STOCHASTIC COALESCENCE MULTI-FRAGMENTATION PROCESSES

EDUARDO CEPEDA

Abstract. We study infinite systems of particles characterized by their masses. Each pair of particles with masses x and y coalesce at a given rate $K(x,y)$ and a particle with mass x fragmentates into $\theta_1 x, \theta_2 x, \dots$ at a rate given by $F(x)\beta(d\theta)$. We assume that K and F satisfy a sort of Hölder property with index $\lambda \in (0,1]$ and $\alpha \in [0,\infty)$, respectively. We show existence of such infinite particle systems, as strong Markov processes, enjoying a Feller property, with values in ℓ_λ, the set of ordered $[0,\infty)$-valued sequences $(m_i)i \geq 1$ such that $\sum_{i\geq 1} m_i^\lambda < \infty$.

This work relies on the use of a Wasserstein-type distance, which has shown to be particularly well-adapted to coalescence phenomena. It was introduced in previous works on coagulation and coalescence.

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1. Introduction

We consider a possibly infinite system of microscopic particles, the coalescence of two particles of mass x and y gives birth a new one of mass $x+y$, $\{x,y\} \to x+y$ with a rate equal to the coagulation kernel $K(x,y)$. On the other hand, the fragmentation of a particle of mass x gives birth a new set of smaller particles $x \to \{\theta_1 x, \theta_2 x, \dots\}$, where $\theta_i x$ represents the fragments of x, with a rate equal to $F(x)\beta(d\theta)$ and where $F:(0,\infty) \to (0,\infty)$ and β is a positive measure on the set $\Theta = \{\theta = (\theta_i)_{i\geq 1} : 1 > \theta_1 \geq \theta_2 \geq \dots \geq 0\}$. This means that the distribution of the ratios of daughter masses to parent mass is only determined by a function of these ratios (and not by the parent mass). Under this framework we will introduce the processes of Coalescence - Fragmentation, which will be defined through its infinitessimal generator and can be composed by an infinite number of particules but with a finite total mass.

The fragmentation part of the model was first introduced by Bertoin [3] and takes into account an infinite measure β and a mechanism of dislocation with a possibly infinite number of fragments.

The microscopic scale version of this model (which is deterministic) is studied in Cepeda [5]. We believe that a *hydrodynamical limit* result concerning this two settings is possible to obtain in the following way. Denoting by $\mu^n = \frac{1}{n}\sum_{i\geq 1} \delta_{m_i}$ the empirical measure associated to the system composed by $(m_1, m_2, \dots)$, then the Coalescence-Fragmentation process associated $(\mu_t^n)_{t\geq 0}$ converges to the solution to the deterministic equation. For a first result concerning convergence in the case where $F \equiv 0$ see Norris [16, 17] and Cepeda-Fournier [6] for a explicit rate of convergence.

In this paper we are mainly interested in a result of general well-posedness, this means, with the less possible assumptions on K, F, β. In particular, the method in this paper is based on the use of the following Wasserstein-like distance: for $m, \tilde{m}$ two non-negative sequences such that $\sum_{i\geq 1}(m_i^\lambda + \tilde{m}_i^\lambda) < \infty$ and let $Perm(\mathbb{N})$ be the set of all finite permutations of $\mathbb{N}$, we set

$$\delta_\lambda(m, \tilde{m}) = \inf_{\pi, \sigma \in Perm(\mathbb{N})} \sum_{i\geq 1} |m_{\pi(i)}^\lambda - \tilde{m}_{\sigma(i)}^\lambda|.$$

We study the existence and uniqueness of a stochastic process of Coalescence - Fragmentation. We extend the result in Founier [8] concerning only coalescence, we follow the same ideas in this paper and in Fournier-Löcherbach [10], and we construct a stochastic particle system undergoing coalescences and fragmentations. In this case the non-increasing total mass property allows us to consider in particular self-similar fragmentation kernels as defined in [3] and more generally unbounded fragmentation kernels.

We have chosen this model for the fragmentation since it is actually more tractable mathematically, see Bertoin [3, 2] and Haas [11, 12] where the properties of the only fragmentation model are extensively studied. Kolokoltsov [13] shows in the discrete case a hydrodynamical limit result for a different model than ours, namely he introduces a mass exchange Markov process. An extensive study of the methods used by the author are given in the books [15, 14], we refer also to Berestycki [1] concerning a similar result to ours for a version of exchangeable processes. Finally, we refer to Eibeck-Wagner [7] where a different model is studied which is used to approach general nonlinear kinetic equations.

The paper is organized as follows: the stochastic Coalescence - Fragmentation processes are studied in Sections 2, 3 and 4 and in Appendix A we give some technical details which are useful in this case.

