Non-local diffusion equations with Lévy-type operators and divergence free drift

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Abstract

In this paper we are interested in some properties related to the solutions of non-local diffusion equations with divergence free drift. Existence, maximum principle and a positivity principle are proved. In order to study Hölder regularity, we apply a method that relies in the Hölder-Hardy spaces duality and in the molecular characterisation of local Hardy spaces. In these equations, the diffusion is given by Lévy-type operators with an associated Lévy measure satisfying some upper and lower bounds.

Keywords: Lévy-type operators, Lévy-Khinchin formula, Hölder regularity, molecular Hardy spaces.

1 Introduction

We study in this article a class of non-local diffusion equations with divergence free drift of the following form:

$$\begin{cases} \partial_t\theta(x,t) - \nabla\cdot(v\,\theta)(x,t) + \mathcal{L}\theta(x,t) = 0, \\ \\ \theta(x,0) = \theta_0(x), \\ \\ \text{with } \ div(v) = 0 \ \text{ and } t\in[0,T]. \end{cases} \tag{1}$$

This type of transport-diffusion equations is a generalization of a well-known equation from fluid dynamics. Indeed, in space dimension $n = 2$ if $\mathcal{L} = (-\Delta)^\alpha$ is the fractional Laplacian, with $0 < \alpha \leq 1$, and if $v = (-R_2\theta, R_1\theta)$ where $R_{1,2}$ are the Riesz Transforms defined in the Fourier level by $\widehat{R_j\theta}(\xi) = -\frac{i\xi_j}{|\xi|}\widehat{\theta}(\xi)$ for $j = 1, 2$, we obtain the quasi-geostrophic equation $(QG)_\alpha$ which has been recently studied by many authors with different approaches and with a variety of results, see [1], [6], [12], [4], [5], [14] and the references there in for more details. In the framework of this equation it is classical to distinguish three regimes: *super-critical* if $0 < \alpha < 1/2$, *critical* if $\alpha = 1/2$ and *sub-critical* if $1/2 < \alpha < 1$, from which only the two first are of interest since in the sub-critical case the regularization effect is in some sense "stronger" than the drift. In the other cases there is an interesting and complex competition between the smoothing term and the transport term, see [5] for more details.

Inspired by the work of Kiselev and Nazarov [12], it is possible to study the Hölder regularity of the solutions of the $(QG)_{1/2}$ equation by a duality-based method. The aim of this article is to generalize this method to a wider family of operators and we will consider here Lévy-type operators under some hypothesis that will be stated in the lines below. This class of operators corresponds to a natural generalization of recent works where some results are obtained for different operators using quite specific techniques: for example see the article [13] where the operator's kernel satisfies some similar bounds to those imposed in our hypothesis.

In this paper we will mainly consider problems of existence of the solutions, a maximum principle, a positivity principle and of course we will study Hölder regularity of the solutions of equation (1).

Let us start by describing our setting in a general way. This framework will be made precise later on.

- In the formula (1) we noted $\theta : \mathbb{R}^n \times [0, T] \longrightarrow \mathbb{R}$ a real-valued function, where $n \geq 2$ is the euclidean dimension.

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- The drift (or velocity) term v is such that $v : \mathbb{R}^n \times [0,T] \longrightarrow \mathbb{R}^n$ and we will always assume that $div(v) = 0$ and that v belongs to $L^\infty([0,T]; bmo(\mathbb{R}^n))$. Recall that local $bmo(\mathbb{R}^n)$ space is defined as locally integrable functions f such that

$$\sup_{|B|\leq 1} \frac{1}{|B|}\int_B |f(x) - f_B|dx < M \qquad \text{and} \qquad \sup_{|B|>1} \frac{1}{|B|}\int_B |f(x)|dx < M \qquad \text{for a constant } M;$$

we noted $B(R)$ a ball of radius $R > 0$ and $f_B = \frac{1}{|B|}\displaystyle\int_{B(R)} f(x)dx$. The norm $\|\cdot\|_{bmo}$ is then fixed as the smallest constant M satisfying these two conditions.

- The operator $\mathcal{L}$ is a Lévy operator which has the following general form called the *Lévy-Khinchin* representation formula:

$$\mathcal{L}(f)(x) = b\cdot\nabla f(x) + \sum_{j,k=1}^n a_{j,k}\frac{\partial^2 f(x)}{\partial x_j \partial x_k} + \int_{\mathbb{R}^n\setminus\{0\}} \big[f(x) - f(x-y) + y\cdot\nabla f(x)\mathbb{1}_{\{|y|\leq 1\}}(y)\big]\Pi(dy),$$

where $b \in \mathbb{R}^n$ is a vector, $a_{j,k}$ are constants (note that the matrix $(a_{j,k})_{1\leq j,k\leq n}$ should be positive semi-definite) and Π is a nonnegative Borel measure on $\mathbb{R}^n$ satisfying $\Pi(\{0\}) = 0$ and

$$\int_{\mathbb{R}^n} \min(1, |y|^2)\Pi(dy) < +\infty. \tag{2}$$

In the Fourier level we have $\widehat{\mathcal{L}f}(\xi) = a(\xi)\widehat{f}(\xi)$ where the symbol $a(\cdot)$ is given by the Lévy-Khinchin formula

$$a(\xi) = ib\cdot\xi + q(\xi) + \int_{\mathbb{R}^n\setminus\{0\}}\Big(1 - e^{-iy\cdot\xi} - iy\cdot\xi\mathbb{1}_{\{|y|<1\}}(y)\Big)\Pi(dy), \qquad \text{where } q(\xi) = \sum_{j,k=1}^n a_{j,k}\xi_j\xi_k. \tag{3}$$

Our main references concerning Lévy operators and the Lévy-Khinchin representation formula are the books [9], [10] and [16]. See also the lecture notes [11] for interesting applications to the PDEs.

We need to make some assumptions for the Lévy operator considered before. First we will set $b = 0$ and $a_{j,k} = 0$. We assume then that the measure Π is absolutely continuous with respect to the Lebesgue measure, so this measure can be written as $\Pi(dy) = \pi(y)dy$, this hypothesis is important as it simplifies considerably the computations. We will also require some symmetry in the following sense: $\pi(y) = \pi(-y)$. Finally, the most crucial issue concerns some estimates on the function π and we will assume the inequalities:

$$c_1|y|^{-n-2\alpha} \leq \quad \pi(y) \quad \leq c_2|y|^{-n-2\beta} \qquad \text{over } |y|\leq 1, \tag{4}$$

$$0 \leq \quad \pi(y) \quad \leq c_3|y|^{-n-2\delta} \qquad \text{over } |y| > 1, \tag{5}$$

where $c_1, c_2, c_3 > 0$ are positive constants. We need to define the values of the parameters α, β, δ and we will study the following cases:

(a) $0 < \alpha \leq \beta < 1/2$ and $0 < \delta < 1/2$,

(b) $0 < \alpha = \beta = \delta < 1/2$,

(c) $\alpha = \beta = 1/2$ and $0 < \delta < 1/2$,

(d) $\alpha = \beta = \delta = 1/2$.

The choice of these bounds is mainly technical: some of our results (Theorem 1, 2, and 3) are valid for the cases **(a)**, **(b)**, **(c)** and **(d)**; but Theorem 4, which is the main novelty of this paper, is valid for cases **(c)** and **(d)** and it is necessary to make this distinction since in some special cases counterexamples can be constructed (see [18]). Finally, the upper bound 1/2 is inherited from the quasi-geostrophic equation regimes as it was explained before.

Note now that these two conditions (4) and (5) imply the next pointwise property which will be useful in the sequel

$$0 \leq \pi(y) \leq c_4(|y|^{-n-2\beta} + |y|^{-n-2\delta}) \qquad \text{for all} \quad y \in \mathbb{R}^n \text{ and } c_4 > 0. \tag{6}$$

We observe also that these assumptions for the function π imply that the operator $\mathcal{L}$ and its symbol $a(\cdot)$ can be rewritten in the following way:

$$\mathcal{L}(f)(x) = \text{v.p.}\int_{\mathbb{R}^n}\big[f(x) - f(x-y)\big]\pi(y)dy \tag{7}$$

and

$$a(\xi) = \int_{\mathbb{R}^n \setminus \{0\}} \big(1 - \cos(\xi \cdot y)\big)\pi(y)dy. \tag{8}$$

As we can see, the properties of the operator $\mathcal{L}$ can be easily read, in the real variable or in the Fourier level, by the properties of the function π.

In order to have a better understanding of these properties it is helpful to consider an important example which is given by the fractional Laplacian $(-\Delta)^\alpha$ defined by the expression

$$(-\Delta)^\alpha f(x) = \text{v.p.} \int_{\mathbb{R}^n} \frac{f(x) - f(x-y)}{|y|^{n+2\alpha}} dy, \quad \text{with } 0 < \alpha \leq 1/2.$$

Note that we have here $\pi(y) = |y|^{-n-2\alpha}$ and π satisfies (4) and (5) with $\alpha = \beta = \delta$, so this example corresponds to the cases **(b)** and **(d)** stated above. Equivalently, we have a Fourier characterisation by the formula $\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\widehat{f}(\xi)$ so the function $a(\xi)$ is equal to $|\xi|^{2\alpha}$.

With this example we observe that the lower bound in (4) guarantees a *diffusion or regularization effect*[1] like $(-\Delta)^\alpha$ and this is an important assumption for the function π. Indeed, in some general sense, only the part of the integral (7) near the origin is critical as π satisfies (5). We note also that the upper bounds given in (4) and (5) imply the property (2) since in any case we have $\beta, \delta \leq 1/2$.

Let us consider more examples: it is shown in Theorem 3.7.7 of [9], that each continuous negative definite function $a(\cdot)$ can be writen in the form (3), so under hypothesis (4) and (5) we can obtain a large class of operators that are in the scope of this work. In the paper [13] another approach is given: the assumptions for the function π are quite similar but they are stated in a different way, furthermore the authors of this article only consider the case $\alpha = \beta = \delta$ in their hypothesis, so our framework is slightly more general. However they allow dependence of the function π in the x variable and in the time variable t. A further work could follow this path, assuming for example in formula (7) that $\pi = \pi(x, y, t)$ instead of $\pi = \pi(y)$. Note that some amount of work is already done in this direction, see chapter 4 and Definition 4.5.10 of [9] for more information.

Presentation of the results

We assume from now on that the operator $\mathcal{L}$ is of the form (7). We will work with a function π satisfying the hypothesis (4) and (5) with the parameters α, β, δ satisfying **(a)-(d)** unless otherwise specified.

In this article we present some results concerning non-local diffusion equation (1). Maybe the three first of them are well known for different mathematical communities, so perhaps the only novelty here is the use of the *bmo* space. Nevertheless we will give the proofs for the sake of completness.

Theorem 1 (Existence and uniqueness for L^p initial data) *If $\theta_0 \in L^p(\mathbb{R}^n)$ with $2 \leq p \leq +\infty$ is an initial data, then equation (1) has a unique weak solution $\theta \in L^\infty([0,T]; L^p(\mathbb{R}^n))$.*

Theorem 2 (Maximum Principle) *Let $\theta_0 \in L^p(\mathbb{R}^n)$ with $2 \leq p \leq +\infty$ be an initial data, then the weak solution of equation (1) satisfies the following maximum principle for all $t \in [0,T]$: $\|\theta(\cdot,t)\|_{L^p} \leq \|\theta_0\|_{L^p}$.*

Theorem 3 (Positivity Principle) *Let β and δ be the parameters given in cases **(a)-(d)**. Let $\frac{n}{2\min(\beta,\delta)} < p \leq +\infty$ and $M > 0$ a constant, if the initial data $\theta_0 \in L^p(\mathbb{R}^n)$ is such that $0 \leq \theta_0 \leq M$ then the weak solution of equation (1) satisfies $0 \leq \theta(x,t) \leq M$ for all $t \in [0,T]$.*

Our main theorem is the following one which is a generalization of a duality method used in the framework of the quasi-geostrophic equation. With this method we obtain a small regularity gain, but for technical reasons we need to consider here the cases **(c)** and **(d)**.

Theorem 4 (Hölder regularity) *Let $\mathcal{L}$ be a Lévy operator of the form (7) with a Lévy measure π satisfying hypothesis (4) and (5) with $\alpha = \beta = 1/2$ and $\delta < 1/2$ or $\alpha = \beta = \delta = 1/2$. Fix a small time $T_0 > 0$. Let θ_0 be a function such that $\theta_0 \in L^\infty(\mathbb{R}^n)$. If $\theta(x,t)$ is a solution for the equation (1), then for all time $T_0 < t < T$, we have that $\theta(\cdot,t)$ belongs to the Hölder space $\mathcal{C}^\gamma(\mathbb{R}^n)$ with $0 < \gamma < 2\delta < 1$ in the case **(c)** or $0 < \gamma < 1$ in the case **(d)**.*

The plan of the article is the following: in the section 2 we study existence and uniqueness of solutions with initial data in L^p with $1 \leq p < +\infty$. In this section we will also prove the maximum principle. Section 3 is devoted to a positivity principle that will be useful in our proofs and section 4 studies existence of solution with $\theta_0 \in L^\infty$. In section 5 we study the Hölder regularity of the solutions of equation (1) by a duality method.

[1] the term "diffusion" must be taken in the sense of the PDEs considered by analysts.

2 Existence and uniqueness with L^p initial data and Maximum Principle.

In this section we will study existence and uniqueness for weak solution of equation (1) with initial data $\theta_0 \in L^p(\mathbb{R}^n)$ where $p \geq 1$. We will start by considering *viscosity solutions* with an approximation of the velocity field v, and we will prove existence and uniqueness for this system. To pass to the limit we will need a further step that is a consequence of the maximum principle.

2.1 Viscosity solutions

We begin our study with the following approximation of the equation (1):

$$\begin{cases} \partial_t \theta(x,t) + \nabla \cdot (v_\varepsilon\, \theta)(x,t) + \mathcal{L}\theta(x,t) = \varepsilon \Delta \theta(x,t) & (\varepsilon > 0) \\ \theta(x,0) = \theta_0(x) & \\ div(v) = 0 \quad \text{ and } v \in L^\infty([0,T]; L^\infty(\mathbb{R}^n)). & \end{cases} \tag{9}$$

where v_ε is defined by $v_\varepsilon = v * \omega_\varepsilon$ with $\omega_\varepsilon(x) = \varepsilon^{-n}\omega(x/\varepsilon)$ and $\omega \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is a function such that $\displaystyle\int_{\mathbb{R}^n} \omega(x)dx = 1$. Here $\mathcal{L}$ is a Lévy operator of the form (7) with hypothesis (4) and (5) with α, β, δ satisfying the bounds given in the cases **(a)-(d)**. Following [6], the solutions of this problem are called *viscosity solutions*.

Remark 2.1 *Since the velocity v is a data of the problem, it is equivalent to consider $-v$ instead of v, thus for simplicity we fix velocity's sign as in equation (9) above. The same proofs are valid for equation (1).*

Observe that we fixed here the velocity v such that $v \in L^\infty([0,T']; L^\infty(\mathbb{R}^n))$. This is not very restrictive since we have the following lemma:

Lemma 2.1 *Let f be a function in $bmo(\mathbb{R}^n)$. For $k \in \mathbb{N}$, define f_k by*

$$f_k(x) = \begin{cases} -k & if \quad f(x) \leq -k \\ f(x) & if \quad -k \leq f(x) \leq k \\ k & if \quad k \leq f(x). \end{cases} \tag{10}$$

Then $(f_k)_{k\in\mathbb{N}}$ converges weakly to f in $bmo(\mathbb{R}^n)$.

A proof of this lemma can be found in [19].

Note now that the problem (9) admits the following equivalent integral representation:

$$\theta(x,t) = e^{\varepsilon t \Delta}\theta_0(x) - \int_0^t e^{\varepsilon(t-s)\Delta}\nabla \cdot (v_\varepsilon\, \theta)(x,s)ds - \int_0^t e^{\varepsilon(t-s)\Delta}\mathcal{L}\theta(x,s)ds, \tag{11}$$

In order to prove Theorem 1, we will first investigate a local result with the following theorem where we will apply the Banach contraction scheme in the space $L^\infty([0,T]; L^p(\mathbb{R}^n))$ with the norm $\|f\|_{L^\infty(L^p)} = \sup\limits_{t\in[0,T]} \|f(\cdot,t)\|_{L^p}$.

Theorem 5 (Local existence for viscosity solutions) *Let $1 \leq p < +\infty$ and let θ_0 and v be two functions such that $\theta_0 \in L^p(\mathbb{R}^n)$, $div(v) = 0$ and $v \in L^\infty([0,T']; L^\infty(\mathbb{R}^n))$. If the initial data satisfies $\|\theta_0\|_{L^p} \leq K$ and if T' is a time small enough, then (11) has a unique solution $\theta \in L^\infty([0,T']; L^p(\mathbb{R}^n))$ on the closed ball $\overline{B}(0,2K) \subset L^\infty([0,T']; L^p(\mathbb{R}^n))$.*

Proof of Theorem 5. We note $L_\varepsilon(\theta)$ and $N_\varepsilon^v(\theta)$ the quantities

$$L_\varepsilon(\theta)(x,t) = \int_0^t e^{\varepsilon(t-s)\Delta}\mathcal{L}\theta(x,s)ds \quad \text{ and } \quad N_\varepsilon^v(\theta)(x,t) = \int_0^t e^{\varepsilon(t-s)\Delta}\nabla \cdot (v_\varepsilon\, \theta)(x,s)ds.$$

We construct now a sequence of functions in the following way

$$\theta_{n+1}(x,t) = e^{\varepsilon t\Delta}\theta_0(x) - L_\varepsilon(\theta_n)(x,t) - N_\varepsilon^v(\theta_n)(x,t),$$

we take the $L^\infty L^p$-norm of this expression to obtain

$$\|\theta_{n+1}\|_{L^\infty(L^p)} \leq \|e^{\varepsilon t\Delta}\theta_0\|_{L^\infty(L^p)} + \|L_\varepsilon(\theta_n)\|_{L^\infty(L^p)} + \|N^v_\varepsilon(\theta_n)\|_{L^\infty(L^p)} \tag{12}$$

and we will treat each one of the terms of the left-hand side separately.

