $BV(\mathbb{Z}_2)$, the space of function of bounded variation, we study in this article some functional relationships between Besov spaces.

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1 Introduction

The starting point of this article is given by the following inequality proved by A. Cohen, W. Dahmen, I. Daubechies & R. De Vore in [4]. For a function $f:\mathbb{R}^n\longrightarrow\mathbb{R}$ such that $f\in BV\cap\dot{B}^{-1,\infty}_\infty$ we have

$$\|f\|^2_{L^2}\leq C\|f\|_{BV}\|f\|_{\dot{B}^{-1,\infty}_\infty} \tag{1}$$

Here BV denotes the space of functions of bounded variation and $\dot{B}^{-1,\infty}_\infty$ stands for an homogeneous Besov space. In the article [3], we proved that in the special setting of the 2-adic group $\mathbb{Z}_2$, the space $BV(\mathbb{Z}_2)$ can be identified to the Besov space $\dot{B}^{1,\infty}_1(\mathbb{Z}_2)$ and therefore, inequality (1) becomes

$$\|f\|^2_{L^2}\leq C\|f\|_{\dot{B}^{1,\infty}_1}\|f\|_{\dot{B}^{-1,\infty}_\infty} \tag{2}$$

Note that the previous estimate is *false* in $\mathbb{Z}_2$, see [3] for a counterexample. The identification between these two functional spaces and the consequences on the inequality (1) are very surprising in the sense that these estimates *depend* on the underlying group structure: compare the topological properties of $\mathbb{R}^n$ to the totally discontinuous setting of $\mathbb{Z}_2$.

However, one may think that the Besov norm $\|\cdot\|_{\dot{B}^{1,\infty}_1}$ in the right hand side of (2) is *too* small to achieve the inequality. Thus, it is a natural question to study the validity of (2) if we replace this norm by a *bigger* one (just think on the inclusion of Besov spaces $\dot{B}^{1,q}_1\subset\dot{B}^{1,\infty}_1$ valid for $q\geq 1$). The answer to this question is given by the next result

Theorem 1 *If $f:\mathbb{Z}_2\longrightarrow\mathbb{R}$ is a function such that $f\in\dot{B}^{1,q}_1\cap\dot{B}^{-1,\infty}_\infty(\mathbb{Z}_2)$ with $q>2$, then the following inequality is false:*

$$\|f\|^2_{L^2}\leq C\|f\|_{\dot{B}^{1,q}_1}\|f\|_{\dot{B}^{-1,\infty}_\infty} \tag{3}$$

*Laboratoire d'Analyse et de Probabilités, Université d'Evry Val d'Essonne, 23 Boulevard de France, 91037 Evry Cedex - France, diego.chamorro@univ-evry.fr

This is the main theorem of this article and we will construct a counterexample in the section 4 below, but before, it would be interesting to compare inequality (3) to the general estimates given by the interpolation theory[1]. Indeed, following this general theory, we can obtain inequalities of the form

$$\|f\|_{L^2}^2 \leq C\|f\|_{\dot{B}^{s_0,q_0}_{p_0}}\|f\|_{\dot{B}^{s_1,q_1}_{p_1}} \tag{4}$$

for some special values of the real parameters $s_0, s_1, p_0, p_1, q_0, q_1$.

Perhaps the most popular case is given by the real method: set $p_0 = p_1 = p$, fix $0 < \theta < 1$ and suppose $s_0 \neq s_1$ with the relationship $s = (1-\theta)s_0 + \theta s_1$. We obtain the following expression

$$\big(\dot{B}^{s_0,q_0}_{p}, \dot{B}^{s_1,q_1}_{p}\big)_{\theta,q} = \dot{B}^{s,q}_{p}$$

which gives us the estimate

$$\|f\|_{\dot{B}^{s,q}_{p}} \leq C\|f\|^{1-\theta}_{\dot{B}^{s_0,q_0}_{p}}\|f\|^{\theta}_{\dot{B}^{s_1,q_1}_{p}} \tag{5}$$

It is very important to remark that in this particular case no relationship between q_0, q_1 and q is asked. Obviously, inequality (3) can not be obtained from (5), since $p_0 \neq p_1$.