2. Notation and Definitions

Let $\mathcal{S}^\downarrow$ the set of non-increasing sequences $m = (m_n)_{n\geq 1}$ with values in $[0, +\infty)$. A state m in $\mathcal{S}^\downarrow$ represents the sequence of the ordered masses of the particles in a particle system. Next, for $\lambda \in (0, 1]$, consider

$$\ell_\lambda = \left\{ m = (m_k)_{k\geq 1} \in \mathcal{S}^\downarrow, \|m\|_\lambda := \sum_{k=1}^\infty m_k^\lambda < \infty \right\}. \tag{2.1}$$

Consider also the sets of finite particle systems, completed for convenience with infinitely many 0-s.

$$\ell_{0+} = \left\{ m = (m_k)_{k\geq 1} \in \mathcal{S}^\downarrow, \inf\{k \geq 1, m_k = 0\} < \infty \right\}.$$

Remark 2.1. *Note that for all $0 < \lambda_1 < \lambda_2$, $\ell_{0+} \subset \ell_{\lambda_1} \subset \ell_{\lambda_2}$. Note also that, since $\|m\|_1 \leq \|m\|_\lambda^{\frac{1}{\lambda}}$ the total mass of $m \in \ell_\lambda$ is always finite.*

Hypothesis 2.2. *We consider a coagulation kernel K bounded on every compact set in $[0, \infty)^2$. There exists $\lambda \in (0, 1]$ such that for all $a > 0$ there exists a constant $\kappa_a > 0$ such that for all x, y, $\tilde{x}$, $\tilde{y} \in (0, a]$,*

$$|K(x, y) - K(\tilde{x}, \tilde{y})| \leq \kappa_a \left[|x^\lambda - \tilde{x}^\lambda| + |y^\lambda - \tilde{y}^\lambda| \right], \tag{2.2}$$

We consider also a fragmentation kernel $F : (0, \infty) \mapsto [0, \infty)$, bounded on every compact set in $[0, \infty)$. There exists $\alpha \in [0, \infty)$ such that for all $a > 0$ there exists a constant $\mu_a > 0$ such that for all x, $\tilde{x} \in (0, a]$,

$$|F(x) - F(\tilde{x})| \leq \mu_a \, |x^\alpha - \tilde{x}^\alpha|. \tag{2.3}$$

We define the set of ratios by

$$\Theta = \{\theta = (\theta_k)_{k\geq 1} : 1 > \theta_1 \geq \theta_2 \geq \ldots \geq 0\}.$$

Hypothesis 2.3 (The β measure.-)**.** *We consider on Θ a measure $\beta(\cdot)$ and assume that it satisfies*

$$\beta\left(\sum_{k\geq 1}\theta_k > 1\right) = 0, \tag{2.4}$$

$$C_\beta^\lambda := \int_\Theta \left[\sum_{k\geq 2}\theta_k^\lambda + (1-\theta_1)^\lambda\right]\beta(d\theta) < \infty, \quad \textit{for some } \lambda \in (0,1]. \tag{2.5}$$

Remark 2.4. *i) The property (2.4) means that there is no gain of mass due to the dislocation of a particle. Nevertheless, it does not exclude a loss of mass due to the dislocation of the particles.*

ii) Note that under (2.4) we have $\sum_{k\geq 1}\theta_k - 1 \leq 0$ β-a.e., and since $\theta_k \in [0,1)$ for all $k \geq 1$, $\theta_k \leq \theta_k^\lambda$, we have

$$\begin{cases} 1-\theta_1^\lambda \leq 1-\theta_1 \leq (1-\theta_1)^\lambda, \ \beta - a.e., \\ \sum_{k\geq 1}\theta_k^\lambda - 1 = \sum_{k\geq 2}\theta_k^\lambda - (1-\theta_1^\lambda) \leq \sum_{k\geq 2}\theta_k^\lambda, \ \beta - a.e. \end{cases} \tag{2.6}$$

implying the following bounds:

$$\begin{cases} \int_\Theta (1-\theta_1)\beta(d\theta) \leq C_\beta^\lambda, \ \int_\Theta \left[\sum_{k\geq 2}\theta_k^\lambda + (1-\theta_1^\lambda)\right]\beta(d\theta) \leq C_\beta^\lambda, \\ \int_\Theta \left(\sum_{k\geq 1}\theta_k^\lambda - 1\right)^+ \beta(d\theta) \leq C_\beta^\lambda. \end{cases} \tag{2.7}$$

We point out that $\int_\Theta \left|\sum_{k\geq 1}\theta_k^\lambda - 1\right|\beta(d\theta) \leq 2C_\beta^\lambda$ but when the term $\sum_{k\geq 1}\theta_k^\lambda - 1$ is negative our calculations can be realized in a simpler way. We will thus use the positive bound given in the last inequality.