For the first term above we note that, since $e^{\varepsilon t\Delta}$ is a contraction operator, the estimate $\|e^{\varepsilon t\Delta}f\|_{L^p} \leq \|f\|_{L^p}$ is valid for all function $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq +\infty$, for all $t > 0$ and all $\varepsilon > 0$. Thus, we have

$$\|e^{\varepsilon t\Delta}f\|_{L^\infty(L^p)} \leq \|f\|_{L^p}. \tag{13}$$

For the second term of (12) we have the following fact: if $f \in L^\infty([0,T'];L^p(\mathbb{R}^n))$, then

$$\|L_\varepsilon(f)\|_{L^\infty(L^p)} \leq C\Phi(T',\varepsilon)\ \|f\|_{L^\infty(L^p)} \tag{14}$$

where $\Phi(T',\varepsilon) = \left(\frac{T'^{1-\beta}}{\varepsilon^\beta} + \frac{T'^{1-\delta}}{\varepsilon^\delta}\right)$; $\left(\frac{T'^{1-\alpha}}{\varepsilon^\alpha}\right)$; $\left(\frac{T'^{1/2}}{\varepsilon^{1/2}} + T' + \frac{T'^{1-\delta}}{\varepsilon^\delta}\right)$ and $\left(\frac{T'^{1/2}}{\varepsilon^{1/2}}\right)$, for the cases **(a)-(d)** respectively.

Indeed, we write

$$\|L_\varepsilon(f)\|_{L^\infty(L^p)} = \sup_{0<t<T'}\left\|\int_0^t e^{\varepsilon(t-s)\Delta}\mathcal{L}f(\cdot,s)ds\right\|_{L^p} = \sup_{0<t<T'}\left\|\int_0^t \mathcal{L}f * h_{\varepsilon(t-s)}(\cdot,s)ds\right\|_{L^p}$$

where h_t is the heat kernel on $\mathbb{R}^n$. By the properties of the Lévy operator $\mathcal{L}$ we can write $\mathcal{L}f * h_{\varepsilon(t-s)} = f * \mathcal{L}h_{\varepsilon(t-s)}$ and then we obtain the estimate

$$\|L_\varepsilon(f)\|_{L^\infty(L^p)} \leq \sup_{0<t<T'}\int_0^t \|f(\cdot,s)\|_{L^p}\|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1}ds \leq \|f\|_{L^\infty(L^p)}\sup_{0<t<T'}\int_0^t \|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1}ds.$$

We need now to study the quantity $\|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1}$, for this we will use Besov spaces and a short lemma. We recall that for $0 < s < 1$ and $1 \leq p < +\infty$, homogeneous Besov spaces $\dot{B}^{s,p}_p(\mathbb{R}^n)$ may be defined as

$$\|f\|_{\dot{B}^{s,p}_p} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(x-y)|^p}{|y|^{n+ps}}dydx\right)^{1/p}.$$

Now, here is the lemma (see a proof in the appendix):

Lemma 2.2 *Let $\mathcal{L}$ be a Lévy operator satisfying the hypothesis stated above.*

- ***(a)*** *If $0 < \alpha \leq \beta < 1/2$ and $0 < \delta < 1/2$ then, for all $f \in \dot{B}^{2\beta,1}_1(\mathbb{R}^n)\cap\dot{B}^{2\delta,1}_1(\mathbb{R}^n)$ we have $\|\mathcal{L}f\|_{L^1} \leq \|f\|_{\dot{B}^{2\beta,1}_1} + \|f\|_{\dot{B}^{2\delta,1}_1}$. In particular we have for the heat kernel $\|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1} \leq C\big([\varepsilon(t-s)]^{-\beta} + [\varepsilon(t-s)]^{-\delta}\big)$.*
- ***(b)*** *If $\alpha = \beta = \delta < 1/2$, we have $\mathcal{L} = (-\Delta)^\alpha$ and thus $\|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1} \leq C[\varepsilon(t-s)]^{-\alpha}$.*
- ***(c)*** *If $\alpha = \beta = 1/2$ and $0 < \delta < 1/2$ we have $\|\mathcal{L}f\|_{L^1} \leq C\big(\|(-\Delta)^{1/2}f\|_{L^1} + \|f\|_{L^1} + \|f\|_{\dot{B}^{2\delta,1}_1}\big)$ where the quantities above are assumed to be bounded. In particular we have $\|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1} \leq C\big([\varepsilon(t-s)]^{-1/2} + 1 + [\varepsilon(t-s)]^{-\delta}\big)$.*
- ***(d)*** *If $\alpha = \beta = \delta = 1/2$, we have $\mathcal{L} = (-\Delta)^{1/2}$ and thus $\|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1} \leq C[\varepsilon(t-s)]^{-1/2}$.*

We will say then that $\|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1} \leq \varphi(t-s,\varepsilon)$.

With these estimates at our disposal for the quantity $\|\mathcal{L}h_{\varepsilon(t-s)}\|_{L^1}$, after an integration in time and following the different cases we obtain the wished inequality (14).

For the last term of (12) we have the following inequality: if $f \in L^\infty([0,T'];L^p(\mathbb{R}^n))$ and if $v \in L^\infty([0,T'];L^\infty(\mathbb{R}^n))$, then

$$\|N^v_\varepsilon(f)\|_{L^\infty(L^p)} \leq C\sqrt{\frac{T'}{\varepsilon}}\ \|v\|_{L^\infty(L^\infty)}\|f\|_{L^\infty(L^p)} \tag{15}$$

Indeed, we write:

$$\begin{aligned}
\|N^v_\varepsilon(f)\|_{L^\infty(L^p)} &= \sup_{0<t<T'}\left\|\int_0^t e^{\varepsilon(t-s)\Delta}\nabla\cdot(v_\varepsilon f)(\cdot,s)ds\right\|_{L^p} = \sup_{0<t<T'}\left\|\int_0^t \nabla\cdot(v_\varepsilon f) * h_{\varepsilon(t-s)}(\cdot,s)ds\right\|_{L^p}\\
&\leq \sup_{0<t<T'}\int_0^t \|v_\varepsilon f(\cdot,s)\|_{L^p}\left\|\nabla h_{\varepsilon(t-s)}\right\|_{L^1}ds \leq \sup_{0<t<T'}\int_0^t \|v_\varepsilon(\cdot,s)\|_{L^\infty}\|f(\cdot,s)\|_{L^p}\,C(\varepsilon(t-s))^{-1/2}ds\\
&\leq \|v\|_{L^\infty(L^\infty)}\|f\|_{L^\infty(L^p)}\sup_{0<t<T'}\int_0^t C(\varepsilon(t-s))^{-1/2}ds \leq C\sqrt{\frac{T'}{\varepsilon}}\|v\|_{L^\infty(L^\infty)}\|f\|_{L^\infty(L^p)}.
\end{aligned}$$

Now, applying the inequalities (13), (14) and (15) to the left-hand side of (12) we have

$$\|\theta_{n+1}\|_{L^\infty(L^p)} \leq \|\theta_0\|_{L^p} + C\left(\Phi(T',\varepsilon) + \frac{T'^{1/2}}{\varepsilon^{1/2}}\|v\|_{L^\infty(L^\infty)}\right)\|\theta_n\|_{L^\infty(L^p)}$$

Thus, if $\|\theta_0\|_{L^p} \leq K$ and if we define the time T' to be such that $C\left(\Phi(T',\varepsilon) + \frac{T'^{1/2}}{\varepsilon^{1/2}}\|v\|_{L^\infty(L^\infty)}\right) \leq 1/2$, we have by iteration that $\|\theta_{n+1}\|_{L^\infty(L^p)} \leq 2K$: the sequence $(\theta_n)_{n\in\mathbb{N}}$ constructed from initial data θ_0 belongs to the closed ball $\overline{B}(0,2K)$. In order to finish this proof, let us show that $\theta_n \longrightarrow \theta$ in $L^\infty([0,T'];L^p(\mathbb{R}^n))$. For this we write

$$\|\theta_{n+1}-\theta_n\|_{L^\infty(L^p)} \leq \|L_\varepsilon(\theta_n-\theta_{n-1})\|_{L^\infty(L^p)} + \|N_\varepsilon^v(\theta_n-\theta_{n-1})\|_{L^\infty(L^p)}$$

and using the previous results we have

$$\|\theta_{n+1}-\theta_n\|_{L^\infty(L^p)} \leq C\left(\Phi(T',\varepsilon) + \frac{T'^{1/2}}{\varepsilon^{1/2}}\|v\|_{L^\infty(L^\infty)}\right)\|\theta_n-\theta_{n-1}\|_{L^\infty(L^p)}$$

so, by iteration we obtain

$$\|\theta_{n+1}-\theta_n\|_{L^\infty(L^p)} \leq \left[C\left(\Phi(T',\varepsilon) + \frac{T'^{1/2}}{\varepsilon^{1/2}}\|v\|_{L^\infty(L^\infty)}\right)\right]^n \|\theta_1-\theta_0\|_{L^\infty(L^p)}$$

hence, with the definition of T' it comes $\|\theta_{n+1}-\theta_n\|_{L^\infty(L^p)} \leq \left(\frac{1}{2}\right)^n \|\theta_1-\theta_0\|_{L^\infty(L^p)}$. Finally, if $n \longrightarrow +\infty$, the sequence $(\theta_n)_{n\in\mathbb{N}}$ convergences towards θ in $L^\infty([0,T'];L^p(\mathbb{R}^n))$. Since it is a Banach space we deduce uniqueness for the solution θ of problem (11). The proof of Theorem 5 is finished. ■

Corollary 2.1 *The solution constructed above depends continuously on the initial value θ_0.*

Proof. Let $\varphi_0, \theta_0 \in L^p(\mathbb{R}^n)$ be two initial values and let φ and θ be the associated solutions. We write

$$\theta(x,t)-\varphi(x,t) = e^{\varepsilon t\Delta}(\theta_0(x)-\varphi_0(x)) - L_\varepsilon(\theta-\varphi)(x,t) - N_\varepsilon^v(\theta-\varphi)(x,t)$$

Taking $L^\infty L^p$-norm in formula above and applying the same previous calculations one obtains

$$\|\theta-\varphi\|_{L^\infty(L^p)} \leq \|\theta_0-\varphi_0\|_{L^p} + C_0\|\theta-\varphi\|_{L^\infty(L^p)}$$

This shows continuous dependence of the solution since $C_0 = C\left(\Phi(T',\varepsilon) + \frac{T'^{1/2}}{\varepsilon^{1/2}}\|v\|_{L^\infty(L^\infty)}\right) \leq 1/2$. ■

Remark 2.2 (From Local to Global) *Once we obtain a local result, global existence easily follows by a simple iteration since problems studied here (equations (1) or (9)) are linear as the velocity v does not depend on θ.*

We study now the regularity of the solutions constructed by this method.

Theorem 6 *Solutions of the approximated problem (9) are smooth.*

Proof. By iteration we will prove that $\theta \in \bigcap\limits_{0<T_0<T_1<t<T_2<T^*} L^\infty([0,t];W^{\frac{k}{2},p}(\mathbb{R}^n))$ for all $k \geq 0$ where we define the Sobolev space $W^{s,p}(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and $1 < p < +\infty$ by $\|f\|_{W^{s,p}} = \|f\|_{L^p} + \|(-\Delta)^{s/2}f\|_{L^p}$. Note that this is true for $k = 0$. So let us assume that it is also true for $k > 0$ and we will show that it is still true for $k+1$.

Set t such that $0 < T_0 < T_1 < t < T_2 < T^*$ and let us consider the next problem

$$\theta(x,t) = e^{\varepsilon(t-T_0)\Delta}\theta(x,T_0) - \int_{T_0}^t e^{\varepsilon(t-s)\Delta}\nabla\cdot(v_\varepsilon\,\theta)(x,s)ds - \int_{T_0}^t e^{\varepsilon(t-s)\Delta}\mathcal{L}\theta(x,s)ds$$

We have then the following estimate

$$\begin{aligned}\|\theta\|_{L^\infty(W^{\frac{k+1}{2},p})} \leq\ & \|e^{\varepsilon(t-T_0)\Delta}\theta(\cdot,T_0)\|_{L^\infty(W^{\frac{k+1}{2},p})}\\ & + \left\|\int_{T_0}^t e^{\varepsilon(t-s)\Delta}\nabla\cdot(v_\varepsilon\,\theta)(\cdot,s)ds\right\|_{L^\infty(W^{\frac{k+1}{2},p})} + \left\|\int_{T_0}^t e^{\varepsilon(t-s)\Delta}\mathcal{L}\theta(\cdot,s)ds\right\|_{L^\infty(W^{\frac{k+1}{2},p})}\end{aligned}$$

Now, we will treat separately each of the previous terms.

(i) For the first one we have

$$\|e^{\varepsilon(t-T_0)\Delta}\theta(\cdot,T_0)\|_{W^{\frac{k+1}{2},p}} = \|\theta(\cdot,T_0)*h_{\varepsilon(t-T_0)}\|_{L^p} + \|\theta(\cdot,T_0)*(-\Delta)^{\frac{k+1}{4}}h_{\varepsilon(t-T_0)}\|_{L^p}$$
$$\leq \|\theta(\cdot,T_0)\|_{L^p} + \|\theta(\cdot,T_0)\|_{L^p}\|(-\Delta)^{\frac{k+1}{4}}h_{\varepsilon(t-T_0)}\|_{L^1}$$

where h_t is the heat kernel, so we can write

$$\|e^{\varepsilon(t-T_0)\Delta}\theta(\cdot,T_0)\|_{L^\infty(W^{\frac{k+1}{2},p})} \leq C\|\theta(\cdot,T_0)\|_{L^p}\sup\left\{[\varepsilon(t-T_0)]^{-\frac{k+1}{4}};1\right\}$$

(ii) For the second term, one has

$$\left\|\int_{T_0}^t e^{\varepsilon(t-s)\Delta}\nabla\cdot(v_\varepsilon\,\theta)(\cdot,s)ds\right\|_{W^{\frac{k+1}{2},p}} \leq \int_{T_0}^t \|\nabla\cdot(v_\varepsilon\,\theta)*h_{\varepsilon(t-s)}\|_{W^{\frac{k+1}{2},p}}ds$$
$$\leq \int_{T_0}^t \|\nabla\cdot(v_\varepsilon\,\theta)*h_{\varepsilon(t-s)}\|_{L^p} + \|(-\Delta)^{\frac{k+1}{4}}\left[\nabla\cdot(v_\varepsilon\,\theta)*h_{\varepsilon(t-s)}\right]\|_{L^p}ds$$
$$\leq \int_{T_0}^t \|v_\varepsilon\,\theta\|_{L^p}\|\nabla h_{\varepsilon(t-s)}\|_{L^1} + \|(-\Delta)^{\frac{k}{4}}(v_\varepsilon\,\theta)\|_{L^p}\|(-\Delta)^{\frac{1}{4}}(\nabla h_{\varepsilon(t-s)})\|_{L^1}ds$$
$$\leq C\int_{T_0}^t \|v_\varepsilon\,\theta(\cdot,s)\|_{W^{\frac{k}{2},p}}\max\left([\varepsilon(t-s)]^{-\frac{1}{2}};[\varepsilon(t-s)]^{-\frac{3}{4}}\right)ds.$$

Note now that we have here the estimations below for $N\geq k/2$

$$\|v_\varepsilon\theta(\cdot,s)\|_{W^{\frac{k}{2},p}} \leq \|v_\varepsilon(\cdot,s)\|_{\mathcal{C}^N}\|\theta(\cdot,s)\|_{W^{\frac{k}{2},p}} \leq C\varepsilon^{-N}\|v(\cdot,s)\|_{L^\infty}\|\theta(\cdot,s)\|_{W^{\frac{k}{2},p}}$$

hence, we can write

$$\left\|\int_{T_0}^t e^{\varepsilon(t-s)\Delta}\nabla\cdot(v_\varepsilon\,\theta)(\cdot,s)ds\right\|_{L^\infty(W^{\frac{k+1}{2},p})} \leq C\|v\|_{L^\infty(L^\infty)}\|\theta\|_{L^\infty(W^{\frac{k}{2},p})}\sup\int_{T_0}^t \varepsilon^{-N}\max\left\{[\varepsilon(t-s)]^{-\frac{1}{2}};[\varepsilon(t-s)]^{-\frac{3}{4}}\right\}ds$$

(iii) Finally, for the last term we have

$$\left\|\int_{T_0}^t e^{\varepsilon(t-s)\Delta}\mathcal{L}\theta(\cdot,s)ds\right\|_{W^{\frac{k+1}{2},p}} \leq \int_{T_0}^t \left\|\theta(\cdot,s)*\mathcal{L}h_{\varepsilon(t-s)}\right\|_{L^p} + \left\|(-\Delta)^{\frac{k}{4}}\theta(\cdot,s)*\mathcal{L}(-\Delta)^{\frac{1}{4}}h_{\varepsilon(t-s)}\right\|_{L^p}ds$$
$$\leq \int_{T_0}^t \|\theta(\cdot,s)\|_{L^p}\left\|\mathcal{L}h_{\varepsilon(t-s)}\right\|_{L^1} + \left\|(-\Delta)^{\frac{k}{4}}\theta(\cdot,s)\|_{L^p}\|\mathcal{L}(-\Delta)^{\frac{1}{4}}h_{\varepsilon(t-s)}\right\|_{L^1}ds$$

now, applying Lemma 2.2 to the function $(-\Delta)^{\frac{1}{4}}h_{\varepsilon(t-s)}$ we obtain by homogeneity that

$$\|\mathcal{L}(-\Delta)^{\frac{1}{4}}h_{\varepsilon(t-s)}\|_{L^1} \leq \phi(\varepsilon(t-s))$$

where $\phi(\varepsilon(t-s)) = \left([\varepsilon(t-s)]^{-\frac{1+4\beta}{4}} + [\varepsilon(t-s)]^{-\frac{1+4\delta}{4}}\right)$; $\left([\varepsilon(t-s)]^{-\frac{1+4\alpha}{4}}\right)$; $\left([\varepsilon(t-s)]^{-\frac{3}{4}} + [\varepsilon(t-s)]^{-\frac{1}{4}} + [\varepsilon(t-s)]^{-\frac{1+4\delta}{4}}\right)$ and $\left([\varepsilon(t-s)]^{-\frac{3}{4}}\right)$ for the cases **(a)-(d)** respectively. So we obtain with the Lemma 2.2:

$$\left\|\int_{T_0}^t e^{\varepsilon(t-s)\Delta}\mathcal{L}\theta(\cdot,s)ds\right\|_{L^\infty(W^{\frac{k+1}{2},p})} \leq C\|\theta\|_{L^\infty(W^{\frac{k}{2},p})}\int_{T_0}^t \sup\left\{\varphi((t-s),\varepsilon);\phi(\varepsilon(t-s))\right\}ds.$$

Now, with formulas (i)-(iii) at our disposal, we have that the norm $\|\theta\|_{L^\infty(W^{\frac{k+1}{2},p})}$ is controlled for all $\varepsilon>0$: we have proven spatial regularity.