The case when $p_0 \neq p_1$ is more restrictive and following the complex method we have for $1 \leq p_0, q_0 \leq +\infty$ and $1 \leq p_1, q_1 < +\infty$ the formula

$$\big[\dot{B}^{s_0,q_0}_{p_0}, \dot{B}^{s_1,q_1}_{p_1}\big]_{\theta} = \dot{B}^{s,q}_{p}$$

which gives us an estimate of the type (4) with $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Note that we have in this case a relationship between q_0, q_1 and q. Again, this method can not be applied to inequality (3).

It seems of course that inequality (3) cannot be obtained by an simple interplation argument -actually this inequality is false in $\mathbb{R}^n$-, but what it would make it plausible in the setting of $\mathbb{Z}_2$ is the special relationship between inequalities (1) and (2) and this is the main reason why Theorem 1 is relevant.

The plan of the article is the following. In section 2 we recall some properties of the p-adic spaces, in section 3 we give the definition of Besov spaces over the 2-adic group $\mathbb{Z}_2$ and in section 4 we prove Theorem 1.

2 p-adic groups

Our main reference here are the books [9], [7] and [1] where more details concerning the topological structure of the p-adic groups can be found.

We write $a|b$ when a divide b or, equivalently, when b is a multiple of a. Let p be any prime number, for $0 \neq x \in \mathbb{Z}$, we define the p-adic valuation of x by $\gamma(x) = \max\{r : p^r|x\} \geq 0$ and, for any rational number $x = \frac{a}{b} \in \mathbb{Q}$, we write $\gamma(x) = \gamma(a) - \gamma(b)$. Furthermore if $x = 0$, we agree to write $\gamma(0) = +\infty$.

[1]see the book [2] for more details.

Let $x \in \mathbb{Q}$ and p be any prime number, with the p-adic valuation of x we can construct a norm by writing

$$|x|_p = \begin{cases} p^{-\gamma} & \text{if} \quad x \neq 0 \\ \\ p^{-\infty} = 0 & \text{if} \quad x = 0. \end{cases} \tag{6}$$

This expression satisfy the following properties

a) $|x|_p \geq 0$, and $|x|_p = 0 \iff x = 0$;

b) $|xy|_p = |x|_p |y|_p$;

c) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$, with equality when $|x|_p \neq |y|_p$.

When a norm satisfy *c)* it is called a non-Archimedean norm and an interesting fact is that over $\mathbb{Q}$ *all* the possible norms are equivalent to $|\cdot|_p$ for some p: this is the so-called Ostrowski theorem, see [1] for a proof.

Definition 2.1 *Let p be a any prime number. We define the field of p-adic numbers $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ when using the norm $|\cdot|_p$.*

We present in the following lines the algebraic structure of the set $\mathbb{Q}_p$. Every p-adic number $x \neq 0$ can be represented in a unique manner by the formula

$$x = p^{\gamma}(x_0 + x_1 p + x_2 p^2 + ...), \tag{7}$$

where $\gamma = \gamma(x)$ is the p-adic valuation of x and x_j are integers such that $x_0 > 0$ and $0 \leq x_j \leq p - 1$ for $j = 1, 2, ...$ Remark that this canonical representation implies the identity $|x|_p = p^{-\gamma}$.