We will use the following conventions

$$K(x,0) = 0 \quad \text{for all } \ x \in [0,\infty),$$
$$F(0) = 0.$$

Remark that this convention is also valid, for example, for $K = 1$. Actually, 0 is a symbol used to refer to a particle that does not exist. For $\theta \in \Theta$ and $x \in (0,\infty)$ we will write $\theta \cdot x$ to say that the particle of mass x of the system splits into $\theta_1 x, \theta_2 x, \ldots$.

Consider $m \in \ell_\lambda$, the dynamics of the process is as follows. A pair of particles m_i and m_j coalesce with rate given by $K(m_i, m_j)$ and is described by the map $c_{ij} : \ell_\lambda \to \ell_\lambda$ (see below). A particle m_i fragmentates following the dislocation configuration $\theta \in \Theta$ with rate given by $F(m_i)\beta(d\theta)$ and is described by the map $f_{i\theta} : \ell_\lambda \to \ell_\lambda$, with

$$\begin{array}{rcl} c_{ij}(m) & = & \text{reorder}(m_1, \ldots, m_{i-1}, m_i + m_j, m_{i+1}, \ldots, m_{j-1}, m_{j+1}, \ldots), \\ f_{i\theta}(m) & = & \text{reorder}(m_1, \ldots, m_{i-1}, \theta \cdot m_i, m_{i+1}, \ldots), \end{array} \tag{2.8}$$

the reordering being in the decreasing order.

Distances on $S^{\downarrow}$

We endow $S^{\downarrow}$ with the pointwise convergence topology, which can be metrized by the distance

$$d(m,\tilde{m}) = \sum_{k\geq 1} 2^{-k}|m_k - \tilde{m}_k|. \tag{2.9}$$

Also, for $\lambda \in (0,1]$ and m, $\tilde{m} \in \ell_\lambda$, we set (we recall that from [8, Lemma 3.1.] we have)

$$\delta_\lambda(m,\tilde{m}) = \sum_{k\geq 1} |m_k^\lambda - \tilde{m}_k^\lambda| \tag{2.10}$$

Infinitesimal generator $\mathcal{L}^{\beta}_{K,F}$

Consider some coagulation and fragmentation kernels K and F and a measure β. We define the infinitesimal generator $\mathcal{L}^{\beta}_{K,F}$ for any $\Phi : \ell_\lambda \to \mathbb{R}$ sufficiently regular and for any $m \in \ell_\lambda$ by

$$\mathcal{L}^{\beta}_{K,F}\Phi(m) = \sum_{1\leq i<j<\infty} K(m_i,m_j)\left[\Phi\left(c_{ij}(m)\right) - \Phi(m)\right] + \sum_{i\geq 1} F(m_i)\int_\Theta \left[\Phi\left(f_{i\theta}(m)\right) - \Phi(m)\right]\beta(d\theta). \tag{2.11}$$

3. Results

We define first the finite coalescence - fragmentation process. In order to prove the existence of this process we need to add two properties to the measure β. Namely, the measure of Θ must be finite and the number of fragments at each fragmentation must be bounded:

$$\begin{cases} \beta(\Theta) & < & \infty, \\ \beta(\Theta\setminus\Theta_k) & = & 0 \qquad \text{for some } k\in\mathbb{N}, \end{cases} \tag{3.1}$$

where

$$\Theta_k = \left\{\theta = (\theta_n)_{n\geq 1} \in \Theta : \theta_{k+1} = \theta_{k+2} = \cdots = 0\right\}.$$

We introduce some notation that will be useful when working with finite processes. We consider a mesure β satisfying Hypotheses 2.3., $n \in \mathbb{N}$ and the set $\Theta(n)$ defined by $\Theta(n) = \left\{\theta \in \Theta : \theta_1 \leq 1 - \frac{1}{n}\right\}$, we consider also the projector

$$\begin{array}{rccl} \psi_n : & \Theta & \to & \Theta_n \\ & \theta & \mapsto & \psi_n(\theta) = (\theta_1,\ldots,\theta_n,0,\ldots), \end{array} \tag{3.2}$$

and we put

$$\beta_n = \mathbb{1}_{\theta\in\Theta(n)}\beta\circ\psi_n^{-1}. \tag{3.3}$$

The measure β_n can be seen as the restriction of β to the projection of $\Theta(n)$ onto Θ_n. Note that $\Theta(n) \subset \Theta(n+1)$ and that since we have excluded the degenerated cases $\theta_1 = 1$ we have $\bigcup_n \Theta(n) = \Theta$.