Time regularity follows since we have

$$\frac{\partial^k}{\partial t^k}\theta(x,t) + \nabla\cdot\left(\frac{\partial^k}{\partial t^k}(v_\varepsilon\,\theta)\right)(x,t) + \mathcal{L}\left(\frac{\partial^k}{\partial t^k}\theta\right)(x,t) = \varepsilon\Delta\left(\frac{\partial^k}{\partial t^k}\theta\right)(x,t).$$

■

Remark 2.3 *The solutions $\theta(\cdot,\cdot)$ constructed above depend on ε.*

2.2 Maximum principle for viscosity solutions

The maximum principle we are studying here will be a consequence of few inequalities, some of them are well known. We will start with the solutions obtained in the previous section:

Proposition 2.1 (Viscosity version of Theorem 2) *Let $\theta_0 \in L^p(\mathbb{R}^n)$ with $1 < p \leq +\infty$ be an initial data, then the associated solution of the viscosity problem (9) satisfies the following maximum principle for all $t \in [0,T]$: $\|\theta(\cdot,t)\|_{L^p} \leq \|\theta_0\|_{L^p}$.*

Proof. We write for $1 < p < +\infty$:

$$\frac{d}{dt}\|\theta(\cdot,t)\|_{L^p}^p = p\int_{\mathbb{R}^n}|\theta|^{p-2}\theta\Big(\varepsilon\Delta\theta - \nabla\cdot(v_\varepsilon\,\theta) - \mathcal{L}\theta\Big)dx = p\varepsilon\int_{\mathbb{R}^n}|\theta|^{p-2}\theta\Delta\theta dx - p\int_{\mathbb{R}^n}|\theta|^{p-1}sgn(\theta)\mathcal{L}\theta dx$$

where we used the fact that $div(v) = 0$. Thus, we have

$$\frac{d}{dt}\|\theta(\cdot,t)\|_{L^p}^p - p\varepsilon\int_{\mathbb{R}^n}|\theta|^{p-2}\theta\Delta\theta dx + p\int_{\mathbb{R}^n}|\theta|^{p-1}sgn(\theta)\mathcal{L}\theta dx = 0,$$

and integrating in time we obtain

$$\|\theta(\cdot,t)\|_{L^p}^p - p\varepsilon\int_0^t\int_{\mathbb{R}^n}|\theta|^{p-2}\theta\Delta\theta dxds + p\int_0^t\int_{\mathbb{R}^n}|\theta|^{p-1}sgn(\theta)\mathcal{L}\theta dxds = \|\theta_0\|_{L^p}^p. \tag{16}$$

To finish, we have that the quantities

$$-p\varepsilon\int_{\mathbb{R}^n}|\theta|^{p-2}\theta\Delta\theta dx \quad \text{and} \quad \int_0^t\int_{\mathbb{R}^n}|\theta|^{p-1}sgn(\theta)\mathcal{L}\theta dxds$$

are both positive. Indeed, for the first expression, since $e^{\varepsilon s\Delta}$ is a contraction semi-group we have $\|e^{\varepsilon s\Delta}f\|_{L^p} \leq \|f\|_{L^p}$ for all $s > 0$ and all $f \in L^p(\mathbb{R}^n)$. Thus $F(s) = \|e^{\varepsilon s\Delta}f\|_{L^p}$ is decreasing in s; taking the derivative in s and evaluating in $s = 0$ we obtain the desired result. The positivity of the second expression above follows immediately from the *Strook-Varopoulos estimate* for general Lévy-type operators given by the following formula (see remark 1.23 of [11] for a proof, more details can be found in [20] and [21]):

$$C\langle\mathcal{L}|\theta|^{p/2}, |\theta|^{p/2}\rangle \leq \langle\mathcal{L}\theta, |\theta|^{p-1}sgn(\theta)\rangle \tag{17}$$

It is enough to note now that $\langle\mathcal{L}|\theta|^{p/2}, |\theta|^{p/2}\rangle = \|\mathcal{L}^{\frac{1}{2}}|\theta|^{p/2}\|_{L^2}^2 \geq 0$, where the operator $\mathcal{L}^{\frac{1}{2}}$ is defined by the formula $(\mathcal{L}^{\frac{1}{2}}f)\widehat{\ }(\xi) = a^{\frac{1}{2}}(\xi)\widehat{f}(\xi)$.

Thus, getting back to (16), we have that all these quantities are bounded and positive and we write for all $1 < p < +\infty$:

$$\|\theta(\cdot,t)\|_{L^p} \leq \|\theta_0\|_{L^p}.$$

Since $\|\theta(\cdot,t)\|_{L^p} \underset{p\to+\infty}{\longrightarrow} \|\theta(\cdot,t)\|_{L^\infty}$, the maximum principle is proven for viscosity solutions. ■

2.3 Besov Regularity and the limit $\varepsilon \longrightarrow 0$ for viscosity solutions

In order to deal with Theorem 1 and Theorem 2 we will need some aditional results that will allow us to pass to the limit. Indeed, a more detailed study of expression (16) above will lead us to a result concerning weak solution's regularity.

Lemma 2.3 *If the function π satisfies the conditions (4) and (5), then we have for the cases* **(a)-(d)** *the following pointwise estimates on the symbol $a(\cdot)$ for all $\xi \in \mathbb{R}^n$:*

1) $a(\xi) \leq |\xi|^{2\beta} + |\xi|^{2\delta}$

2) $|\xi|^{2\alpha} \leq a(\xi) + C$.

Proof. We use the Lévy-Khinchin formula to obtain $|\xi|^{2\alpha} = \displaystyle\int_{\mathbb{R}^n\setminus\{0\}}\big(1-\cos(y\cdot\xi)\big)|y|^{-n-2\alpha}dy$. It is enough to apply the hypothesis (4), (5) and to use the inequality (6) to conclude. ■

We state now an useful result for passing to the limit $\varepsilon \longrightarrow 0$ which is interesting for its own sake:

Theorem 7 (Besov Regularity) *Let $\mathcal{L}$ be a Lévy-type operator of the form (7) with hypothesis (4) and (5) for the measure π with α, β, δ satisfying the bounds given in the cases* **(a)-(d)**. *Let $2 \leq p < +\infty$ and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function such that $f \in L^p(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} |f(x)|^{p-2} f(x) \mathcal{L} f(x) dx < +\infty, \qquad \textit{then} \qquad f \in \dot{B}_p^{2\alpha/p,p}(\mathbb{R}^n).$$

Proof. We will prove the following estimates valid for a positive function f:

$$\|f\|^p_{\dot{B}_p^{2\alpha/p,p}} \leq C \|f^{p/2}\|^2_{\dot{B}_2^{\alpha,2}} \leq \|f^{p/2}\|^2_{L^2} + \int_{\mathbb{R}^n} |f(x)|^{p-2} f(x) \mathcal{L} f(x) dx \tag{18}$$

The first inequality can be found in [2], so we only need to focus on the right-hand side of the previous estimate. For this, we will start assuming that the function f is positive. Using Plancherel's formula, the characterisation of $\mathcal{L}^{\frac{1}{2}}$ via the symbol $a^{\frac{1}{2}}(\xi)$ and Lemma 2.3 we write

$$\|f^{p/2}\|^2_{\dot{B}_2^{\alpha,2}} \; = \; \|f^{p/2}\|^2_{\dot{H}^\alpha} = \int_{\mathbb{R}^n} |\xi|^{2\alpha} |\widehat{f^{p/2}}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^n} (a^{\frac{1}{2}}(\xi) + C)^2 |\widehat{f^{p/2}}(\xi)|^2 d\xi \leq c \left(\|f^{p/2}\|^2_{L^2} + \|\mathcal{L}^{\frac{1}{2}} f^{p/2}\|^2_{L^2} \right).$$

Now, using the Strook-Varopoulos inequality (17) we have

$$\|f^{p/2}\|^2_{L^2} + \|\mathcal{L}^{\frac{1}{2}} f^{p/2}\|^2_{L^2} \leq \|f^{p/2}\|^2_{L^2} + c \int_{\mathbb{R}^n} f^{p-2} f \mathcal{L} f dx$$

So inequality (18) is proven for positive functions. For the general case we write $f(x) = f_+(x) - f_-(x)$ where $f_\pm(x)$ are positive functions with disjoint support and we have:

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x)|^{p-2} f(x) \mathcal{L} f(x) dx \; &= \; \int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \mathcal{L} f_+(x) dx + \int_{\mathbb{R}^n} f_-(x)^{p-2} f_-(x) \mathcal{L} f_-(x) dx \\ &- \; \int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \mathcal{L} f_-(x) dx - \int_{\mathbb{R}^n} f_-(x)^{p-2} f_-(x) \mathcal{L} f_+(x) dx \end{aligned} \tag{19}$$

We only need to treat the two last integrals, and in fact we just need to study one of them since the other can be treated in a similar way. So, for the third integral we have

$$\begin{aligned} \int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \mathcal{L} f_-(x) dx \; &= \; \int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \int_{\mathbb{R}^n} [f_-(x) - f_-(y)] \pi(x-y) dy dx \\ &= \; \int_{\mathbb{R}^n} f_+(x)^{p-2} \int_{\mathbb{R}^n} [f_+(x) f_-(x) - f_+(x) f_-(y)] \pi(x-y) dy dx \end{aligned}$$

However, since f_+ and f_- have disjoint supports we obtain the following estimate:

$$\int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \mathcal{L} f_-(x) dx = - \int_{\mathbb{R}^n} f_+(x)^{p-2} \int_{\mathbb{R}^n} [f_+(x) f_-(y)] \pi(x-y) dy dx \leq 0$$

Recalling that π is a positive function we obtain that this quantity is negative as all the terms inside the integral are positive. With this observation we see that the last terms of (19) are positive and we have

$$\int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \mathcal{L} f_+(x) dx + \int_{\mathbb{R}^n} f_-(x)^{p-2} f_-(x) \mathcal{L} f_-(x) dx \leq \int_{\mathbb{R}^n} |f(x)|^{p-2} f(x) \mathcal{L} f(x) dx < +\infty$$

Then, using the first part of the proof we have $f_\pm \in \dot{B}_p^{2\alpha/p,p}(\mathbb{R}^n)$ and since $f = f_+ - f_-$ we conclude that f belongs to the Besov space $\dot{B}_p^{2\alpha/p,p}(\mathbb{R}^n)$. ■

Remark 2.4 *The lower bound $p \geq 2$ in Theorems 1 and 2 is a consequence of Theorem 7 above.*

Proof of Theorem 1 and Theorem 2. We have obtained with the previous results in sections 2.1 and 2.2 a family of regular functions $(\theta^{(\varepsilon)})_{\varepsilon>0} \in L^\infty([0,T]; L^p(\mathbb{R}^n))$ which are solutions of (9) and satisfy the uniform bound $\|\theta^{(\varepsilon)}(\cdot,t)\|_{L^p} \leq \|\theta_0\|_{L^p}$; in order to conclude we need to pass to the limit $\varepsilon \longrightarrow 0$.

Since $L^\infty([0,T]; L^p(\mathbb{R}^n)) = \left(L^1([0,T]; L^q(\mathbb{R}^n))\right)'$, with $\frac{1}{p} + \frac{1}{q} = 1$, we can extract from those solutions $\theta^{(\varepsilon)}$ a subsequence $(\theta_k)_{k\in\mathbb{N}}$ which is $*$-weakly convergent to some function θ in the space $L^\infty([0,T]; L^p(\mathbb{R}^n))$, which implies convergence in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^n)$. However, this weak convergence is not sufficient to assure the convergence of $(v_\varepsilon \, \theta_k)$ to

$v\,\theta$. For this we use the remarks that follow.

First, using the Lemma 2.1 we can consider a sequence $(v_k)_{k\in\mathbb{N}}$ with v_k as in formula (10) such that $v_k \longrightarrow v$ weakly in $bmo(\mathbb{R}^n)$. Secondly, combining Proposition 2.1 and Theorem 7 we obtain that solutions θ_k belongs to the space $L^\infty([0,T];L^p(\mathbb{R}^n))\cap L^1([0,T];\dot{B}^{2\alpha/p,p}_p(\mathbb{R}^n))$ for all $k\in\mathbb{N}$.

To finish, fix a function $\varphi\in\mathcal{C}^\infty_0([0,T]\times\mathbb{R}^n)$. Then we have the fact that $\varphi\theta_k\in L^1([0,T];\dot{B}^{2\alpha/p,p}_p(\mathbb{R}^n))$ and $\partial_t\varphi\theta_k\in L^1([0,T];\dot{B}^{-N,p}_p(\mathbb{R}^n))$. This implies the local inclusion, in space as well as in time, $\varphi\theta_k\in \dot{W}^{2\alpha/p,p}_{t,x}\subset \dot{W}^{2\alpha/p,2}_{t,x}$ so we can apply classical results such as the Rellich's theorem to obtain convergence of $v_k\,\theta_k$ to $v\,\theta$.

Thus, we obtain existence and uniqueness of weak solutions for the problem (1) with an initial data in $\theta_0\in L^p(\mathbb{R}^n)$, $2\leq p<+\infty$ that satisfy the maximum principle. Moreover, we have that these solutions $\theta(x,t)$ belong to the space $L^\infty([0,T];L^p(\mathbb{R}^n))\cap L^p([0,T];\dot{B}^{2\alpha/p,p}_p(\mathbb{R}^n))$. ■

Remark 2.5 *These lines explain how to obtain weak solutions from viscosity ones and it will be used freely in the sequel.*

3 Positivity principle

We prove in this section Theorem 3. Recall that by hypothesis we have $0\leq\psi_0\leq M$ an initial datum for the equation (1) with $\psi_0\in L^p(\mathbb{R}^n)$ and $\frac{n}{2\min(\beta,\delta)}<p\leq+\infty$. We will show here that the associated solution $\psi(x,t)$ satisfies the bounds $0\leq\psi(x,t)\leq M$.

To begin with, we fix two constants, ρ, R such that $R>2\rho>0$. Then we set $A_{0,R}(x)$ a function equals to $M/2$ over $|x|\leq 2R$ and equals to $\psi_0(x)$ over $|x|>2R$ and we write $B_{0,R}(x)=\psi_0(x)-A_{0,R}(x)$, so by construction we have

$$\psi_0(x)=A_{0,R}(x)+B_{0,R}(x)$$

with $\|A_{0,R}\|_{L^\infty}\leq M$ and $\|B_{0,R}\|_{L^\infty}\leq M/2$. Remark that $A_{0,R}, B_{0,R}\in L^p(\mathbb{R}^n)$.

Now fix $v\in L^\infty([0,T];bmo(\mathbb{R}^n))$ such that $div(v)=0$ and consider the equations

$$\begin{cases}\partial_t A_R(x,t)+\nabla\cdot(v\,A_R)(x,t)+\mathcal{L}A_R(x,t)=0,\\[2ex] A_R(x,0)=A_{0,R}(x)\end{cases}\quad\text{and}\quad\begin{cases}\partial_t B_R(x,t)+\nabla\cdot(v\,B_R)(x,t)+\mathcal{L}B_R(x,t)=0\\[2ex] B_R(x,0)=B_{0,R}(x).\end{cases}\tag{20}$$

Using the maximum principle and by construction we have the following estimates for $t\in[0,T]$:

$$\begin{aligned}\|A_R(\cdot,t)\|_{L^p} &\leq \|A_{0,R}\|_{L^p}\leq\|\psi_0\|_{L^p}+CMR^{n/p}\quad(1<p<+\infty)\qquad(21)\\[2ex] \|A_R(\cdot,t)\|_{L^\infty} &\leq \|A_{0,R}\|_{L^\infty}\leq M.\\[2ex] \|B_R(\cdot,t)\|_{L^\infty} &\leq \|B_{0,R}\|_{L^\infty}\leq M/2.\end{aligned}$$

where $A_R(x,t)$ and $B_R(x,t)$ are the weak solutions of the systems (20). Then, the function $\psi(x,t)=A_R(x,t)+B_R(x,t)$ is the unique solution for the problem

$$\begin{cases}\partial_t\psi(x,t)+\nabla\cdot(v\,\psi)(x,t)+\mathcal{L}\psi(x,t)=0\\[2ex] \psi(x,0)=A_{0,R}(x)+B_{0,R}(x).\end{cases}\tag{22}$$

Indeed, using hypothesis for $A_R(x,t)$ and $B_R(x,t)$ and the linearity of equation (22) we have that the function $\psi_R(x,t)=A_R(x,t)+B_R(x,t)$ is a solution for this equation. Uniqueness is assured by the maximum principle and by the continuous dependence from initial data given in corollary 2.1, thus we can write $\psi(x,t)=\psi_R(x,t)$.