Let $x, y \in \mathbb{Q}_p$, using the formula (7) we define the sum of x and y by $x + y = p^{\gamma(x+y)}(c_0 + c_1 p + c_2 p^2 + ...)$ with $0 \leq c_j \leq p - 1$ and $c_0 > 0$, where $\gamma(x + y)$ and c_j are the unique solution of the equation

$$p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...) + p^{\gamma(y)}(y_0 + y_1 p + y_2 p^2 + ...) = p^{\gamma(x+y)}(c_0 + c_1 p + c_2 p^2 + ...).$$

Furthermore, for $a, x \in \mathbb{Q}_p$, the equation $a + x = 0$ has a unique solution in $\mathbb{Q}_p$ given by $x = -a$. In the same way, the equation $ax = 1$ has a unique solution in $\mathbb{Q}_p$: $x = 1/a$.

We take now a closer look at the topological structure of $\mathbb{Q}_p$. With the norm $|\cdot|_p$ we construct a distance over $\mathbb{Q}_p$ by writing

$$d(x, y) = |x - y|_p \tag{8}$$

and we define the balls $B_\gamma(x) = \{y \in \mathbb{Q}_p : d(x, y) \leq p^\gamma\}$ with $\gamma \in \mathbb{Z}$. Remark that, from the properties of the p-adic valuation, this distance has the *ultra-metric* property (*i.e.* $d(x, y) \leq \max\{d(x, z), d(z, y)\} \leq |x|_p + |y|_p$).

We gather with the next proposition some important facts concerning the balls in $\mathbb{Q}_p$.

Proposition 2.1 *Let γ be an integer, then we have*

1) the ball $B_\gamma(x)$ is a open and a closed set for the distance (8).

2) every point of $B_\gamma(x)$ is its center.

3) $\mathbb{Q}_p$ endowed with this distance is a complete Hausdorff metric space.

4) $\mathbb{Q}_p$ is a locally compact set.

5) the p-adic group $\mathbb{Q}_p$ is a totally discontinuous space.

3 Functional spaces

In this article, we will work with the subset $\mathbb{Z}_2$ of $\mathbb{Q}_2$ which is defined by $\mathbb{Z}_2 = \{x \in \mathbb{Q}_2 : |x|_2 \leq 1\}$, and we will focus on real-valued functions over $\mathbb{Z}_2$. Since $\mathbb{Z}_2$ is a locally compact commutative group, there exists a Haar measure dx which is translation invariant *i.e.*: $d(x+a) = dx$, furthermore we have the identity $d(xa) = |a|_2 dx$ for $a \in \mathbb{Z}_2^*$. We will normalize the measure dx by setting

$$\int_{\{|x|_2 \leq 1\}} dx = 1.$$

This measure is then unique and we will note $|E|$ the measure for any subset E of $\mathbb{Z}_2$.

Lebesgue spaces $L^p(\mathbb{Z}_2)$ are thus defined in a natural way: $\|f\|_{L^p} = \left(\int_{\mathbb{Z}_2} |f(x)|^p dx\right)^{1/p}$ for $1 \leq p < +\infty$, with the usual modifications when $p = +\infty$.

See more details about function spaces in [5], [6] and [9].

Let us now introduce the Littlewood-Paley decomposition in $\mathbb{Z}_2$. We note $\mathcal{F}_j$ the Boole algebra formed by the equivalence classes $E \subset \mathbb{Z}_2$ modulo the sub-group $2^j\mathbb{Z}_2$. Then, for any function $f \in L^1(\mathbb{Z}_2)$, we call $S_j(f)$ the conditionnal expectation of f with respect to $\mathcal{F}_j$:

$$S_j(f)(x) = \frac{1}{|B_j(x)|} \int_{B_j(x)} f(y)dy.$$

The dyadic blocks are thus defined by the formula $\Delta_j(f) = S_{j+1}(f) - S_j(f)$ and the Littlewood-Paley decomposition of a function $f : \mathbb{Z}_2 \longrightarrow \mathbb{R}$ is given by

$$f = S_0(f) + \sum_{j=0}^{+\infty} \Delta_j(f) \qquad \text{where } S_0(f) = \int_{\mathbb{Z}_2} f(x)dx. \tag{9}$$