Proposition 3.1 (Finite Coalescence - Fragmentation processes)**.** *Consider $\lambda \in (0,1]$, $\alpha \geq 0$ and $m \in \ell_{0+}$. Assume that the coagulation kernel K, the fragmentation kernel F and a measure β satisfy Hypotheses 2.2. Furthermore, suppose that β satisfies (3.1).*

Then, there exists a unique (in law) strong Markov process $(M(m,t))_{t\geq 0}$ starting at $M(m,0) = m$ and with infinitesimal generator $\mathcal{L}^{\beta}_{K,F}$.

We wish to extend this process to the case where the initial condition consists of infinitely many particles and for more general fragmentation measures β. For this, we will build a particular sequence of finite coalescence - fragmentation processes, the result will be obtained by passing to the limit.

Lemma 3.2 (Definition.- The finite process $M^n(m,t)$)**.** *Consider $\lambda \in (0,1]$, $\alpha \geq 0$ and $m \in \ell_{0+}$. Assume that the coagulation kernel K, the fragmentation kernel F and the measure β satisfy Hypotheses 2.2. Furthermore, recall β_n as defined by (3.3).*

Then, there exists a unique (in law) strong Markov process $(M^n(m,t))_{t\geq 0}$ starting at m and with infinitesimal generator $\mathcal{L}_{K,F}^{\beta_n}$.

This lemma is straightforward, it suffices to note that β_n satisfies (3.1) and to use Proposition 3.1. Indeed, recall (2.7), for $n \geq 1$

$$\beta_n(\Theta) = \int_\Theta \mathbb{1}_{\{1-[\psi_n(\theta)]_1 \geq \frac{1}{n}\}}\, \beta(d\theta) \leq n \int_\Theta (1-\theta_1)\, \beta(d\theta) \leq n\, C_\beta^\lambda < \infty.$$

We have chosen an explicit sequence of measure $(\beta_n)_{n\geq 1}$ because it will be easier to manipulate when coupling two coalescence-fragmentation processes. Nevertheless, more generally, taking any sequence of measures β_n satisfying (3.1) and converging towards β in a suitable sense as n tends to infinity should provide the same result.

Our main result concerning stochastic Coalescence-Fragmentation processes is the following.

Theorem 3.3. *Consider $\lambda \in (0,1]$, $\alpha \geq 0$. Assume that the coagulation K and the fragmentation F kernels and that a measure β satisfy Hypotheses 2.2. Endow ℓ_λ with the distance δ_λ.*

i) For any $m \in \ell_\lambda$, there exists a (necessarily unique in law) strong Markov process $(M(m,t))_{t\geq 0} \in \mathbb{D}([0,\infty), \ell_\lambda)$ satisfying the following property.

For any sequence $m^n \in \ell_{0+}$ such that $\lim_{n\to\infty} \delta_\lambda(m^n, m) = 0$, the sequence $(M^n(m^n,t))_{t\geq 0}$ defined in Lemma 3.2, converges in law, in $\mathbb{D}([0,\infty), \ell_\lambda)$, to $(M(m,t))_{t\geq 0}$.

ii) The obtained process is Feller in the sense that for all $t \geq 0$, the map $m \mapsto Law\,(M(m,t))$ is continuous from ℓ_λ into $\mathcal{P}(\ell_\lambda)$ (endowed with the distance δ_λ).

iii) Recall the expression of d (2.9). For all bounded $\Phi : \ell_\lambda \to \mathbb{R}$ satisfying $|\Phi(m) - \Phi(\tilde{m})| \leq a\, d(m, \tilde{m})$ for some $a > 0$, the process

$$\Phi\left(M(m,t)\right) - \Phi\left(m\right) - \int_0^t \mathcal{L}_{K,F}^{\beta}\left(M(m,s)\right) ds$$

is a local martingale.

This result extends those of Fournier [8] concerning only coalescence and Bertoin [3, 2] concerning only fragmentation. We point out that in [3] is not assumed $C_\beta^\lambda < \infty$ but only $\int_\Theta (1-\theta_1)\beta(d\theta) < \infty$. However, we believe that in presence of coalescence our hypotheses on β are optimal. We refer to [4] for an extensive study of coagulation and fragmentation systems.

Theorem 3.3. will be proved in two steps, the first step consists in proving existence and uniqueness of the Finite Coalescence-Fragmentation process, finite in the sense that it is composed by a finite number of particles for all $t \geq 0$. Next, we will use a sequence of finite processes to build a process, as its limit, where the system is composed by an infinite number of particles. The construction of such processes uses a Poissonian representation which is introduced in the next section.