To continue, we will need an auxiliary function $\phi\in\mathcal{C}^\infty_0(\mathbb{R}^n)$ such that $\phi(x)=0$ for $|x|\geq1$ and $\phi(x)=1$ if $|x|\leq1/2$ and we set $\varphi(x)=\phi(x/R)$. Now, we will estimate the L^p-norm of $\varphi(x)(A_R(x,t)-M/2)$ with $p>n/2\min(\beta,\delta)$, where β and δ are the parameters of the hypothesis for the function π in the cases **(a)-(d)**. We write:

$$\begin{aligned}\partial_t\|\varphi(\cdot)(A_R(\cdot,t)-M/2)\|^p_{L^p} &= p\int_{\mathbb{R}^n}\big|\varphi(x)(A_R(x,t)-M/2)\big|^{p-2}\big(\varphi(x)(A_R(x,t)-M/2)\big)\\ &\qquad\times\,\partial_t\big(\varphi(x)(A_R(x,t)-M/2)\big)dx\end{aligned}\tag{23}$$

We observe that we have the following identity for the last term above

$$\begin{aligned}\partial_t(\varphi(x)(A_R(x,t)-M/2)) &= -\nabla\cdot(\varphi(x)\,v(A_R(x,t)-M/2))-\mathcal{L}(\varphi(x)(A_R(x,t)-M/2))\\ &+ (A_R(x,t)-M/2)v\cdot\nabla\varphi(x)+[\mathcal{L},\varphi]A_R(x,t)-M/2\mathcal{L}\varphi(x)\end{aligned}$$

where we noted $[\mathcal{L},\varphi]$ the commutator between $\mathcal{L}$ and φ. Thus, using this identity in (23) and the fact that $div(v)=0$ we have

$$\begin{aligned}\partial_t\|\varphi(\cdot)(A_R(\cdot,t)-M/2)\|_{L^p}^p &= -p\int_{\mathbb{R}^n}\big|\varphi(x)(A_R(x,t)-M/2)\big|^{p-2}\big(\varphi(x)(A_R(x,t)-M/2)\big)\\ &\quad\times\mathcal{L}(\varphi(x)(A_R(x,t)-M/2))dx \qquad (24)\\ &+ p\int_{\mathbb{R}^n}\big|\varphi(x)(A_R(x,t)-M/2)\big|^{p-2}\big(\varphi(x)(A_R(x,t)-M/2)\big)\\ &\quad\times\left([\mathcal{L},\varphi]A_R(x,t)-M/2\mathcal{L}\varphi(x)\right)dx\end{aligned}$$

Remark that the integral (24) is positive so one has

$$\begin{aligned}\partial_t\|\varphi(\cdot)(A_R(\cdot,t)-M/2)\|_{L^p}^p &\leq p\int_{\mathbb{R}^n}\big|\varphi(x)(A_R(x,t)-M/2)\big|^{p-2}\big(\varphi(x)(A_R(x,t)-M/2)\big)\\ &\quad\times\left([\mathcal{L},\varphi]A_R(x,t)-M/2\mathcal{L}\varphi(x)\right)dx\end{aligned}$$

Using Hölder's inequality and integrating in time the previous expression we have

$$\|\varphi(\cdot)(A_R(\cdot,t)-M/2)\|_{L^p}^p \leq \|\varphi(\cdot)(A_R(\cdot,0)-M/2)\|_{L^p}^p+\int_0^t\left(\|[\mathcal{L},\varphi]A_R(\cdot,s)\|_{L^p}+\|M/2\mathcal{L}\varphi\|_{L^p}\right)ds \qquad (25)$$

The first term of the right side is null since over the support of φ we have identity $A_R(x,0)=M/2$. For the term $\|[\mathcal{L},\varphi]A_R(\cdot,s)\|_{L^p}$ we will need the following lemma (see the proof in the appendix):

Lemma 3.1 *For $1\leq p\leq+\infty$ we have for the cases **(a)-(d)** the following inequality:*

$$\|[\mathcal{L},\varphi]A_R(\cdot,s)\|_{L^p}\leq C(R^{-2\beta}+R^{-2\delta})\|A_{0,R}\|_{L^p}.$$

Now, getting back to the last term of (25) we have by the definition of φ and the properties of the operator $\mathcal{L}$ the estimate:

$$\|M/2\mathcal{L}\varphi\|_{L^p}\leq CMR^{n/p}(R^{-2\beta}+R^{-2\delta}).$$

We thus have

$$\|\varphi(\cdot)(A_R(\cdot,t)-M/2)\|_{L^p}^p\leq C(R^{-2\beta}+R^{-2\delta})\int_0^t\left(\|A_{0,R}\|_{L^p}+MR^{n/p}\right)ds.$$

Observe that we have at our disposal estimate (21), so we can write

$$\|\varphi(\cdot)(A_R(\cdot,t)-M/2)\|_{L^p}^p\leq Ct(R^{-2\beta}+R^{-2\delta})\left(\|\psi_0\|_{L^p}+MR^{n/p}\right)$$

Using again the definition of φ one has $\displaystyle\int_{B(0,\rho)}|A_R(\cdot,t)-M/2|^p dx\leq Ct(R^{-2\beta}+R^{-2\delta})\left(\|\psi_0\|_{L^p}+MR^{n/p}\right)$. Thus, if $R\longrightarrow+\infty$ and since $p>\frac{n}{2\min(\beta,\delta)}$, we have $A(x,t)=M/2$ over $B(0,\rho)$.

Hence, by construction we have $\psi(x,t)=A_R(x,t)+B_R(x,t)$ where ψ is a solution of (22) with initial data $\psi_0=A_{0,R}+B_{0,R}$, but, since over $B(0,\rho)$ we have $A(x,t)=M/2$ and $\|B(\cdot,t)\|_{L^\infty}\leq M/2$, one finally has the desired estimate $0\leq\psi(x,t)\leq M$. ■

4 Existence of solutions with a L^∞ initial data

The proof given before for the positivity principle allows us to obtain the existence of solutions for the fractional diffusion transport equation (1) when the initial data θ_0 belongs to the space $L^\infty(\mathbb{R}^n)$. The utility of this fact will appear clearly in the next section as it will be used in Theorem 4.

Let us fix $\theta_0^R = \theta_0 \mathbb{1}_{B(0,R)}$ with $R > 0$ so we have $\theta_0^R \in L^p(\mathbb{R}^n)$ for all $1 \leq p \leq +\infty$. Following section 2, there is a unique solution θ^R for the problem

$$\begin{cases} \partial_t \theta^R + \nabla \cdot (v\theta^R) + \mathcal{L}\theta^R = 0 \\ \\ \theta^R(x,0) = \theta_0^R(x) \\ \\ div(v) = 0 \quad \text{ and } v \in L^\infty([0,T]; bmo(\mathbb{R}^n)). \end{cases}$$

such that $\theta^R \in L^\infty([0,T]; L^p(\mathbb{R}^n))$. By the maximum principle we have $\|\theta^R(\cdot,t)\|_{L^p} \leq \|\theta_0^R\|_{L^p} \leq v_n \|\theta_0\|_{L^\infty} R^{n/p}$. Taking the limit $p \longrightarrow +\infty$ and making $R \longrightarrow +\infty$ we finally get

$$\|\theta(\cdot,t)\|_{L^\infty} \leq C\|\theta_0\|_{L^\infty}.$$

This shows that for an initial data $\theta_0 \in L^\infty(\mathbb{R}^n)$ there exists an associated solution $\theta \in L^\infty([0,T]; L^\infty(\mathbb{R}^n))$.

5 Hölder Regularity

In this section we are going to prove Theorem 4. It is very important to note that we will work only with the cases **(c)** and **(d)**: from now on the operator $\mathcal{L}$ is assumed to be of the form (7) with an associated Lévy measure π satisfiying the hypothesis (4) and (5) with $\alpha = \beta = 1/2$ and $0 < \delta < 1/2$ or $\alpha = \beta = \delta = 1/2$.

We will now study Hölder regularity by duality using Hardy spaces. These spaces have several equivalent characterizations (see [3], [7] and [19] for a detailed treatment). In this paper we are interested mainly in the molecular approach that defines *local* Hardy spaces.

Definition 5.1 (Local Hardy spaces h^σ) *Let $0 < \sigma < 1$. The local Hardy space $h^\sigma(\mathbb{R}^n)$ is the set of distributions f that admits the following molecular decomposition:*

$$f = \sum_{j\in\mathbb{N}} \lambda_j \psi_j \tag{26}$$

where $(\lambda_j)_{j\in\mathbb{N}}$ is a sequence of complex numbers such that $\sum_{j\in\mathbb{N}} |\lambda_j|^\sigma < +\infty$ and $(\psi_j)_{j\in\mathbb{N}}$ is a family of r-molecules in the sense of definition 5.2 below. The h^σ-norm[2] is then fixed by the formula

$$\|f\|_{h^\sigma} = \inf \left\{ \left(\sum_{j\in\mathbb{N}} |\lambda_j|^\sigma \right)^{1/\sigma} : f = \sum_{j\in\mathbb{N}} \lambda_j \psi_j \right\}$$

where the infimum runs over all possible decompositions (26).

Local Hardy spaces have many remarquable properties and we will only stress here, before passing to duality results concerning h^σ spaces, the fact that Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is dense in $h^\sigma(\mathbb{R}^n)$.

Now, let us take a closer look at the dual space of the local Hardy spaces. In [7] D. Goldberg proved the following important theorem:

Theorem 8 (Hardy-Hölder duality) *Let $\frac{n}{n+1} < \sigma < 1$ and fix $\gamma = n(\frac{1}{\sigma} - 1)$. Then the dual of local Hardy space $h^\sigma(\mathbb{R}^n)$ is the Hölder space $\mathcal{C}^\gamma(\mathbb{R}^n)$ fixed by the norm*

$$\|f\|_{\mathcal{C}^\gamma} = \|f\|_{L^\infty} + \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\gamma}.$$

This result allows us to study the Hölder regularity of functions in terms of Hardy spaces and it will be applied to the solutions of the equation (1).

Remark 5.1 *Since $\frac{n}{n+1} < \sigma < 1$, we have $\sum_{j\in\mathbb{N}} |\lambda_j| \leq \left(\sum_{j\in\mathbb{N}} |\lambda_j|^\sigma\right)^{1/\sigma}$ thus for testing Hölder continuity of a function f it is enough to study the quantities $|\langle f, \psi_j \rangle|$ where ψ_j is an r-molecule.*

[2]it is not actually a *norm* since $0 < \sigma < 1$. More details can be found in [7] and [19].

Since we are going to work with local Hardy spaces, we will introduce a size treshold in order to distinguish *small* molecules from *big* ones in the following way:

Definition 5.2 (r-molecules) *Set $\frac{n}{n+1} < \sigma < 1$, define $\gamma = n(\frac{1}{\sigma} - 1)$ and fix a real number ω such that $0 < \gamma < \omega < 1$. An integrable function ψ is an r-molecule if we have*

- *Small molecules $(0 < r < 1)$:*

$$\int_{\mathbb{R}^n} |\psi(x)||x - x_0|^\omega dx \leq r^{\omega-\gamma}, \text{ for } x_0 \in \mathbb{R}^n \qquad \textit{(concentration condition)} \tag{27}$$

$$\|\psi\|_{L^\infty} \leq \frac{1}{r^{n+\gamma}} \qquad \textit{(height condition)} \tag{28}$$

$$\int_{\mathbb{R}^n} \psi(x)dx = 0 \qquad \textit{(moment condition)} \tag{29}$$

- *Big molecules $(1 \leq r < +\infty)$:*

  In this case we only require conditions (27) and (28) for the r-molecule ψ while the moment condition (29) is dropped.

Remark 5.2

1) *Note that the point $x_0 \in \mathbb{R}^n$ can be considered as the "center" of the molecule.*
2) *Conditions (27) and (28) imply the estimate $\|\psi\|_{L^1} \leq C\, r^{-\gamma}$ thus every r-molecule belongs to $L^p(\mathbb{R}^n)$ with $1 < p < +\infty$.*
3) *In this definition, we find convenient to show explicitely the dependence on the Hölder parameter γ instead of σ.*

The main interest of using molecules relies in the possibility of *transfering* the regularity problem to the evolution of such molecules:

Proposition 5.1 (Transfer property) *Let $\psi(x,s)$ be a solution of the backward problem*

$$\begin{cases} \partial_s\psi(x,s) = & -\nabla\cdot[v(x,t-s)\psi(x,s)] - \mathcal{L}\psi(x,s) \\ \psi(x,0) = & \psi_0(x) \in L^1 \cap L^\infty(\mathbb{R}^n) \\ div(v) = & 0 \quad \textit{and} \quad v \in L^\infty([0,T]; bmo(\mathbb{R}^n)) \end{cases} \tag{30}$$

If $\theta(x,t)$ is a solution of (1) with $\theta_0 \in L^\infty(\mathbb{R}^n)$ then we have the identity

$$\int_{\mathbb{R}^n} \theta(x,t)\psi(x,0)dx = \int_{\mathbb{R}^n} \theta(x,0)\psi(x,t)dx.$$

Proof. We first consider the expression

$$\partial_s \int_{\mathbb{R}^n} \theta(x,t-s)\psi(x,s)dx = \int_{\mathbb{R}^n} -\partial_s\theta(x,t-s)\psi(x,s) + \partial_s\psi(x,s)\theta(x,t-s)dx.$$

Using equations (1) and (30) we obtain

$$\begin{aligned} \partial_s \int_{\mathbb{R}^n} \theta(x,t-s)\psi(x,s)dx &= \int_{\mathbb{R}^n} -\nabla\cdot[(v(x,t-s)\theta(x,t-s)]\,\psi(x,s) + \mathcal{L}\theta(x,t-s)\psi(x,s) \\ &- \nabla\cdot[(v(x,t-s)\psi(x,s))]\,\theta(x,t-s) - \mathcal{L}\psi(x,s)\theta(x,t-s)dx. \end{aligned}$$

Now, using the fact that v is divergence free and the symmetry of the operator $\mathcal{L}$ we have that the expression above is equal to zero, so the quantity

$$\int_{\mathbb{R}^n} \theta(x,t-s)\psi(x,s)dx$$

remains constant in time. We only have to set $s = 0$ and $s = t$ to conclude. ■

This proposition says, that in order to control $\langle\theta(\cdot,t),\psi_0\rangle$, it is enough (and much simpler) to study the bracket $\langle\theta_0,\psi(\cdot,t)\rangle$.

Proof of Theorem 4. Once we have the transfer property proven above, the proof of Theorem 4 is quite direct and it reduces to a L^1 estimate for molecules. Indeed, assume that for *all* molecular initial data ψ_0 we have a L^1 control for $\psi(\cdot,t)$ a solution of (30), then Theorem 4 follows easily: applying Proposition 5.1 with the fact that $\theta_0\in L^\infty(\mathbb{R}^n)$ we have

$$|\langle\theta(\cdot,t),\psi_0\rangle| = \left|\int_{\mathbb{R}^n}\theta(x,t)\psi_0(x)dx\right| = \left|\int_{\mathbb{R}^n}\theta(x,0)\psi(x,t)dx\right| \leq \|\theta_0\|_{L^\infty}\|\psi(\cdot,t)\|_{L^1} < +\infty. \tag{31}$$

From this, we obtain that $\theta(\cdot,t)$ belongs to the Hölder space $\mathcal{C}^\gamma(\mathbb{R}^n)$.

Now we need to study the control of the L^1 norm of $\psi(\cdot,t)$ and we divide our proof in two steps following the molecule's size. For the initial big molecules, *i.e.* if $r\geq 1$, the needed control is straightforward: apply the maximum principle and the remark 5.2-2) above to obtain

$$\|\theta_0\|_{L^\infty}\|\psi(\cdot,t)\|_{L^1} \leq \|\theta_0\|_{L^\infty}\|\psi_0\|_{L^1} \leq C\frac{1}{r^\gamma}\|\theta_0\|_{L^\infty},$$

but, since $r\geq 1$, we have that $|\langle\theta(\cdot,t),\psi_0\rangle| < +\infty$ for all *big* molecules.

In order to finish the proof of this theorem, it only remains to treat the L^1 control for *small* molecules. This is the most complex part of the proof and it is treated in the following theorem:

Theorem 9 *For all small r-molecules (i.e. $0<r<1$), there exists a time $T_0>0$ such that we have the following control of the L^1-norm.*

$$\|\psi(\cdot,t)\|_{L^1} \leq CT_0^{-\gamma} \qquad (T_0<t<T),$$

with $0<\gamma<2\delta\leq 1$.

Accepting for a while this result, we have then a good control over the quantity $\|\psi(\cdot,t)\|_{L^1}$ for all $0<r<1$ and getting back to (31) we obtain that $|\langle\theta(\cdot,t),\psi_0\rangle|$ is always bounded for $T_0<t<T$ and for any molecule ψ_0: we have proven by a duality argument the Theorem 4. ■

Let us now briefly explain the main steps of Theorem 9. We need to construct a suitable control in time for the L^1-norm of the solutions $\psi(\cdot,t)$ of the backward problem (30) where the inital data ψ_0 is a *small* r-molecule. This will be achieved by iteration in two different steps:

- The first step explains the molecules' deformation after a very small time $s_0>0$, which is related to the size r by the bounds $0<s_0\leq\epsilon r$ with ϵ a small constant. This will be done in section 5.1.
- In order to obtain a control of the L^1 norm for larger times we need to perform a second step which takes as a starting point the results of the first step and gives us the deformation for another small time s_1, which is also related to the original size r. This part is treated in section 5.2.

To conclude it is enough to iterate the second step as many times as necessary to get rid of the dependence of the times $s_0,s_1,...$ from the molecule's size. This way we obtain the L^1 control needed for all time $T_0<t<T$.

5.1 Small time molecule's evolution: First step

The following theorem shows how the molecular properties are deformed with the evolution for a small time s_0.

Theorem 10 *Set σ, γ and ω three real numbers such that $\frac{n}{n+1}<\sigma<1$, $\gamma=n(\frac{1}{\sigma}-1)$ and $0<\gamma<\omega<2\delta<1$ in the case **(c)** or $0<\gamma<\omega<1$ in the case **(d)**. Let $\psi(x,s_0)$ be a solution of the problem*

$$\begin{cases} \partial_{s_0}\psi(x,s_0) = & -\nabla\cdot(v\,\psi)(x,s_0)-\mathcal{L}\psi(x,s_0)\\ \psi(x,0) = & \psi_0(x)\\ div(v) = & 0 \quad \textit{and} \quad v\in L^\infty([0,T];bmo(\mathbb{R}^n)) \quad \textit{with} \sup\limits_{s_0\in[0,T]}\|v(\cdot,s_0)\|_{bmo}\leq\mu \end{cases} \tag{32}$$

If ψ_0 is a small r-molecule in the sense of definition 5.2 for the local Hardy space $h^\sigma(\mathbb{R}^n)$, then there exists a positive constant $K=K(\mu)$ big enough and a positive constant ϵ such that for all $0<s_0\leq\epsilon r$ small we have the following

estimates

$$\int_{\mathbb{R}^n} |\psi(x,s_0)||x-x(s_0)|^{\omega}dx \le (r+Ks_0)^{\omega-\gamma} \tag{33}$$

$$\|\psi(\cdot,s_0)\|_{L^\infty} \le \frac{1}{(r+Ks_0)^{n+\gamma}} \tag{34}$$

$$\|\psi(\cdot,s_0)\|_{L^1} \le \frac{v_n}{(r+Ks_0)^{\gamma}} \tag{35}$$

where v_n denotes the volume of the n-dimensional unit ball.