We will need in the sequel some very special sets noted $Q_{j,k}$. Here is the definition and some properties:

Proposition 3.1 *Let $j \in \mathbb{N}$ and $k = \{0, 1, ..., 2^j - 1\}$. Define the subset $Q_{j,k}$ of $\mathbb{Z}_2$ by*

$$Q_{j,k} = \left\{k + 2^j\mathbb{Z}_2\right\}. \tag{10}$$

Then

1) We have the identity $\mathcal{F}_j = \bigcup_{0 \leq k < 2^j} Q_{j,k}$,

2) For $k = \{0, 1, ..., 2^j - 1\}$ the sets $Q_{j,k}$ are mutually disjoint,

3) $|Q_{j,k}| = 2^{-j}$ for all k,

4) the 2-adic valuation is constant over $Q_{j,k}$.

The verifications are easy and left to the reader.

With the Littlewood-Paley decomposition given in (9), we obtain the following equivalence for the Lebesgue spaces $L^p(\mathbb{Z}_2)$ with $1 < p < +\infty$:

$$\|f\|_{L^p} \simeq \|S_0(f)\|_{L^p} + \left\| \left(\sum_{j\in\mathbb{N}} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p}.$$

See the book [8], chapter IV, for a general proof.

For Besov spaces we will define them by the norm

$$\|f\|_{B^{s,q}_p} \simeq \|S_0 f\|_{L^p} + \left(\sum_{j\in\mathbb{N}} 2^{jsq} \|\Delta_j f\|^q_{L^p} \right)^{1/q} \tag{11}$$

where $s \in \mathbb{R}$, $1 \leq p, q < +\infty$ with the necessary modifications when $p, q = +\infty$.

Remark 1 For homogeneous functional spaces $\dot{B}^{s,q}_p$, we drop out the term $\|S_0 f\|_{L^p}$ in (11).

4 Proof of the Theorem 1

To begin the construction of the counterexample we consider $0 < j_0 < j_1$ two integers and we fix $\alpha, \beta \in \mathbb{R}$ such that

$$2^{2j_0} \leq \frac{\beta}{\alpha}. \tag{12}$$

Take now a decreasing sequence $(\varepsilon_j)_{j\in\mathbb{N}} \in \ell^q(\mathbb{N})$ with $q > 2$ such that $\varepsilon_0 = 1$ and $(\varepsilon_j)_{j\in\mathbb{N}} \notin \ell^2(\mathbb{N})$.

Define N_j in the following form

$$N_j = \begin{cases} 2^j & \text{if} \quad 0 < j < j_0, \\ \\ 2^{-j}\frac{\beta}{\alpha} & \text{if} \quad j_0 \leq j \leq j_1. \end{cases} \tag{13}$$

We construct a function $f : \mathbb{Z}_2 \longrightarrow \mathbb{R}$ by considering his values over the dyadic blocs and we will use for this the sets $Q_{j,k}$ defined in (10):

$$\Delta_j f(x) = \begin{cases} \varepsilon_j \alpha 2^j & \text{over} \quad Q_{j+1,0}, \\ -\varepsilon_j \alpha 2^j & \text{over} \quad Q_{j+1,1}, \\ \varepsilon_j \alpha 2^j & \text{over} \quad Q_{j+1,2}, \\ -\varepsilon_j \alpha 2^j & \text{over} \quad Q_{j+1,3}, \\ & \quad \vdots \\ \varepsilon_j \alpha 2^j & \text{over} \quad Q_{j+1,2N_j-2}, \\ -\varepsilon_j \alpha 2^j & \text{over} \quad Q_{j+1,2N_j-1}, \\ 0 & \text{elsewhere.} \end{cases}$$

Remark that, with this definition of $\Delta_j f(x)$ we have the identities

- $\|\Delta_j f\|_{L^\infty} = \varepsilon_j \alpha 2^j$,
- $\|\Delta_j f\|_{L^1} = \varepsilon_j \alpha N_j$,

- $\|\Delta_j f\|_{L^2}^2 = \varepsilon_j^2 \alpha^2 2^j N_j$.