4. A Poisson-driven S.D.E.

We now introduce a representation of the stochastic processes of coagulation - fragmentation in terms of Poisson measures, in order to couple two of these processes with different initial data.

Definition 4.1. *Assume that a coagulation kernel K, a fragmentation kernel F and a measure β satisfy Hypotheses 2.2.*

a) For the coagulation, we consider a Poisson measure $N(dt, d(i,j), dz)$ on $[0,\infty)\times\{(i,j)\in\mathbb{N}^2, i<j\}\times[0,\infty)$ with intensity measure $dt\left[\sum_{k<l}\delta_{(k,l)}(d(i,j))\right] dz$, and denote by $(\mathcal{F}_t)_{t\geq 0}$ the associated canonical filtration.

b) For the fragmentation, we consider $M(dt, di, d\theta, dz)$ a Poisson measure on $[0,\infty)\times\mathbb{N}\times\Theta\times[0,\infty)$ with intensity measure $dt\left(\sum_{k\geq 1}\delta_k(di)\right)\beta(d\theta)\, dz$, and denote by $(\mathcal{G}_t)_{t\geq 0}$ the associated canonical filtration. M is independent of N.

Finally, we consider $m\in\ell_\lambda$. A càdlàg $(\mathcal{H}_t)_{t\geq 0}=(\sigma(\mathcal{F}_t,\mathcal{G}_t))_{t\geq 0}$-adapted process $(M(m,t))_{t\geq 0}$ is said to be a solution to $SDE(K,F,m,N,M)$ if it belongs a.s. to $\mathbb{D}([0,\infty),\ell_\lambda)$ and if for all $t\geq 0$, a.s.

$$\begin{aligned} M(m,t) \quad = \quad & m+\int_0^t\int_{i<j}\int_0^\infty \left[c_{ij}\left(M(m,s-)\right)-M(m,s-)\right]\mathbb{1}_{\{z\leq K(M_i(m,s-),M_j(m,s-))\}} \\ & \qquad\qquad\qquad\qquad\qquad\qquad N(dt,d(i,j),dz) \\ & +\int_0^t\int_i\int_\Theta\int_0^\infty \left[f_{i\theta}\left(M(m,s-)\right)-M(m,s-)\right]\mathbb{1}_{\{z\leq F(M_i(m,s-))\}} \\ & \qquad\qquad\qquad\qquad\qquad\qquad M(dt,di,d\theta,dz). \end{aligned} \tag{4.1}$$

Remark that due to the independence of the Poisson measures only a coagulation or a fragmentation mechanism occurs at each instant t.

We begin by checking that the integrals in (4.1) always make sense.

Lemma 4.2. *Let $\lambda\in(0,1]$ and $\alpha\geq 0$, consider K, F, β and the Poisson measures N and M as in Definition 4.1. For any $(\mathcal{H}_t)_{t\geq 0}$-adapted process $(M(t))_{t\geq 0}$ belonging a.s. to $\mathbb{D}([0,\infty),\ell_\lambda)$, a.s.*

$$\begin{aligned} I_1 \quad &= \quad \int_0^t\int_{i<j}\int_0^\infty \left[c_{ij}\left(M(s-)\right)-M(s-)\right]\mathbb{1}_{\{z\leq K(M_i(s-),M_j(s-))\}}N(dt,d(i,j),dz), \\ I_2 \quad &= \quad \int_0^t\int_i\int_\Theta\int_0^\infty \left[f_{i\theta}\left(M(s-)\right)-M(s-)\right]\mathbb{1}_{\{z\leq F(M_i(s-))\}}M(dt,di,d\theta,dz), \end{aligned}$$

are well-defined and finite for all $t\geq 0$.

Proof. The processes in the integral being càdlàg and adapted, it suffices to check the compensators are a.s. finite. We have to show that a.s., for all $k\geq 1$, all $t\geq 0$,

$$\begin{aligned} C_k(t) \quad = \quad & \int_0^t ds\sum_{i<j}K(M_i(s),M_j(s))\left|[c_{ij}(M(s))]_k-M_k(s)\right| \\ & +\int_0^t ds\int_\Theta\beta(d\theta)\sum_{i\geq 1}F(M_i(s))\left|[f_{i\theta}(M(s))]_k-M_k(s)\right|<\infty. \end{aligned}$$