The new molecule's center $x(s_0)$ used in formula (33) is fixed by

$$\begin{cases} x'(s_0) = & \overline{v}_{B_r} = \frac{1}{|B_r|}\displaystyle\int_{B_r} v(y,s_0)dy \quad \text{where } B_r = B(x(s_0),r). \\ x(0) = & x_0. \end{cases} \tag{36}$$

Remark 5.3

1) *The definition of the point $x(s_0)$ given by (36) reflects the molecule's center transport using velocity v.*
2) *Remark that it is enough to treat the case $0<(r+Ks_0)<1$ since s_0 is small: otherwise the L^1 control will be trivial for time s_0 and beyond: we only need to apply the maximum principle.*

Proof of the Theorem 10. We will follow the next scheme: first we prove the small Concentration condition (33) and then we prove the Height condition (34). Once we have these two conditions, the L^1 estimate (35) will follow easily.

1) Small time Concentration condition

Let us write $\Omega_0(x) = |x-x(s_0)|^{\omega}$ and $\psi(x) = \psi_+(x) - \psi_-(x)$ where the functions $\psi_\pm(x) \ge 0$ have disjoint support. We will note $\psi_\pm(x,s_0)$ two solutions of (32) with $\psi_\pm(x,0) = \psi_\pm(x)$. At this point, we use the positivity principle, thus by linearity we have

$$|\psi(x,s_0)| = |\psi_+(x,s_0) - \psi_-(x,s_0)| \le \psi_+(x,s_0) + \psi_-(x,s_0)$$

and we can write

$$\int_{\mathbb{R}^n} |\psi(x,s_0)|\Omega_0(x)dx \le \int_{\mathbb{R}^n} \psi_+(x,s_0)\Omega_0(x)dx + \int_{\mathbb{R}^n} \psi_-(x,s_0)\Omega_0(x)dx$$

so we only have to treat one of the integrals on the right side above. We have:

$$\begin{aligned} I &= \left|\partial_{s_0}\int_{\mathbb{R}^n} \Omega_0(x)\psi_+(x,s_0)dx\right| \\ &= \left|\int_{\mathbb{R}^n} \partial_{s_0}\Omega_0(x)\psi_+(x,s_0) + \Omega_0(x)\left[-\nabla\cdot(v\,\psi_+(x,s_0)) - \mathcal{L}\psi_+(x,s_0)\right]dx\right| \\ &= \left|\int_{\mathbb{R}^n} -\nabla\Omega_0(x)\cdot x'(s_0)\psi_+(x,s_0) + \Omega_0(x)\left[-\nabla\cdot(v\,\psi_+(x,s_0)) - \mathcal{L}\psi_+(x,s_0)\right]dx\right| \end{aligned}$$

Using the fact that v is divergence free, we obtain

$$I = \left|\int_{\mathbb{R}^n} \nabla\Omega_0(x)\cdot(v - x'(s_0))\psi_+(x,s_0) - \Omega_0(x)\mathcal{L}\psi_+(x,s_0)dx\right|.$$

Since the operator $\mathcal{L}$ is symmetric and using the definition of $x'(s_0)$ given in (36) we have

$$I \le \underbrace{c\int_{\mathbb{R}^n} |x-x(s_0)|^{\omega-1}|v-\overline{v}_{B_r}||\psi_+(x,s_0)|dx}_{I_1} + \underbrace{c\int_{\mathbb{R}^n} \left|\mathcal{L}\Omega_0(x)\right||\psi_+(x,s_0)|dx}_{I_2}. \tag{37}$$

We will study separately each of the integrals I_1 and I_2:

Lemma 5.1 *For integral I_1 above we have the estimate $I_1 \le C\mu\, r^{\omega-1-\gamma}$.*

Lemma 5.2 *For integral I_2 in inequality (37) we have the inequality $I_2 \leq Cr^{\omega-1-\gamma}$.*

Using these lemmas and getting back to estimate (37) we have

$$\left|\partial_{s_0}\int_{\mathbb{R}^n}\Omega_0(x)\psi_+(x,s_0)dx\right| \leq C(\mu+1)\, r^{\omega-1-\gamma}$$

This last estimation is compatible with the estimate (33) for $0 \leq s_0 \leq \epsilon r$ small enough: just fix K such that

$$C\,(\mu+1) \leq K(\omega-\gamma). \tag{38}$$

Indeed, since the time s_0 is very small, we can linearize the formula $(r+Ks_0)^{\omega-\gamma}$ in the right-hand side of (33) in order to obtain

$$\phi = (r+Ks_0)^{\omega-\gamma} \approx r^{\omega-\gamma}\left(1+[K(\omega-\gamma)]\frac{s_0}{r}\right).$$

Finally, taking the derivative with respect to s_0 in the above expression we have $\phi' \approx r^{\omega-1-\gamma}K(\omega-\gamma)$ and with condition (38), the small time Concentration condition (33) follows.

We prove now the Lemmas 5.1 and 5.2; but before, we will need the following result

Lemma 5.3 *Let $f \in bmo(\mathbb{R}^n)$, then*

1) *for all $1 < p < +\infty$, f is locally in L^p and $\frac{1}{|B|}\int_B |f(x)-f_B|^p dx \leq C\|f\|_{bmo}^p$*
2) *for all $k \in \mathbb{N}$, we have $|f_{2^kB} - f_B| \leq Ck\|f\|_{bmo}$ where $2^kB = B(x, 2^kR)$ is a ball centered at a point x of radius 2^kR.*

For a proof of these results see [19].

Proof the Lemma 5.1. We begin by considering the space $\mathbb{R}^n$ as the union of a ball with dyadic coronas centered around $x(s_0)$, more precisely we set $\mathbb{R}^n = B_r \cup \bigcup_{k\geq 1} E_k$ where

$$B_r = \{x \in \mathbb{R}^n : |x - x(s_0)| \leq r\} \quad \text{and} \quad E_k = \{x \in \mathbb{R}^n : r2^{k-1} < |x-x(s_0)| \leq r2^k\} \quad \text{for } k \geq 1, \tag{39}$$

(i) Estimations over the ball B_r. Applying Hölder's inequality to the integral I_{1,B_r} we obtain

$$\begin{aligned} I_{1,B_r} = \int_{B_r} |x-x(s_0)|^{\omega-1}|v-\overline{v}_{B_r}||\psi_+(x,s_0)|dx \quad &\leq \quad \underbrace{\||x-x(s_0)|^{\omega-1}\|_{L^p(B_r)}}_{(1)} \\ &\times \quad \underbrace{\|v-\overline{v}_{B_r}\|_{L^z(B_r)}}_{(2)}\underbrace{\|\psi_+(\cdot,s_0)\|_{L^q(B_r)}}_{(3)} \end{aligned} \tag{40}$$

where $\frac{1}{p}+\frac{1}{z}+\frac{1}{q} = 1$ and $p, z, q > 1$. We treat each of the previous terms separately:

- First observe that for $1 < p < n/(1-\omega)$ we have for the term (1) above:

$$\||x-x(s_0)|^{\omega-1}\|_{L^p(B_r)} \leq Cr^{n/p+\omega-1}.$$

- By hypothesis $v(\cdot,s_0) \in bmo$ and applying the Lemma 5.3 we have $\|v-\overline{v}_{B_r}\|_{L^z(B_r)} \leq C|B_r|^{1/z}\|v(\cdot,s_0)\|_{bmo}$. Since $\sup\limits_{s_0\in[0,T]}\|v(\cdot,s_0)\|_{bmo} \leq \mu$, we write for the term (2):

$$\|v-\overline{v}_{B_r}\|_{L^z(B_r)} \leq C\mu\, r^{n/z}.$$

- Finally for (3) by the maximum principle we have $\|\psi_+(\cdot,s_0)\|_{L^q(B_r)} \leq \|\psi_+(\cdot,0)\|_{L^q}$; hence using the fact that ψ_0 is an r-molecule and remark 5.2-2) we obtain

$$\|\psi_+(\cdot,s_0)\|_{L^q(B_r)} \leq C\left[r^{-\gamma}\right]^{1/q}\left[\frac{1}{r^{n+\gamma}}\right]^{1-1/q}.$$

We combine all these inequalities together in order to obtain the following estimation for (40):

$$I_{1,B_r} \leq C\mu\, r^{\omega-1-\gamma}. \tag{41}$$

(ii) Estimations for the dyadic corona E_k. Let us note I_{1,E_k} the integral

$$I_{1,E_k} = \int_{E_k} |x - x(s_0)|^{\omega-1}|v - \overline{v}_{B_r}||\psi_+(x, s_0)|dx.$$

Since over E_k we have[3] $|x - x(s_0)|^{\omega-1} \leq C2^{k(\omega-1)}r^{\omega-1}$ we write

$$I_{1,E_k} \leq C2^{k(\omega-1)}r^{\omega-1}\left(\int_{E_k} |v - \overline{v}_{B_{r2^k}}||\psi_+(x, s_0)|dx + \int_{E_k} |\overline{v}_{B_r} - \overline{v}_{B_{r2^k}}||\psi_+(x, s_0)|dx\right)$$

where we noted $B_{r2^k} = B(x(s_0), r2^k)$, then

$$I_{1,E_k} \leq C2^{k(\omega-1)}r^{\omega-1}\left(\int_{B_{r2^k}} |v - \overline{v}_{B_{r2^k}}||\psi_+(x, s_0)|dx + \int_{B_{r2^k}} |\overline{v}_{B_r} - \overline{v}_{B_{r2^k}}||\psi_+(x, s_0)|dx\right).$$

Now, since $v(\cdot, s_0) \in bmo(\mathbb{R}^n)$, using the Lemma 5.3 we have $|\overline{v}_{B_r} - \overline{v}_{B_{r2^k}}| \leq Ck\|v(\cdot, s_0)\|_{bmo} \leq Ck\mu$ and we can write

$$\begin{aligned}
I_{1,E_k} &\leq C2^{k(\omega-1)}r^{\omega-1}\left(\int_{B_{r2^k}} |v - \overline{v}_{B_{r2^k}}||\psi_+(x, s_0)|dx + Ck\mu\|\psi_+(\cdot, s_0)\|_{L^1}\right)\\
&\leq C2^{k(\omega-1)}r^{\omega-1}\left(\|\psi_+(\cdot, s_0)\|_{L^{a_0}}\|v - \overline{v}_{B_{r2^k}}\|_{L^{\frac{a_0}{a_0-1}}} + Ck\mu\, r^{-\gamma}\right)
\end{aligned}$$

where we used Hölder's inequality with $1 < a_0 < \frac{n}{n+(\omega-1)}$ and maximum principle for the last term above. Using again the properties of bmo spaces we have

$$I_{1,E_k} \leq C2^{k(\omega-1)}r^{\omega-1}\left(\|\psi_+(\cdot, 0)\|_{L^1}^{1/a_0}\|\psi_+(\cdot, 0)\|_{L^\infty}^{1-1/a_0}|B_{r2^k}|^{1-1/a_0}\|v(\cdot, s)\|_{bmo} + Ck\mu r^{-\gamma}\right).$$

Let us now apply the estimates given by hypothesis for $\|\psi_+(\cdot, 0)\|_{L^1}$, $\|\psi_+(\cdot, 0)\|_{L^\infty}$ and $\|v(\cdot, s_0)\|_{bmo}$ to obtain

$$I_{1,E_k} \leq C2^{k(n-n/a_0+\omega-1)}r^{\omega-1-\gamma}\mu + C2^{k(\omega-1)}k\mu\, r^{\omega-1-\gamma}.$$

Since $1 < a_0 < \frac{n}{n+(\omega-1)}$, we have $n - n/a_0 + (\omega - 1) < 0$, so that, summing over each dyadic corona E_k, we have

$$\sum_{k\geq 1} I_{1,E_k} \leq C\mu\, r^{\omega-1-\gamma}. \tag{42}$$

Finally, gathering together the estimations (41) and (42) we obtain the desired conclusion. ■

Proof of the Lemma 5.2. As for the Lemma 5.1, we consider $\mathbb{R}^n$ as the union of a ball with dyadic coronas centered on $x(s_0)$ (cf. (39)).

(i) Estimations over the ball B_r. We write, using the maximum principle and the hypothesis for $\|\psi_+(\cdot, 0)\|_{L^\infty}$:

$$\begin{aligned}
I_{2,B_r} &= \int_{B_r} \big|\mathcal{L}(|x - x(s_0)|^\omega)\big||\psi_+(x, s_0)|dx \leq \|\psi_+(\cdot, s_0)\|_{L^\infty}\int_{B_r} |\mathcal{L}|x - x(s_0)|^\omega|dx\\
&\leq \|\psi_+(\cdot, 0)\|_{L^\infty}\int_{\{|x|\leq r\}} \left|\text{v.p.}\int_{\mathbb{R}^n} [|x|^\omega - |x - y|^\omega]\pi(y)dy\right| dx\\
&\leq r^{-n-\gamma}\int_{\{|x|\leq r\}} \left|\text{v.p.}\int_{\mathbb{R}^n} [|x|^\omega - |x - y|^\omega]\pi(y)dy\right| dx.
\end{aligned}$$

We use now the hypothesis (4) and (5) for the function π in the case **(c)**, *i.e.* $\alpha = \beta = 1/2$ and $0 < \delta < 1/2$, in order to obtain

$$\begin{aligned}
I_{2,B_r} &\leq r^{-n-\gamma}\int_{\{|x|\leq r\}} \left|\text{v.p.}\int_{\{|y|\leq 1\}} \frac{|x|^\omega - |x - y|^\omega}{|y|^{n+1}}dy\right| dx + r^{-n-\gamma}\int_{\{|x|\leq r\}}\int_{\mathbb{R}^n} \frac{||x|^\omega - |x - y|^\omega|}{|y|^{n+2\delta}}dydx\\
&\leq r^{-n-\gamma}\big(I_{2,B_r^1} + I_{2,B_r^\delta}\big).
\end{aligned} \tag{43}$$

[3] recall that $0 < \gamma < \omega < 2\delta \leq 1$.

We start studying the first term I_{2,B_r^1} above. Recalling that

$$(-\Delta)^{1/2}(|x|^\omega) = \text{v.p.}\int_{\mathbb{R}^n} \frac{|x|^\omega - |x-y|^\omega}{|y|^{n+1}}dy = |x|^{\omega-1}, \tag{44}$$

by homogeneity and using the fact that $0 < r < 1$ we obtain:

$$I_{2,B_r^1} \leq r^{\omega+n-1}\left(\int_{\{|x|\leq 1\}} |x|^{\omega-1}dx + \int_{\{|x|\leq 1\}}\int_{\{|y|>1/r\}} \frac{||x|^\omega - |x-y|^\omega|}{|y|^{n+1}}dydx\right) = Cr^{\omega+n-1}.$$

For the second term I_{2,B_r^δ} we will proceed as follows. First, by homogeneity we obtain

$$I_{2,B_r^\delta} = r^{\omega+n-2\delta}\underbrace{\int_{\{|x|\leq 1\}}\int_{\mathbb{R}^n} \frac{||x|^\omega - |x-y|^\omega|}{|y|^{n+2\delta}}dydx}_{I}.$$

Then we decompose this integral I in the following way

$$\begin{aligned} I &= \int_{\{|x|\leq 1\}}\int_{\{|y|\leq 1\}} \frac{||x|^\omega - |x-y|^\omega|}{|y|^{n+2\delta}}dydx + \int_{\{|x|\leq 1\}}\int_{\{|y|>1\}} \frac{||x|^\omega - |x-y|^\omega|}{|y|^{n+2\delta}}dydx \\ &\leq \int_{\{|x|\leq 1\}}\left(\sup_{0<|y|<1} \frac{||x|^\omega - |x-y|^\omega|}{|y|}\right)\left(\int_{\{|y|\leq 1\}} |y|^{1-n-2\delta}dy\right)dx + \int_{\{|x|\leq 1\}}\left(\int_{\{|y|>1\}} |y|^{\omega-n-2\delta}dy\right)dx \end{aligned}$$

Since $0 < \gamma < \omega < 2\delta < 1$, it is not complicated to see that

$$I \leq C\int_{\{|x|\leq 1\}}\left(\sup_{0<|y|<1} \frac{||x|^\omega - |x-y|^\omega|}{|y|}\right)dx + C \tag{45}$$

and that this latter quantity is bounded. Then, getting back to (43) we write $I_{2,B_r} \leq C(r^{\omega-\gamma-1} + r^{\omega-\gamma-2\beta})$. Recalling that we are working with small molecules, *i.e.* that $0 < r < 1$, we obtain $r^{\omega-2\beta-\gamma} \leq r^{\omega-1-\gamma}$, so we finally have

$$I_{2,B_r} \leq Cr^{\omega-\gamma-1}.$$

The case **(d)**, when $\alpha = \beta = \delta = 1/2$, is easier since $(-\Delta)^{1/2}(|x|^\omega) = |x|^{\omega-1}$. Thus, in any case we can write:

$$I_{2,B_r} = \int_{B_r} |\mathcal{L}(|x - x(s_0)|^\omega)||\psi_+(x, s_0)|dx \leq Cr^{\omega-1-\gamma}. \tag{46}$$

(ii) Estimations for the dyadic corona E_k. We start with the case **(c)** when $\alpha = \beta = 1/2$ and $0 < \delta < 1/2$:

$$\begin{aligned} I_{2,E_k} &= \int_{E_k} |\mathcal{L}(|x - x(s_0)|^\omega)||\psi_+(x, s_0)|dx \leq \|\psi_+(\cdot, s_0)\|_{L^1} \sup_{x\in E_k} |\mathcal{L}(|x - x(s_0)|^\omega)| \\ &\leq r^{-\gamma}\Bigg(\underbrace{\sup_{r2^{k-1}<|x|\leq r2^k}\left|\text{v.p.}\int_{\{|y|\leq 1\}} \frac{|x|^\omega - |x-y|^\omega}{|y|^{n+1}}dy\right|}_{I_{2,E_k^1}} + \underbrace{\sup_{r2^{k-1}<|x|\leq r2^k}\int_{\mathbb{R}^n} \frac{||x|^\omega - |x-y|^\omega|}{|y|^{n+2\delta}}dy}_{I_{2,E_k^\delta}}\Bigg) \end{aligned}$$

Let us start with I_{2,E_k^1}, by homogeneity and using the formula (44) we obtain

$$I_{2,E_k^1} \leq \sup_{r2^{k-1}<|x|\leq r2^k} |x|^{\omega-1} + C(r2^k)^{\omega-1}\left(\sup_{1<|x|\leq 2}\int_{\{|y|>1/r2^{k-1}\}} \frac{||x|^\omega - |x-y|^\omega|}{|y|^{n+1}}dy\right)$$

We only need to study the last term of this expression. If $0 < r2^{k-1} \leq 1$, the integral above is immediately bounded by a constant. The case when $1 < r2^{k-1}$ is treated as follows:

$$\begin{aligned} \sup_{1<|x|\leq 2}\int_{\{|y|>1/r2^{k-1}\}} \frac{||x|^\omega - |x-y|^\omega|}{|y|^{n+1}}dy &= \sup_{1<|x|\leq 2}\left(\int_{\{1/r2^{k-1}<|y|<1\}} \frac{||x|^\omega - |x-y|^\omega|}{|y|^{n+1}}dy + \int_{\{1<|y|\}} \frac{||x|^\omega - |x-y|^\omega|}{|y|^{n+1}}dy\right) \\ &\leq \sup_{1<|x|\leq 2}\left(\sup_{0<|y|<1} \frac{||x|^\omega - |x-y|^\omega|}{|y|}\right)\ln(2^{k-1}) + C \end{aligned}$$

Thus we obtain $I_{2,E_k^1} \leq C(r2^k)^{\omega-1}\big(1+\ln(2^{k-1})\big)$.