From this quantities we construct the following norms

(a) for the Besov space $\dot{B}^{-1,\infty}_{\infty}$ we have

$$\|f\|_{\dot{B}^{-1,\infty}_{\infty}} = \sup_{j\in\mathbb{N}} 2^{-j}\|\Delta_j f\|_{L^\infty} = \alpha, \text{ since the sequence } (\varepsilon_j)_{j\in\mathbb{N}} \text{ is decreasing and } \varepsilon_0 = 1.$$

(b) for the Besov space $\dot{B}^{1,q}_{1}$ we write

$$\|f\|^q_{\dot{B}^{1,q}_1} = \sum_{j=0}^{j_1} \left(2^j\|\Delta_j f\|_{L^1}\right)^q = \sum_{j=0}^{j_1} 2^{jq}\varepsilon_j^q \alpha^q N_j^q = \alpha^q \left(\sum_{j=0}^{j_0} 2^{jq}\varepsilon_j^q N_j^q + \sum_{j>j_0}^{j_1} 2^{jq}\varepsilon_j^q N_j^q\right)$$

We use now the values of N_j given in (13) and the relationship (12) to obtain

$$= \alpha^q \left(\sum_{j=0}^{j_0} 2^{2jq}\varepsilon_j^q + \sum_{j>j_0}^{j_1} \varepsilon_j^q \frac{\beta^q}{\alpha^q}\right) = \beta^q \left(\sum_{j=0}^{j_0} 2^{2jq}\frac{\alpha^q}{\beta^q}\varepsilon_j^q + \sum_{j>j_0}^{j_1} \varepsilon_j^q\right) \simeq \beta^q \left(\sum_{j=0}^{j_0} 2^{q(2j-2j_0)}\varepsilon_j^q + \sum_{j>j_0}^{j_1} \varepsilon_j^q\right).$$

Then we have $\|f\|_{\dot{B}^{1,q}_1} \simeq \beta\left(C_1 + \sum_{j>j_0}^{j_1} \varepsilon_j^q\right)^{1/q}$.

(c) For the Lebesgue space L^2 we use the same arguments above to obtain

$$\|f\|^2_{L^2} = \sum_{j=0}^{j_1} \varepsilon_j^2 \alpha^2 2^j N_j = \alpha^2 \left(\sum_{j=0}^{j_0} 2^{2j}\varepsilon_j^2 + \sum_{j>j_0}^{j_1} \varepsilon_j^2 \frac{\beta}{\alpha}\right) \simeq \alpha\beta \left(C_2 + \sum_{j>j_0}^{j_1} \varepsilon_j^2\right).$$

Once these norms are computed, we go back to the inequality

$$\|f\|^2_{L^2} \leq C\|f\|_{\dot{B}^{1,q}_1}\|f\|_{\dot{B}^{-1,\infty}_{\infty}}$$

and we have

$$\alpha\beta\left(C_2 + \sum_{j>j_0}^{j_1}\varepsilon_j^2\right) \leq C\times\alpha\times\beta\left(C_1 + \sum_{j>j_0}^{j_1}\varepsilon_j^q\right)^{1/q}.$$

But, by hypothesis, we have $(\varepsilon_j)_{j\in\mathbb{N}} \notin \ell^2(\mathbb{N})$ and $(\varepsilon_j)_{j\in\mathbb{N}} \in \ell^q(\mathbb{N})$, thus, for j_1 big enough it is impossible to find an universal constant C such that the above inequality is true.

■

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Diego Chamorro

Laboratoire d'Analyse et de Probabilités
Université d'Evry Val d'Essonne

23 Boulevard de France
91037 Evry Cedex

diego.chamorro@univ-evry.fr