Note first that for all $s\in[0,t]$, $\sup_i M_i(s)\leq\sup_{[0,t]}\|M(s)\|_1\leq\sup_{[0,t]}\|M(s)\|_\lambda^{1/\lambda}=: a_t<\infty$ a.s. since M belongs a.s. to $\mathbb{D}([0,\infty),\ell_\lambda)$. Next, let

$$\overline{K}_t=\sup_{(x,y)\in[0,a_t]^2}K(x,y) \qquad\text{and}\qquad \overline{F}_t=\sup_{x\in[0,a_t]}F(x), \tag{4.2}$$

which are $a.s.$ finite since K and F are bounded on every compact in $[0,\infty)^2$ and $[0,\infty)$, respectively. Then using (A.15) and (A.17) with (2.6) and (2.7), we write:

$$\begin{aligned}
\sum_{k\geq 1} 2^{-k} C_k(t) &= \int_0^t ds \sum_{i<j} K(M_i(s), M_j(s))\, d\left(c_{ij}(M(s)), M(s)\right) \\
&\quad + \int_0^t ds \int_\Theta \beta(d\theta) \sum_{i\geq 1} F(M_i(s))\, d\left(f_{i\theta}(M(s)), M(s)\right) \\
&\leq \overline{K}_t \int_0^t ds \sum_{i<j} \frac{3}{2} 2^{-i} M_j(s) + C_\beta^\lambda \overline{F}_t \int_0^t ds \sum_{i\geq 1} 2^{-i} M_i(s) \\
&\leq \left(\frac{3}{2}\overline{K}_t + C_\beta^\lambda \overline{F}_t\right) \int_0^t \|M(s)\|_1 ds \\
&\leq t \left(\frac{3}{2}\overline{K}_t + C_\beta^\lambda \overline{F}_t\right) \sup_{[0,t]} \|M(s)\|_\lambda^{1/\lambda} \;<\; \infty.
\end{aligned}$$

□

4.1. **Existence and uniqueness for *SDE*: finite case.** The aim of this paragraph is to prove Proposition 3.1, this proposition is a consequence of Proposition 4.3. bellow. We will first prove existence and uniqueness of the Finite Coalescence - Fragmentation processes satisfiying (SDE) and then some fundamental inequalities.

Proposition 4.3. *Let $m \in \ell_{0+}$. Consider the coagulation kernel K, the fragmentation kernel F, the measure β and the Poisson measures N and M as in Definition 4.1, suppose furthermore that β satisfies (3.1).*

Then there exists a unique process $(M(m,t))_{t\geq 0}$ which solves $SDE(K,F,m,N,M)$. This process is a finite Coalescence-Fragmentation process in the sense of Proposition 3.1.

4.1.1. *A Gronwall type inequality.* We will also check a fundamental inequality, which shows that the distance between two coagulation-fragmentation processes introduced in Proposition 4.3. cannot increase excessively while their moments of order λ remain finite.

Proposition 4.4. *Let $\lambda \in (0,1]$, $\alpha \geq 0$ and $m, \tilde{m} \in \ell_{0+}$. Consider K, F, β and the Poisson measures N and M as in Definition 4.1, we furthermore suppose that β satisfies (3.1). Consider the unique solutions $M(m,t)$ and $M(\tilde{m},t)$ to $SDE(K,F,m,N,M)$ and $SDE(K,F,\tilde{m},N,M)$ constructed in Proposition 4.3. and recall C_β^λ (2.5).*

i) The map $t \mapsto \|M(m,t)\|_1$ is a.s. non-increasing. Futhermore, for all $t \geq 0$

$$\mathbb{E}\left[\sup_{s\in[0,t]} \|M(m,s)\|_\lambda\right] \leq \|m\|_\lambda\, e^{\overline{F}_m C_\beta^\lambda\, t},$$

where $\overline{F}_m = \sup_{[0,\|m\|_1]} F(x)$.

ii) We define, for all $x > 0$, the stopping time $\tau(m,x) = \inf\{t \geq 0, \|M(m,t)\|_\lambda \geq x\}$. Then for all $t \geq 0$ and all $x > 0$,

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau(m,x)\wedge\tau(\tilde{m},x)]} \delta_\lambda\left(M(m,s), M(\tilde{m},s)\right)\right] \leq \delta_\lambda\left(m,\tilde{m}\right) e^{C(x+1)\,t}.$$

where C is a positive constant depending on K, F, C_β^λ, $\|m\|_1$ and $\|\tilde{m}\|_1$.

This proposition will be useful to construct a process in the sense of Definition 4.1. as the limit of a sequence of approximations. It will provide some important uniform bounds not depending on the approximations but only on the initial conditions and C_β^λ.

4.1.2. *Proofs.* In this section we give the proves to propositions 4.3., 3.1. and 4.4.

Proof of Proposition 4.3. This proposition will be proved using that in such a system the number of particles remains finite, we will then use that the total rate of jumps of the system is bounded by the number of particles to conclude.