The term I_{2,E_k^δ} is easier: applying essentially the same ideas used in the formulas (43)-(45) above and by homogeneity we have $I_{2,E_k^\delta} \leq C(r2^k)^{\omega-2\delta}$.

Finally, we obtain the following inequality for I_{2,E_k}:

$$I_{2,E_k} \quad \leq \quad Cr^{-\gamma}\left((r2^k)^{\omega-1}\big(1+\ln(2^{k-1})\big)+(r2^k)^{\omega-2\delta}\right)$$

Since $0<\gamma<\omega<2\delta<1$, summing over $k\geq 1$, we obtain $\sum\limits_{k\geq 1} I_{2,E_k} \leq Cr^{-\gamma}\left(r^{\omega-1}+r^{\omega-2\delta}\right)$. Repeating the same argument used before (*i.e.* the fact that $0<r<1$), we finally obtain

$$\sum_{k\geq 1} I_{2,E_k} \leq Cr^{\omega-1-\gamma}. \tag{47}$$

The case **(d)** is straightforward since we have $\mathcal{L}=(-\Delta)^{1/2}$ and $(-\Delta)^{1/2}(|x|^\omega)=|x|^{\omega-1}$. In order to finish the proof of Lemma 5.2 we combine together the estimates (46) and (47). ■

2) Small time Height condition

We treat now the Height condition (34) and for this we will give a sligthly different proof of the maximum principle of A. Córdoba & D. Córdoba. Indeed, the following proof only relies on the Concentration condition.

Assume that molecules we are working with are smooth enough. Following an idea of [6] (section 4 p.522-523) (see also [9] p. 346), we will note $\overline{x}$ the point of $\mathbb{R}^n$ such that $\psi(\overline{x},s_0)=\|\psi(\cdot,s_0)\|_{L^\infty}$. Thus we can write, by the properties of the function π (recall that we assumed $\alpha=\beta=1/2$ and $0<\delta<1/2$ or $\alpha=\beta=\delta=1/2$):

$$\frac{d}{ds_0}\|\psi(\cdot,s_0)\|_{L^\infty} \leq -\int_{\mathbb{R}^n}[\psi(\overline{x},s_0)-\psi(\overline{x}-y,s_0)]\pi(y)dy \leq -\int_{\{|\overline{x}-y|<1\}}\frac{\psi(\overline{x},s_0)-\psi(y,s_0)}{|\overline{x}-y|^{n+1}}dy \leq 0. \tag{48}$$

Let us consider the corona centered in $\overline{x}$ defined by

$$\mathcal{C}(R_1,R_2)=\{y\in\mathbb{R}^n : R_1\leq|\overline{x}-y|\leq R_2\}$$

where $1>R_2=\rho R_1$ with $\rho>2$ and where R_1 will be fixed later. Then:

$$\int_{\{|\overline{x}-y|<1\}}\frac{\psi(\overline{x},s_0)-\psi(y,s_0)}{|\overline{x}-y|^{n+1}}dy \geq \int_{\mathcal{C}(R_1,R_2)}\frac{\psi(\overline{x},s_0)-\psi(y,s_0)}{|\overline{x}-y|^{n+1}}dy.$$

Define the sets B_1 and B_2 by $B_1=\{y\in\mathcal{C}(R_1,R_2) : \psi(\overline{x},s_0)-\psi(y,s_0)\geq\frac{1}{2}\psi(\overline{x},s_0)\}$ and $B_2=\{y\in\mathcal{C}(R_1,R_2) : \psi(\overline{x},s_0)-\psi(y,s_0)<\frac{1}{2}\psi(\overline{x},s_0)\}$ such that $\mathcal{C}(R_1,R_2)=B_1\cup B_2$.

We obtain the inequalities

$$\int_{\mathcal{C}(R_1,R_2)}\frac{\psi(\overline{x},s_0)-\psi(y,s_0)}{|\overline{x}-y|^{n+1}}dy \quad\geq\quad \int_{B_1}\frac{\psi(\overline{x},s_0)-\psi(y,s_0)}{|\overline{x}-y|^{n+1}}dy \geq \frac{\psi(\overline{x},s_0)}{2R_2^{n+1}}|B_1| = \frac{\psi(\overline{x},s_0)}{2R_2^{n+1}}\left(|\mathcal{C}(R_1,R_2)|-|B_2|\right).$$

Since $R_2=\rho R_1$ one has

$$\int_{\mathcal{C}(R_1,R_2)}\frac{\psi(\overline{x},s_0)-\psi(y,s_0)}{|\overline{x}-y|^{n+1}}dy \geq \frac{\psi(\overline{x},s_0)}{2\rho^{n+1}R_1^{n+1}}\left(v_n(\rho^n-1)R_1^n-|B_2|\right) \tag{49}$$

where v_n denotes the volume of the n-dimensional unit ball.

To continue, we need to estimate the quantity $|B_2|$ in the right-hand side of (49) in terms of $\psi(\overline{x},s_0)$ and R_1. We will distinguish two cases:

1) if $|\overline{x}-x(s_0)|>2R_2$ or $|\overline{x}-x(s_0)|<R_1/2$ then

$$C_1(r+Ks_0)^{\omega-\gamma}\psi(\overline{x},s_0)^{-1}R_1^{-\omega} \geq |B_2| \tag{50}$$

2) if $R_1/2\leq|\overline{x}-x(s_0)|\leq 2R_2$ then

$$\left(C_2(r+Ks_0)^{\omega-\gamma}R_1^{n-\omega}\psi(\overline{x},s_0)^{-1}\right)^{1/2} \geq |B_2|. \tag{51}$$

For these two estimates, our starting point is the Concentration condition :

$$
\begin{aligned}
(r+Ks_0)^{\omega-\gamma} &\geq \int_{\mathbb{R}^n} |\psi(y,s_0)||y-x(s_0)|^{\omega}dy \\
&\geq \int_{B_2} |\psi(y,s_0)||y-x(s_0)|^{\omega}dy \geq \frac{\psi(\overline{x},s_0)}{2}\int_{B_2} |y-x(s_0)|^{\omega}dy. \qquad (52)
\end{aligned}
$$

We just need to estimate the last integral following the cases given above. Indeed, if $|\overline{x}-x(s_0)|>2R_2$ then we have

$$
\min_{y\in B_2\subset \mathcal{C}(R_1,R_2)} |y-x(s_0)|^{\omega} \geq R_2^{\omega} = \rho^{\omega} R_1^{\omega}
$$

while if $|\overline{x}-x(s_0)|<R_1/2$, one has

$$
\min_{y\in B_2\subset \mathcal{C}(R_1,R_2)} |y-x(s_0)|^{\omega} \geq \frac{R_1^{\omega}}{2^{\omega}}.
$$

Applying these results to (52) we obtain $(r+Ks_0)^{\omega-\gamma} \geq \frac{\psi(\overline{x},s_0)}{2}\rho^{\omega}R_1^{\omega}|B_2|$ and $(r+Ks_0)^{\omega-\gamma} \geq \frac{\psi(\overline{x},s_0)}{2}\frac{R_1^{\omega}}{2^{\omega}}|B_2|$, and since $\rho>2$ we have the first desired estimate

$$
\frac{C_1(r+Ks_0)^{\omega-\gamma}}{\psi(\overline{x},s_0)R_1^{\omega}} \geq \frac{2(r+Ks_0)^{\omega-\gamma}}{\rho^{\omega}\psi(\overline{x},s_0)R_1^{\omega}} \geq |B_2| \qquad \text{with } C_1 = 2^{1+\omega}.
$$

For the second case, since $R_1/2 \leq |\overline{x}-x(s_0)| \leq 2R_2$, we can write using the Cauchy-Schwarz inequality

$$
\int_{B_2} |y-x(s_0)|^{\omega}dy \geq |B_2|^2\left(\int_{B_2} |y-x(s_0)|^{-\omega}dy\right)^{-1} \qquad (53)
$$

Now, observe that in this case we have $B_2\subset B(x(s_0),5R_2)$ and then

$$
\int_{B_2} |y-x(s_0)|^{-\omega}dy \leq \int_{B(x(s_0),5R_2)} |y-x(s_0)|^{-\omega}dy \leq v_n(5\rho R_1)^{n-\omega}.
$$

Getting back to(53) we have

$$
\int_{B_2} |y-x(s_0)|^{\omega}dy \geq |B_2|^2 v_n^{-1}(5\rho R_1)^{-n+\omega}
$$

and we use this estimate in (52) to obtain

$$
\frac{C_2(r+Ks_0)^{\omega/2-\gamma/2}R_1^{n/2-\omega/2}}{\psi(\overline{x},s_0)^{1/2}} \geq |B_2|, \quad \text{where } C_2 = (2\times 5^{n-\omega}v_n\rho^{n-\omega})^{1/2}.
$$

Now, with estimates (50) and (51) at our disposal we can write

(i) if $|\overline{x}-x(s_0)|>2R_2$ or $|\overline{x}-x(s_0)|<R_1/2$ then

$$
\int_{\mathcal{C}(R_1,R_2)} \frac{\psi(\overline{x},s_0)-\psi(y,s_0)}{|\overline{x}-y|^{n+1}}dy \geq \frac{\psi(\overline{x},s_0)}{2\rho^{n+1}R_1^{n+1}}\left(v_n(\rho^n-1)R_1^n - \frac{C_1(r+Ks_0)^{\omega-\gamma}}{\psi(\overline{x},s_0)}R_1^{-\omega}\right)
$$

(ii) if $R_1/2 \leq |\overline{x}-x(s_0)| \leq 2R_2$

$$
\int_{\mathcal{C}(R_1,R_2)} \frac{\psi(\overline{x},s_0)-\psi(y,s_0)}{|\overline{x}-y|^{n+1}}dy \geq \frac{\psi(\overline{x},s_0)}{2\rho^{n+1}R_1^{n+1}}\left(v_n(\rho^n-1)R_1^n - \frac{C_2(r+Ks_0)^{\omega/2-\gamma/2}R_1^{n/2-\omega/2}}{\psi(\overline{x},s_0)^{1/2}}\right)
$$

If we set $R_1=(r+Ks_0)^{\frac{(\omega-\gamma)}{n+\omega}}\psi(\overline{x},s_0)^{\frac{-1}{n+\omega}}$ and if ρ is big enough such that the expressions in brackets above are positive, we obtain for cases (i) and (ii) the following estimate for (49):

$$
\int_{\mathcal{C}(R_1,R_2)} \frac{\psi(\overline{x},s_0)-\psi(y,s_0)}{|\overline{x}-y|^{n+1}}dy \geq C(r+Ks_0)^{-\frac{(\omega-\gamma)}{n+\omega}}\psi(\overline{x},s_0)^{1+\frac{1}{n+\omega}}
$$

where $C=C(n,\rho)=\frac{v_n(\rho^n-1)-\sqrt{2v_n}(5\rho)^{\frac{n-\omega}{2}}}{2\rho^{n+1}}<1$ is a small positive constant. Now, and for all possible cases considered before, we have the following estimate for (48):

$$
\frac{d}{ds_0}\|\psi(\cdot,s_0)\|_{L^{\infty}} \leq -C(r+Ks_0)^{-\frac{(\omega-\gamma)}{n+\omega}}\|\psi(\cdot,s_0)\|_{L^{\infty}}^{1+\frac{1}{n+\omega}}.
$$

In order to solve this problem, it is enough to remark that if $\|\psi(\cdot,s_0)\|_{L^\infty} \leq (r+Ks_0)^{-(n+\gamma)}$, then $\|\psi(\cdot,s_0)\|_{L^\infty}$ satisfies the previous inequality. Indeed, we have

$$\begin{aligned}
\frac{d}{ds_0}\|\psi(\cdot,s_0)\|_{L^\infty} &\leq -K(n+\gamma)(r+Ks_0)^{-(n+\gamma)-1}\\
&\leq -C(r+Ks_0)^{-(n+\gamma)-1} = -C(r+Ks_0)^{-\frac{(\omega-\gamma)}{n+\omega}}(r+Ks_0)^{-(n+\gamma)(1+\frac{1}{n+\omega})}\\
&\leq -C(r+Ks_0)^{-\frac{(\omega-\gamma)}{n+\omega}}\|\psi(\cdot,s_0)\|_{L^\infty}^{1+\frac{1}{n+\omega}}
\end{aligned}$$

Furthermore with the initial data $\|\psi(\cdot,0)\|_{L^\infty} \leq r^{-n-\gamma}$, we obtain that this solution is unique.

The proof of the Height condition is finished for regular molecules. In order to obtain the global result, remark that, for viscosity solutions (9), we have that $\Delta\theta(\overline{x},s_0) \leq 0$ at the points $\overline{x}$ where $\theta(\cdot,s_0)$ reaches its maximum value. See [6] for more details.

3) Small time L^1 estimate

This last condition is an easy consequence of the previous computations. Indeed: we write

$$\begin{aligned}
\int_{\mathbb{R}^n}|\psi(x,s_0)|dx &= \int_{\{|x-x(s_0)|<D\}}|\psi(x,s_0)|dx + \int_{\{|x-x(s_0)|\geq D\}}|\psi(x,s_0)|dx\\
&\leq v_n D^n\|\psi(\cdot,s_0)\|_{L^\infty} + D^{-\omega}\int_{\mathbb{R}}|\psi(x,s_0)||x-x(s_0)|^\omega dx
\end{aligned}$$

Now using the Concentration condition and the Height condition one has:

$$\int_{\mathbb{R}^n}|\psi(x,s_0)|dx \leq v_n\frac{D^n}{(r+Ks_0)^{n+\omega}} + D^{-\omega}(r+Ks_0)^{\omega-\gamma}$$

where v_n denotes the volume of the unit ball. To continue, it is enough to choose correctly the real parameter D to obtain

$$\|\psi(\cdot,s_0)\|_{L^1} \leq \frac{v_n}{(r+Ks_0)^\gamma}.$$

Theorem 10 is now completely proven. ■

5.2 Molecule's evolution: Second step

In the previous section we have obtained deformed molecules after a very small time s_0. The next theorem shows us how to obtain similar profiles in the inputs and the outputs in order to perform an iteration in time.

Recall that we consider here a Lévy-type operator $\mathcal{L}$ of the form (7) with an associate Lévy measure π that satisfies hypothesis (4) and (5) with the following values of the parameters α,β,δ:

(c) $\alpha=\beta=1/2$ and $0<\delta<1/2$,

(d) $\alpha=\beta=\delta=1/2$.

Theorem 11 *Set γ and ω two real numbers such that $0<\gamma<\omega<2\delta<1$ in the case **(c)** or $0<\gamma<\omega<1$ in the case **(d)**. Let $0<s_1\leq T$ and let $\psi(x,s_1)$ be a solution of the problem*

$$\begin{cases}
\partial_{s_1}\psi(x,s_1) = -\nabla\cdot(v\,\psi)(x,s_1) - \mathcal{L}\psi(x,s_1)\\
\psi(x,0) = \psi(x,s_0) \qquad \textit{with } s_0>0\\
div(v) = 0 \quad \textit{and} \quad v\in L^\infty([0,T];bmo(\mathbb{R}^n)) \quad \textit{with} \sup\limits_{s_1\in[s_0,T]}\|v(\cdot,s_1)\|_{bmo}\leq\mu
\end{cases} \tag{54}$$

If $\psi(x,s_0)$ satisfies the three following conditions

$$\int_{\mathbb{R}^n}|\psi(x,s_0)||x-x(s_0)|^\omega dx \leq (r+Ks_0)^{\omega-\gamma};\quad \|\psi(\cdot,s_0)\|_{L^\infty}\leq\frac{1}{(r+Ks_0)^{n+\gamma}};\quad \|\psi(\cdot,s_0)\|_{L^1}\leq\frac{v_n}{(r+Ks_0)^\gamma}$$

where $K = K(\mu)$ is given by (38) and s_0 is such that $(r + Ks_0) < 1$. Then for all $0 < s_1 \leq \epsilon r$ small, we have the following estimates

$$\int_{\mathbb{R}^n} |\psi(x,s_1)||x - x(s_1)|^{\omega}dx \leq (r + K(s_0 + s_1))^{\omega-\gamma} \tag{55}$$

$$\|\psi(\cdot,s_1)\|_{L^\infty} \leq \frac{1}{(r + K(s_0 + s_1))^{n+\gamma}} \tag{56}$$

$$\|\psi(\cdot,s_1)\|_{L^1} \leq \frac{v_n}{\big(r + K(s_0 + s_1)\big)^{\gamma}} \tag{57}$$

Remark 5.4

1) *Since s_1 is small and $(r+Ks_0) < 1$, we can without loss of generality assume that $(r+K(s_0+s_1)) < 1$: otherwise, by the maximum principle there is nothing to prove.*
2) *The new molecule's center $x(s_1)$ used in formula (55) is fixed by*

$$\begin{cases} x'(s_1) = & \overline{v}_{B_{f_1}} = \frac{1}{|B_{f_1}|}\displaystyle\int_{B_{f_1}} v(y,s_1)dy \\ x(0) = & x(s_0). \end{cases} \tag{58}$$

And here we noted $B_{f_1} = B(x(s_1), f_1)$ with f_1 a real valued function given by

$$f_1 = (r + Ks_0). \tag{59}$$

Note that by remark 1) *above we have $0 < f_1 < 1$.*

Proof of the Theorem 11. We will follow the same scheme as before: we first prove the Concentration condition (55), with this estimate at hand we will control the L^∞ decay and then we will obtain the suitable L^1 control.