Lemma 4.5. *Let $m \in \ell_{0+}$, consider K, F, β and the Poisson measures N and M as in Definition 4.1. and assume that β satisfies (3.1). Assume that there exists $(M(m,t))_{t\geq 0}$ solution to $SDE(K,F,m,N,M)$.*

i) The number of particles in the system remains a.s. bounded,

$$\sup_{s\in[0,t]} N_s < \infty, \text{ a.s. for all } t \geq 0,$$

where $N_t = card\{M_i(m,t) : M_i(m,t) > 0\} = \sum_{i\geq 1} \mathbb{1}_{\{M_i(m,t)>0\}}$.

ii) The coalescence and fragmentation jump rates of the process $(M(m,t))_{t\geq 0}$ are a.s. bounded, this is

$$\sup_{s\in[0,t]} (\rho_c(s) + \rho_f(s)) < \infty, \text{ a.s. for all } t \geq 0,$$

where $\rho_c(t) := \sum_{i<j} K(M_i(m,t), M_j(m,t))$ and $\rho_f(t) := \beta(\Theta)\sum_{i\geq 1} F(M_i(m,t))$.

Proof. First, denoting $\overline{K}_m := \sup_{[0,\|m\|_1]^2} K(x,y)$ and $\overline{F}_m := \sup_{[0,\|m\|_1]} F(x)$, note that we have $\rho_c(0) \leq \overline{K}_m N_0^2$ and $\rho_f(0) \leq \beta(\Theta)\overline{F}_m N_0$, which shows that the initial total jump intensity of the system is finite and that the first jump time is strictly positive $T_1 > 0$. We can thus prove by recurrence that there exists a sequence $0 < T_1 < \ldots < T_j < \ldots < T_\infty$ of jumping times with $T_\infty = \lim_{j\to\infty} T_j$. We now prove that $T_\infty = \infty$.

Let $L^f(t) := card\{j \geq 1 : T_j \leq t \text{ and } T_j \text{ is a jump of } M\}$ be the number of fragmentations in the system until the instant $t \geq 0$. Recall that the measure β satisfies (3.1), since k is the maximal number of fragments, it is easy to see that

$$N_t \leq N_0 + (k-1)L^f(t) < \infty \ a.s., \text{ for all}\, t < T_\infty.$$

Applying now (2.11) with $\Psi(m) = \sum_{n\geq 1} m_n$ and since that $\Psi(c_{ij}(m)) - \Psi(m) = 0$ and $\Psi(f_{i\theta}(m)) - \Psi(m) = m_i\left(\sum_{i=1}^k \theta_i - 1\right) \leq 0$, $\beta - a.e.$, we obtain

$$\sup_{s\in[0,t]} \|M(m,s)\|_1 \leq \|m\|_1, a.s., \text{ for all } t < T_\infty,$$

which implies, $a.s.$ for all $t < T_\infty$,

$$\left\{\begin{array}{lll} \rho_c(t) & \leq & \overline{K}_m N_{t-}^2, \\ \rho_f(t) & \leq & \beta(\Theta)\overline{F}_m N_{t-}. \end{array}\right. \tag{4.3}$$

Next, define $\Phi(m) = \sum_{n\geq 1} \mathbb{1}_{\{m_n>0\}}$, recall (2.11) and use $\Phi(c_{ij}(m)) - \Phi(m) \leq 0$, to obtain

$$\begin{aligned}
\mathcal{L}^{\beta}_{K,F}\Phi(m) &\leq \sum_{i\geq 1}\int_{\Theta} F(m_i)\left[\Phi\left(f_{i\theta}(m)\right) - \Phi(m)\right]\beta(d\theta) \\
&\leq \overline{F}_m \sum_{i\geq 1}\int_{\Theta}\left[\sum_{n\geq 1}\mathbb{1}_{\{\theta_n m_i>0\}} - \mathbb{1}_{\{m_i>0\}}\right]\beta(d\theta) \\
&\leq (k-1)\,\overline{F}_m\,\beta(\Theta)\Phi(m),
\end{aligned}$$

we used $\theta_j m_i = 0$ for all $j \geq k+1$.

Hence, we have for all $t \geq 0$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge T_\infty)} N_s\right] &\leq N_0 + (k-1)\,\overline{F}_m\,\beta(\Theta)\mathbb{E}\left[\int_0^{t\wedge T_\infty} N_{s-}ds\right] \\
&\leq N_0 + (k-1)\,\overline{F}_m\,\beta(\Theta)\int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge T_\infty)} N_u\right]du.
\end{aligned}$$

We use the Gronwall Lemma to obtain

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge T_\infty)} N_s\right] \leq N_0\, e^{(k-1)\,\overline{F}_m\,\beta(\Theta)t},$$

for all $t \geq 0$. We thus deduce,

$$\sup_{s\in[0,t\wedge T_\infty)} N_s < \infty,\ a.s., \tag{4.4}$$

for all $t \geq 0$.