1) The Concentration condition

The calculations are very similar of those performed before: the only difference stems from the initial data and the definition of the center $x(s_1)$. So, let us write $\Omega_1(x) = |x - x(s_1)|^{\omega}$ and $\psi(x) = \psi_+(x) - \psi_-(x)$ where the functions $\psi_\pm(x) \geq 0$ have disjoint support. Thus, by linearity and using the positivity principle we have

$$|\psi(x,s_1)| = |\psi_+(x,s_1) - \psi_-(x,s_1)| \leq \psi_+(x,s_1) + \psi_-(x,s_1)$$

and we can write

$$\int_{\mathbb{R}^n} |\psi(x,s_1)\Omega_1(x)dx \leq \int_{\mathbb{R}^n} \psi_+(x,s_1)\Omega_1(x)dx + \int_{\mathbb{R}^n} \psi_-(x,s_1)\Omega_1(x)dx$$

so we only have to treat one of the integrals on the right-hand side above. We have:

$$I = \left|\partial_{s_1}\int_{\mathbb{R}^n} \Omega_1(x)\psi_+(x,s_1)dx\right| = \left|\int_{\mathbb{R}^n} -\nabla\Omega_1(x)\cdot x'(s_1)\psi_+(x,s_1) + \Omega_1(x)\left[-\nabla\cdot(v\,\psi_+(x,s_1)) - \mathcal{L}\psi_+(x,s_1)\right]dx\right|$$

Using the fact that v is divergence free, we obtain

$$I = \left|\int_{\mathbb{R}^n} \nabla\Omega_1(x)\cdot(v - x'(s_1))\psi_+(x,s_1) - \Omega_1(x)\mathcal{L}\psi_+(x,s_1)dx\right|.$$

Finally, using the definition of $x'(s_1)$ given in (58) and replacing $\Omega_1(x)$ by $|x - x(s_1)|^{\omega}$ in the first integral we obtain

$$I \leq \underbrace{c\int_{\mathbb{R}^n} |x - x(s_1)|^{\omega-1}|v - \overline{v}_{B_{f_1}}||\psi_+(x,s_1)|dx}_{I_1} + \underbrace{c\int_{\mathbb{R}^n} |\mathcal{L}\Omega_1(x)||\psi_+(x,s_1)|dx}_{I_2}. \tag{60}$$

We will study separately each of the integrals I_1 and I_2 in the next lemmas:

Lemma 5.4 *For integral I_1 we have the estimate $I_1 \leq C\mu\big(r + Ks_0\big)^{\omega-\gamma-1}$.*

Lemma 5.5 *For integral I_2 in the inequality (60) we have the following estimate $I_2 \leq C\big(r + Ks_0\big)^{\omega-\gamma-1}$.*

Using these lemmas and getting back to the estimate (60) we have

$$\left|\partial_{s_1}\int_{\mathbb{R}^n}\Omega_1(x)\psi_+(x,s_1)dx\right|\leq C\,(\mu+1)\,\big(r+Ks_0\big)^{\omega-\gamma-1} \tag{61}$$

This estimation is compatible with the estimate (55) for $0\leq s_1\leq \epsilon r$ small enough. Indeed, we can write $\phi=(r+K(s_0+s_1))^{\omega-\gamma}$ and we linearize this expression with respect to s_1:

$$\phi\approx(r+s_0)^{\omega-\gamma}\left(1+K(\omega-\gamma)\frac{s_1}{(r+s_0)}\right)$$

Taking the derivative of ϕ with respect to s_1 we have $\phi'\approx K(\omega-\gamma)\big(r+Ks_0\big)^{\omega-\gamma-1}$ and with the condition (38) on $K(\omega-\gamma)$ we obtain that (61) is bounded by ϕ' and the Concentration condition follows.

Proof of the Lemma 5.4. We begin by considering the space $\mathbb{R}^n$ as the union of a ball with dyadic coronas centered on $x(s_1)$, more precisely we set $\mathbb{R}^n=B_{f_1}\cup\bigcup_{k\geq 1}E_k$ where

$$\begin{aligned} B_{f_1} &= \{x\in\mathbb{R}^n: |x-x(s_1)|\leq f_1\}, \\ E_k &= \{x\in\mathbb{R}^n: f_1 2^{k-1}<|x-x(s_1)|\leq f_1 2^k\}\quad\text{for } k\geq 1. \end{aligned} \tag{62}$$

(i) Estimations over the ball B_{f_1}. Applying Hölder's inequality on integral I_1 we obtain

$$\begin{aligned} I_{1,B_{f_1}}=\int_{B_{f_1}}|x-x(s_1)|^{\omega-1}|v-\overline{v}_{B_{f_1}}||\psi_+(x,s_1)|dx &\leq \underbrace{\||x-x(s_1)|^{\omega-1}\|_{L^p(B_{f_1})}}_{(1)} \\ &\times \underbrace{\|v-\overline{v}_{B_{f_1}}\|_{L^z(B_{f_1})}}_{(2)}\underbrace{\|\psi_+(\cdot,s_1)\|_{L^q(B_{f_1})}}_{(3)} \end{aligned}$$

where $\frac{1}{p}+\frac{1}{z}+\frac{1}{q}=1$ and $p,z,q>1$. We treat each of the previous terms separately:

- Observe that for $1<p<n/(1-\omega)$ we have

$$\||x-x(s_1)|^{\omega-1}\|_{L^p(B_{f_1})}\leq Cf_1^{n/p+\omega-1}.$$

- We have $v(\cdot,s_1)\in bmo(\mathbb{R}^n)$, thus $\|v-\overline{v}_{B_{f_1}}\|_{L^z(B_{f_1})}\leq C|B_{f_1}|^{1/z}\|v(\cdot,s_1)\|_{bmo}$. Since $\sup\limits_{s_1\in[s_0,T]}\|v(\cdot,s_1)\|_{bmo}\leq\mu$ we write

$$\|v-\overline{v}_{B_{f_1}}\|_{L^z(B_{f_1})}\leq Cf_1^{n/z}\mu.$$

- Finally, by the maximum principle for L^q norms we have $\|\psi_+(\cdot,s_1)\|_{L^q(B_{f_1})}\leq\|\psi(\cdot,s_0)\|_{L^q}$; hence we obtain

$$\|\psi_+(\cdot,s_1)\|_{L^q(B_{f_1})}\leq\|\psi(\cdot,s_0)\|_{L^1}^{1/q}\|\psi(\cdot,s_0)\|_{L^\infty}^{1-1/q}.$$

We combine all these inequalities in order to obtain the following estimation for $I_{1,B_{f_1}}$:

$$I_{1,B_{f_1}}\leq C\mu f_1^{n(1-1/q)+\omega-1}\|\psi(\cdot,s_0)\|_{L^1}^{1/q}\|\psi(\cdot,s_0)\|_{L^\infty}^{1-1/q}.$$

(ii) Estimations for the dyadic corona E_k. Let us note I_{1,E_k} the integral

$$I_{1,E_k}=\int_{E_k}|x-x(s_1)|^{\omega-1}|v-\overline{v}_{B_{f_1}}||\psi_+(x,s_1)|dx.$$

Since over E_k we have $|x-x(s_1)|^{\omega-1}\leq C2^{k(\omega-1)}f_1^{\omega-1}$ we write

$$\begin{aligned} I_{1,E_k} &\leq C2^{k(\omega-1)}f_1^{\omega-1}\left(\int_{E_k}|v-\overline{v}_{B(f_12^k)}||\psi_+(x,s_1)|dx+\int_{E_k}|\overline{v}_{B_{f_1}}-\overline{v}_{B(f_12^k)}||\psi_+(x,s_1)|dx\right) \\ &\leq C2^{k(\omega-1)}f_1^{\omega-1}\left(\int_{B(f_12^k)}|v-\overline{v}_{B(f_12^k)}||\psi_+(x,s_1)|dx\right. \\ &\qquad\left.+\int_{B(f_12^k)}|\overline{v}_{B_{f_1}}-\overline{v}_{B(f_12^k)}||\psi_+(x,s_1)|dx\right). \end{aligned}$$

where $B(f_1 2^k)=\{x\in\mathbb{R}^n : |x-x(s_1)|\leq f_1 2^k\}$.

Now, since $v(\cdot,s_1)\in bmo(\mathbb{R}^n)$, using the Lemma 5.3 we have $|\overline{v}_{B_{f_1}}-\overline{v}_{B(f_1 2^k)}|\leq Ck\|v(\cdot,s_1)\|_{bmo}\leq Ck\mu$. We write

$$\begin{aligned}
I_{1,E_k} &\leq C2^{k(\omega-1)}f_1^{\omega-1}\left(\int_{B(f_1 2^k)}|v-\overline{v}_{B(f_1 2^k)}||\psi_+(x,s_1)|dx+Ck\mu\|\psi_+(\cdot,s_1)\|_{L^1}\right)\\
&\leq C2^{k(\omega-1)}f_1^{\omega-1}\left(\|\psi_+(\cdot,s_1)\|_{L^{a_0}}\|v-\overline{v}_{B(f_1 2^k)}\|_{L^{\frac{a_0}{a_0-1}}}+Ck\mu\,\|\psi_+(\cdot,s_0)\|_{L^1}\right)
\end{aligned}$$

where we used Hölder's inequality with $1<a_0<\frac{n}{n+(\omega-1)}$ and maximum principle for the last term above. Using again the properties of bmo spaces we have

$$I_{1,E_k}\leq C2^{k(\omega-1)}f_1^{\omega-1}\left(\|\psi_+(\cdot,s_0)\|_{L^1}^{1/a_0}\|\psi_+(\cdot,s_0)\|_{L^\infty}^{1-1/a_0}|B(f_1 2^k)|^{1-1/a_0}\|v(\cdot,s_1)\|_{bmo}+Ck\mu\|\psi(\cdot,s_0)\|_{L^1}\right).$$

Since $\|v(\cdot,s_1)\|_{bmo}\leq\mu$ and since $1<a_0<\frac{n}{n+(\omega-1)}$, we have $n(1-1/a_0)+(\omega-1)<0$, so that, summing over each dyadic corona E_k, we obtain

$$\sum_{k\geq1}I_{1,E_k}\leq C\mu\left(f_1^{n(1-1/a_0)+\omega-1}\|\psi(\cdot,s_0)\|_{L^1}^{1/a_0}\|\psi(\cdot,s_0)\|_{L^\infty}^{1-1/a_0}+f_1^{\omega-1}\|\psi(\cdot,s_0)\|_{L^1}\right).$$

We finally obtain the following inequalities:

$$\begin{aligned}
I_1 &= I_{1,B_{f_1}}+\sum_{k\geq1}I_{1,E_k} \qquad (63)\\
&\leq C\mu\underbrace{f_1^{n(1-1/q)+\omega-1}\|\psi(\cdot,s_0)\|_{L^1}^{1/q}\|\psi(\cdot,s_0)\|_{L^\infty}^{1-1/q}}_{(a)}\\
&\quad+C\mu\left(\underbrace{f_1^{n(1-1/a_0)+\omega-1}\|\psi(\cdot,s_0)\|_{L^1}^{1/a_0}\|\psi(\cdot,s_0)\|_{L^\infty}^{1-1/a_0}}_{(b)}+\underbrace{f_1^{\omega-1}\|\psi(\cdot,s_0)\|_{L^1}}_{(c)}\right)
\end{aligned}$$

Now we will prove that each of the terms (a), (b) and (c) above is bounded by the quantity $\big(r+Ks_0\big)^{\omega-\gamma-1}$:

- for the first term (a) by the hypothesis on the initial data $\psi(\cdot,s_0)$ and the definition of f_1 given in (59) we have:

$$f_1^{n(1-1/q)+\omega-1}\|\psi(\cdot,s_0)\|_{L^1}^{1/q}\|\psi(\cdot,s_0)\|_{L^\infty}^{1-1/q}\ \leq\ \big(r+Ks_0\big)^{[n(1-1/q)+\omega-1]-\frac{\gamma}{q}-(n+\gamma)(1-1/q)}=\big(r+Ks_0\big)^{\omega-\gamma-1}.$$

- For the second term (b) we have, by the same arguments:

$$f_1^{n(1-1/a_0)+\omega-1}\|\psi(\cdot,s_0)\|_{L^1}^{1/a_0}\|\psi(\cdot,s_0)\|_{L^\infty}^{1-1/a_0}\ \leq\ \big(r+Ks_0\big)^{[n(1-1/a_0)+\omega-1]-\frac{\gamma}{a_0}-(n+\gamma)(1-1/a_0)}=\big(r+Ks_0\big)^{\omega-\gamma-1}.$$

- Finally, for the last term (c) we write

$$f_1^{\omega-1}\|\psi(\cdot,s_0)\|_{L^1}\ \leq\ f_1^{\omega-1}(r+Ks_0)^{-\gamma}=\big(r+Ks_0\big)^{\omega-\gamma-1}.$$

Gathering these estimates on $(a),(b)$ and (c), and getting back to (63) we finally obtain

$$I_1\leq C\mu\big(r+Ks_0\big)^{\omega-\gamma-1}.$$

The Lemma 5.4 is proven. ■

Proof of the Lemma 5.5. As for the Lemma 5.4, we consider $\mathbb{R}^n$ as the union of a ball with dyadic coronas centered on $x(s_1)$ (cf. (62)).

(i) Estimations over the ball B_{f_1}. We will follow closely the computations of the Lemma 5.2. We write:

$$\begin{aligned}
I_{2,B_{f_1}} &= \int_{B_{f_1}}|\mathcal{L}(|x-x(s_1)|^\omega)|\,|\psi_+(x,s_1)|dx\leq\|\psi_+(\cdot,s_1)\|_{L^\infty}\int_{B_{f_1}}|\mathcal{L}(|x-x(s_1)|^\omega)|dx\\
&\leq \|\psi_+(\cdot,s_0)\|_{L^\infty}\int_{\{|x|\leq f_1\}}\left|\text{v.p.}\int_{\mathbb{R}^n}[|x|^\omega-|x-y|^\omega]\pi(y)dy\right|dx.
\end{aligned}$$

In the case **(c)** when $\alpha = \beta = 1/2$ and $\delta < 1/2$ we write:

$$I_{2,B_{f_1}} \leq \|\psi_+(\cdot, s_0)\|_{L^\infty} \left(\int_{\{|x|\leq f_1\}} \left| \text{v.p.} \int_{\{|y|\leq 1\}} \frac{|x|^\omega - |x-y|^\omega}{|y|^{n+1}} dy \right| dx + \int_{\{|x|\leq f_1\}} \int_{\mathbb{R}^n} \frac{||x|^\omega - |x-y|^\omega|}{|y|^{n+2\delta}} dy dx \right)$$

Following exactly the same arguments used in Lemma 5.2 with the formulas (43)-(45), *i.e.* essentially by homogeneity, we have

$$I_{2,B_{f_1}} \leq C\|\psi_+(\cdot, s_0)\|_{L^\infty}(f_1^{n+\omega-1} + f_1^{n+\omega-2\delta})$$

Since $0 < 2\delta < 1$, recalling that by the definition of the function f_1 we have the estimate $0 < f_1 < 1$, we obtain $f_1^{\omega-2\delta-\gamma} \leq f_1^{\omega-1-\gamma}$. The case **(d)** is straightforward since $\mathcal{L} = (-\Delta)^{1/2}$ and $(-\Delta)^{1/2}(|x|^\omega) = |x|^{\omega-1}$.

Thus, in any case, we can write:

$$I_{2,B_{f_1}} \leq C f_1^{n+\omega-1}\|\psi_+(\cdot, s_0)\|_{L^\infty}. \tag{64}$$

(ii) Estimations for the dyadic corona E_k. Here we have

$$I_{2,E_k} = \int_{E_k} |\mathcal{L}(|x - x(s_1)|^\omega)|\, |\psi_+(x, s_1)| dx \leq \|\psi_+(\cdot, s_0)\|_{L^1} \sup_{f_1 2^{k-1}<|x|\leq f_1 2^k} \left| \text{v.p.} \int_{\mathbb{R}^n} [|x|^\omega - |x-y|^\omega]\pi(y)dy \right|.$$

In the case **(c)** we have:

$$I_{2,E_k} \leq \|\psi_+(\cdot, s_0)\|_{L^1} \sup_{f_1 2^{k-1}<|x|\leq f_1 2^k} \left(\left| \text{v.p.} \int_{\{|y|\leq 1\}} \frac{|x|^\omega - |x-y|^\omega}{|y|^{n+1}} dy \right| + \int_{\mathbb{R}^n} \frac{||x|^\omega - |x-y|^\omega|}{|y|^{n+2\delta}} dy \right).$$

Again, by homogeneity and following the same lines of the Lemma 5.2 above, we have

$$I_{2,E_k} \leq C\|\psi_+(\cdot, s_0)\|_{L^1} \left((f_1 2^k)^{\omega-1}\big(1 + \ln(2^{k-1})\big) + (f_1 2^k)^{\omega-2\delta}\right)$$

Since $0 < \gamma < \omega < 2\delta < 1$ we have $\omega - 1 < 0$ and $\omega - 2\delta < 0$ and thus, summing over $k \geq 1$, we obtain

$$\sum_{k\geq 1} I_{2,E_k} \leq C\left(f_1^{\omega-1} + f_1^{\omega-2\delta}\right)\|\psi(\cdot, s_0)\|_{L^1}.$$

Repeating the same argument used before (*i.e.* the fact that $0 < f_1 < 1$), we finally get

$$\sum_{k\geq 1} I_{2,E_k} \leq C f_1^{\omega-1}\|\psi(\cdot, s_0)\|_{L^1}. \tag{65}$$

For the case **(d)**, we obtain the same inequality by homogeneity.

To finish the proof of the Lemma 5.5 we combine (64) and (65) and we obtain

$$I_2 = I_{2,B_{f_1}} + \sum_{k\geq 1} I_{2,E_k} \leq C \left(\underbrace{f_1^{n+\omega-1}\|\psi_+(\cdot, s_0)\|_{L^\infty}}_{(d)} + \underbrace{f_1^{\omega-1}\|\psi(\cdot, s_0)\|_{L^1}}_{(e)} \right)$$

Now, we prove that the quantities (d) and (e) can be bounded by $\big(r + Ks_0\big)^{\omega-\gamma-1}$.

- For the term (d) we write $f_1^{n+\omega-1}\|\psi(\cdot, s_0)\|_{L^\infty} \leq f_1^{n+\omega-1}(r + Ks_0)^{-(n+\gamma)} = \big(r + Ks_0\big)^{\omega-\gamma-1}$.
- To treat the term (e) it is enough to apply the same arguments used to prove the part (c) above.

Finally, we obtain

$$I_2 = I_{2,B_{f_1}} + \sum_{k\geq 1} I_{2,E_k} \leq C\big(r + Ks_0\big)^{\omega-\gamma-1}$$

The Lemma 5.5 is proven. ■

2) The Height condition

Now we write down the maximum principle for a small time s_1 but with a initial condition $\psi(\cdot, s_0)$, with $s_0 > 0$. The proof follows essentially the same ideas explained in the previous step pages 19-21. Indeed, since we have assumed that the Concentration condition (55) is bounded by $(r + K(s_0 + s_1))^{\omega-\gamma}$, we obtain in the same manner and with the same constants:

$$\frac{d}{ds_1}\|\psi(\cdot, s_1)\|_{L^\infty} \leq -C(r + K(s_0 + s_1))^{-\frac{(\omega-\gamma)}{n+\omega}}\|\psi(\cdot, s_1)\|_{L^\infty}^{1+\frac{1}{n+\omega}}.$$

To conclude, it is enough to solve the previous differential inequality with initial data $\|\psi(\cdot, 0)\|_{L^\infty} \leq (r + Ks_0)^{-(n+\gamma)}$ to obtain that $\|\psi(\cdot, s_1)\|_{L^\infty} \leq (r + K(s_0 + s_1))^{-(n+\gamma)}$.

3) The L^1 condition

The L^1-norm condition is a direct consequence of the previous concentration condition (55) and of the height condition (56).

We have the estimates (55), (56) and (57) and the Theorem 11 is thus proven. ■

5.3 The iteration

In sections 5.1 and 5.2 we studied respectively the evolution of small molecules from time 0 to a small time s_0 and from this time s_0 to a larger time s_0+s_1 and we obtained a good L^1 control for such molecules. It is now possible to reapply the previous Theorem 11 in order to obtain a larger time control of the L^1 norm. The calculus of the N-th iteration will be essentially the same.

Theorem 12 *Set γ and ω two real numbers such that $0<\gamma<\omega<2\delta<1$ in the case **(c)** or $0<\gamma<\omega<1$ in the case **(d)**. Let $0<s_N\leq T$ and let $\psi(x,s_N)$ be a solution of the problem*

$$\begin{cases} \partial_{s_N}\psi(x,s_N) = & -\nabla\cdot(v\,\psi)(x,s_N)-\mathcal{L}\psi(x,s_N)\\ \psi(x,0) = & \psi(x,s_{N-1}) \qquad\qquad \textit{with } s_{N-1}>0\\ div(v) = & 0 \quad \textit{and} \quad v\in L^\infty([0,T];bmo(\mathbb{R}^n)) \quad \textit{with} \sup_{s_N\in[s_{N-1},T]}\|v(\cdot,s_N)\|_{bmo}\leq\mu \end{cases} \tag{66}$$

If $\psi(x,s_{N-1})$ satisfies the three following conditions

$$\int_{\mathbb{R}^n}|\psi(x,s_{N-1})||x-x(s_{N-1})|^\omega dx \leq (r+K(s_0+\cdots+s_{N-1}))^{\omega-\gamma}$$

$$\|\psi(\cdot,s_{N-1})\|_{L^\infty}\leq\frac{1}{(r+K(s_0+\cdots+s_{N-1}))^{n+\gamma}} \quad ; \quad \|\psi(\cdot,s_{N-1})\|_{L^1}\leq\frac{v_n}{\big(r+K(s_0+\cdots+s_{N-1})\big)^\gamma}$$

where $K=K(\mu)$ is given by (38) and s_N is such that $(r+K(s_0+\cdots+s_N))<1$. Then for all $0<s_N\leq\epsilon r$ small, we have the following estimates

$$\int_{\mathbb{R}^n}|\psi(x,s_N)||x-x(s_N)|^\omega dx \leq (r+K(s_0+\cdots+s_N))^{\omega-\gamma} \tag{67}$$

$$\|\psi(\cdot,s_N)\|_{L^\infty} \leq \frac{1}{(r+K(s_0+\cdots+s_N))^{n+\gamma}}$$

$$\|\psi(\cdot,s_N)\|_{L^1} \leq \frac{v_n}{\big(r+K(s_0+\cdots+s_N)\big)^\gamma} \tag{68}$$

Remark 5.5

1) Again, since s_N is small and $(r+K(s_0+\cdots+s_{N-1}))<1$, we can without loss of generality assume that $(r+K(s_0+\cdots+s_N))<1$: otherwise, by the maximum principle there is nothing to prove.

2) The new molecule's center $x(s_N)$ used in formula (67) is fixed by

$$\begin{cases} x'(s_N) = & \overline{v}_{B_{f_N}} = \frac{1}{|B_{f_N}|}\int_{B_{f_N}} v(y,s_N)dy\\ x(0) = & x(s_{N-1}). \end{cases} \tag{69}$$

And here we noted $B_{f_N}=B(x(s_N),f_N)$ with f_N a real valued function given by

$$f_N=(r+K(s_0+\cdots+s_{N-1})). \tag{70}$$

Note that by remark 1) above we have $0<f_N<1$.

Proof of the Theorem 12. The proof will follow again the same scheme: we start with the Concentration condition, we continue with the Height condition and the L^1 bound will be an easy consequence of these two estimates.

1) The Concentration condition

Write $\Omega_N(x) = |x - x(s_N)|^\omega$ and $\psi(x) = \psi_+(x) - \psi_-(x)$, by linearity and using the positivity principle we have $|\psi(x, s_N)| = |\psi_+(x, s_N) - \psi_-(x, s_N)| \leq \psi_+(x, s_N) + \psi_-(x, s_N)$ and we may consider the formula:

$$I \quad = \quad \left|\partial_{s_N} \int_{\mathbb{R}^n} \Omega_N(x)\psi_+(x, s_N)dx\right| = \left|\int_{\mathbb{R}^n} -\nabla\Omega_N(x) \cdot x'(s_N)\psi_+(x, s_N) + \Omega_N(x)\left[-\nabla \cdot (v\,\psi_+(x, s_N)) - \mathcal{L}\psi_+(x, s_N)\right] dx\right|$$

Using the definition of $x'(s_N)$ given in (69) and replacing $\Omega_N(x)$ by $|x - x(s_N)|^\omega$ in the first integral we obtain

$$I \leq \underbrace{c\int_{\mathbb{R}^n} |x - x(s_N)|^{\omega-1}|v - \overline{v}_{B_f}||\psi_+(x, s_N)|dx}_{I_1} + \underbrace{c\int_{\mathbb{R}^n} |\mathcal{L}\Omega_N(x)||\psi_+(x, s_N)|dx}_{I_2}\,. \tag{71}$$

We will study each of the integrals I_1 and I_2 in the next lemmas:

Lemma 5.6 *For integral I_1 we have $I_1 \leq C\mu\big(r + K(s_0 + \cdots + s_{N-1})\big)^{\omega-\gamma-1}$.*

Proof. It is enough to repeat the same steps of the previous Lemma 5.4, just consider $\mathbb{R}^n = B_{f_N} \cup \bigcup_{k\geq 1} E_k$ where

$$B_{f_N} = \{x \in \mathbb{R}^n : |x - x(s_N)| \leq f_N\}, \qquad E_k = \{x \in \mathbb{R}^n : f_N 2^{k-1} < |x - x(s_N)| \leq f_N 2^k\} \quad \text{for } k \geq 1. \tag{72}$$

In order to obtain the desired inequality, use exactly the same arguments, the maximum principle and the hypothesis of Theorem 12. ■

Lemma 5.7 *For integral I_2 in inequality (71) we have the following estimate*

$$I_2 = \int_{\mathbb{R}^n} |\mathcal{L}\Omega_N(x)||\psi_+(x, s_N)|dx \leq C\big(r + K(s_0 + \cdots + s_{N-1})\big)^{\omega-\gamma-1}.$$

Proof. As for Lemma 5.6, we consider $\mathbb{R}^n$ as the union of a ball with dyadic coronas centered on $x(s_N)$ (cf. (72)). It is then enough to repeat the corresponding estimates of the s_1-case given in Lemma 5.5. ■

Now using the Lemmas 5.6 and 5.7 and getting back to the estimate (71) we have

$$\left|\partial_{s_N} \int_{\mathbb{R}^n} \Omega_N(x)\psi_+(x, s_N)dx\right| \leq C\,(\mu + 1)\,\big(r + K(s_0 + \cdots + s_{N-1})\big)^{\omega-\gamma-1} \tag{73}$$

This estimation is compatible with the estimate (67) for $0 \leq s_N \leq \epsilon r$ small enough. Indeed, we can write $\phi = (r + K(s_0 + \cdots + s_N))^{\omega-\gamma}$ and we linearize this expression with respect to s_N:

$$\phi \approx (r + K(s_0 + \cdots + s_{N-1}))^{\omega-\gamma}\left(1 + K(\omega - \gamma)\frac{s_N}{(r + K(s_0 + \cdots + s_{N-1}))}\right)$$

Taking the derivative of ϕ with respect to s_N we have $\phi' \approx K(\omega - \gamma)\big(r + K(s_0 + \cdots + s_{N-1})\big)^{\omega-\gamma-1}$ and with the condition (38) on $K(\omega - \gamma)$ we obtain that (73) is bounded by ϕ' and we have proven the Concentration condition.

2) The Height condition

Since we have that Concentration condition (67) is bounded by $(r + K(s_0 + \cdots + s_N))^{\omega-\gamma}$, following the previous computations we obtain in the same manner and with the same constants:

$$\frac{d}{ds_N}\|\psi(\cdot, s_N)\|_{L^\infty} \leq -C(r + K(s_0 + \cdots + s_N))^{-\frac{(\omega-\gamma)}{n+\omega}}\|\psi(\cdot, s_N)\|_{L^\infty}^{1+\frac{1}{n+\omega}}.$$

Solving this differencial inequality we obtain $\|\psi(\cdot, s_N)\|_{L^\infty} \leq (r + K(s_0 + \cdots + s_N))^{-(n+\gamma)}$.

3) The L^1-norm estimate

Again this is a direct consequence of the Concentration condition and of the previous Height condition.

Theorem 12 is completely proven. ■

End of the proof of Theorem 9

We see with the Theorem 10 that is possible to control the L^1 behavior of the molecules ψ from 0 to a small time s_0, from time s_0 to time s_1 with Theorem 11, and by iteration from time s_{N-1} to time s_N with Theorem 12. We recall that we have $s_i \sim \epsilon r$ for all $0 \leq i \leq N$, so the bound obtained in (68) depends mainly on the size of the molecule r and the number of iterations N.

We observe now that the smallness of r and of the times $s_0, ..., s_N$ can be compensated by the number of iterations N in the following sense: fix a small $0 < r < 1$ and iterate as explained before. Since each small time $s_0, ..., s_N$ is of order ϵr, we have $s_0 + \cdots + s_N \sim N\epsilon r$. Thus, we will stop the iterations as soon as $Nr \geq T_0$.

Of course, the number of iterations $N = N(r)$ will depend on the smallness of the molecule's size r, and more specifically it is enough to set $N(r) \sim \frac{T_0}{r}$ in order to obtain this lower bound for Nr.

Proceeding this way we will obtain $\|\psi(\cdot, s_N)\|_{L^1} \leq CT_0^{-\gamma} < +\infty$, for all molecules of size r. Note in particular that, once this estimate is available, for bigger times it is enough to apply the maximum principle.

Finally, and for all $r > 0$, we obtain after a time T_0 a L^1 control for small molecules and we finish the proof of the Theorem 9. $\blacksquare$

Appendix

Proof of the Lemma 2.2. By homogeneity the cases **(b)** and **(d)** are straightforward. If $0 < \alpha \leq \beta < 1/2$ and $0 < \delta < 1/2$, using (4) and (5) we obtain

$$\|\mathcal{L}f\|_{L^1} \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(x-y)|}{|y|^{n+2\beta}}dydx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(x-y)|}{|y|^{n+2\delta}}dydx = \|f\|_{\dot{B}^{2\beta,1}_1} + \|f\|_{\dot{B}^{2\delta,1}_1}.$$

If $\alpha = \beta = 1/2$ and $\delta < 1/2$, we simply write

$$\begin{aligned}\|\mathcal{L}f\|_{L^1} &\leq \int_{\mathbb{R}^n}\left|\text{v.p.}\int_{\{|y|\leq 1\}}\big[f(x)-f(x-y)\big]\pi(y)dy\right|dx + \int_{\mathbb{R}^n}\left|\int_{\{|y|>1\}}[f(x)-f(x-y)]\pi(y)dy\right|dx\\ &\leq \int_{\mathbb{R}^n}\left|\text{v.p.}\int_{\{|y|\leq 1\}}\frac{f(x)-f(x-y)}{|y|^{n+1}}dy\right|dx + \|f\|_{\dot{B}^{2\delta,1}_1}.\end{aligned}$$

Now, since $(-\Delta)^{1/2}f(x) = \text{v.p.}\displaystyle\int_{\mathbb{R}^n}\frac{f(x)-f(x-y)}{|y|^{n+1}}dy$ it is easy to obtain that

$$\int_{\mathbb{R}^n}\left|\text{v.p.}\int_{\{|y|\leq 1\}}\frac{f(x)-f(x-y)}{|y|^{n+1}}dy\right|dx \leq \|(-\Delta)^{1/2}f\|_{L^1} + C\|f\|_{L^1}.$$

Finally, by homogeneity and since the heat kernel h_t is a smooth function, we obtain the wished estimates for this function. $\blacksquare$

Proof of the Lemma 3.1. We have $[\mathcal{L},\varphi]A_R(x,s) = \displaystyle\int_{\mathbb{R}^n}\big(\varphi(x)-\varphi(x-y)\big)A_R(x-y,s)\pi(y)dy$ and we divide our study following the different cases **(a)-(d)**.

For the case **(a)**, where $0 < \alpha \leq \beta < 1/2$ and $0 < \delta < 1/2$, or in the case **(b)** where $0 < \alpha = \beta = \delta < 1/2$, we proceed as follows. We begin with the case $p = +\infty$ and we write:

$$|[\mathcal{L},\varphi]A_R(x,s)| \leq \int_{\mathbb{R}^n}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+2\beta}}|A_R(y,s)|dy + \int_{\mathbb{R}^n}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+2\delta}}|A_R(y,s)|dy \tag{74}$$

Again, it is enough to study one of these two integrals since the other can be treated in a totally similar way. We write:

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+2\beta}}|A_R(y,s)|dy &= \int_{\{|x-y|>R\}}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+2\beta}}|A_R(y,s)|dy + \int_{\{|x-y|\leq R\}}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+2\beta}}|A_R(y,s)|dy\\ &\leq 2\|\varphi\|_{L^\infty}\int_{\{|x-y|>R\}}\frac{|A_R(y,s)|}{|x-y|^{n+2\beta}}dy + \int_{\{|x-y|\leq R\}}\frac{\|\nabla\varphi\|_{L^\infty}|x-y|}{|x-y|^{n+2\beta}}|A_R(y,s)|dy\\ &\leq 2\|\varphi\|_{L^\infty}\|A_R(\cdot,s)\|_{L^\infty}\int_{\{|x-y|>R\}}\frac{1}{|x-y|^{n+2\beta}}dy + CR^{-1}\int_{\{|x-y|\leq R\}}\frac{|A_R(y,s)|}{|x-y|^{n+2\beta-1}}dy\\ &\leq 2C\|\varphi\|_{L^\infty}\|A_R(\cdot,s)\|_{L^\infty}R^{-2\beta} + C\|A_R(\cdot,s)\|_{L^\infty}R^{-2\beta} \leq CR^{-2\beta}\|A_{0,R}\|_{L^\infty}.\end{aligned}$$

Then, with the δ-part in inequality (74) we have

$$\|[\mathcal{L},\varphi]A_R(\cdot,s)\|_{L^\infty} \leq C(R^{-2\beta}+R^{-2\delta})\|A_{0,R}\|_{L^\infty}.$$

The case $p=1$ is very similar. Using inequality (74) we have

$$\int_{\mathbb{R}^n}|[\mathcal{L},\varphi]A_R(x,s)|dx \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+2\beta}}|A_R(y,s)|dydx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+2\delta}}|A_R(y,s)|dydx$$

We only estimate one of the previous integrals.

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+2\beta}}|A_R(y,s)|dydx &\leq C\|\varphi\|_{L^\infty}\int_{\mathbb{R}^n}\int_{\{|x-y|>R\}}\frac{|A_R(y,s)|}{|x-y|^{n+2\beta}}dydx\\ &\quad +R^{-1}\int_{\mathbb{R}^n}\int_{\{|x-y|\leq R\}}\frac{|A_R(y,s)|}{|x-y|^{n+2\beta-1}}dydx\\ &\leq C\|\varphi\|_{L^\infty}\|A_R(\cdot,s)\|_{L^1}R^{-2\beta}+C\|A_R(\cdot,s)\|_{L^1}R^{-2\beta}\leq CR^{-2\beta}\|A_{0,R}\|_{L^1}.\end{aligned}$$

With the other integral, we obtain

$$\|[\mathcal{L},\varphi]A_R(\cdot,s)\|_{L^1} \leq C(R^{-2\beta}+R^{-2\delta})\|A_{0,R}\|_{L^1}.$$

Finally, the case $1<p<+\infty$ is obtained by interpolation. See [8] or [19] for more details about interpolation.

For the remaining cases **(c)** and **(d)** (*i.e.* if $\alpha=\beta=1/2$ and $0<\delta<1/2$ or $\alpha=\beta=\delta=1/2$), the result will be a consequence of the Calderón's commutator inequality (see [8]) and the maximum principle. ■

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