Suppose now that $T_\infty < \infty$, then from (4.4) we deduce that $\sup_{t\in[0,T_\infty)} N_t < \infty,\ a.s.$. which means that, using (4.3), $\sup_{t\in[0,T_\infty)}(\rho_c(t)+\rho_f(t)) < \infty,\ a.s.$ This is in contradiction with $T_\infty < \infty$ since the total jump intensity necessarily explodes to infinity on T_∞ when $T_\infty < \infty$.

We deduce that,

$$\mathbb{E}\left[\sup_{s\in[0,t]} N_s\right] \leq N_0\, e^{(k-1)\,\overline{F}_m\,\beta(\Theta)t},$$

for all $t \geq 0$, and *i)* readily follows. Finally, *ii)* follows easily from *i)* and (4.3).

This ends the proof of Lemma 4.5. □

From Lemma 4.5. we deduce that the total rate of jumps of the system is uniformly bounded. Thus, pathwise existence and uniqueness holds for $(M(m,t))_{t\geq 0}$ solution to $SDE(K,F,m,N,M)$.

This ends the proof of Proposition 4.3. □

Proof of Proposition 3.1. Let $\lambda \in (0,1]$, $\alpha \geq 0$ and $m \in \ell_{0+}$, and consider K, F, β and the Poisson measures N and M as in Proposition 3.1.

Consider the process $(M(m,t))_{t\geq 0}$, the unique solution to $SDE(K,F,m,N,M)$ built in Proposition 4.3. The system $(M(m,t))_{t\geq 0}$ is a strong Markov process in continuous time with infinitesimal generator $\mathcal{L}^{\beta}_{K,F}$ and Proposition 3.1. follows. □

Proof of Proposition 4.4. Let $\lambda \in (0,1]$, $\alpha \geq 0$ and $m \in \ell_{0+}$, and consider $(M(m,t))_{t\geq 0}$ the solution to $SDE(K,F,m,N,M)$ constructed in Proposition 4.3. We begin studying the behavior of the moments of this solution.

First, we will see that under our assumptions the total mass $\|\cdot\|_1$ does $a.s.$ not increase in time. This property is fundamental in this approach since that we will use the bound $\sup_{[0,\|M(m,0)\|_1]} F(x)$, which is finite whenever $\|M(m,0)\|_\lambda$ is. This will allows us to bound lower moments of $M(m,t)$ for $t \geq 0$.

Next, we will prove that the λ-moment remains finite in time. Finally, we will show that the distance δ_λ between two solutions to (4.1) are bounded in time while theirs λ-moments remain finite.

We point out that in these paragraphs we will use more general estimates for $m \in \ell_\lambda$ and β satisfying Hypotheses 2.3. and not necessarily (3.1). This will provide uniform bound when dealing with finite processes.

Moments Estimates.- The aim of this paragraph is to prove *i)*.

The solution to $SDE(K,F,m,N,M)$ will be written $M(t) := M(m,t)$ for simplicity. From Lemma 4.5. *i)*, we know that the number of particles in the system is $a.s.$ finite and thus the following sums are obviously well-defined.

First, from (4.1) we have for $k \geq 1$,

$$\begin{aligned} M_k(t) \quad = \quad & M_k(0) + \int_0^t \int_{i<j} \int_0^\infty \left[[c_{ij}\left(M(s-)\right)]_k - M_k(s-) \right] \mathbb{1}_{\{z \leq K(M_i(s-),M_j(s-))\}} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad N(dt,d(i,j),dz) \\ & + \int_0^t \int_i \int_\Theta \int_0^\infty \left[[f_{i\theta}\left(M(s-)\right)]_k - M(s-)_k \right] \mathbb{1}_{\{z \leq F(M_i(s-))\}} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad M(dt,di,d\theta,dz), \end{aligned} \tag{4.5}$$

and summing on k, we deduce

$$\begin{aligned} \|M(t)\|_1 \quad = \quad & \|m\|_1 + \int_0^t \int_{i<j} \int_0^\infty \left[\|c_{ij}\left(M(s-)\right)\|_1 - \|M(s-)\|_1 \right] \mathbb{1}_{\{z \leq K(M_i(s-),M_j(s-))\}} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad N(dt,d(i,j),dz) \\ & + \int_0^t \int_i \int_0^\infty \left[\|f_{i\theta}\left(M(s-)\right)\|_1 - \|M(s-)\|_1 \right] \mathbb{1}_{\{z \leq F(M_i(s-))\}} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad M(dt,di,d\theta,dz). \end{aligned} \tag{4.6}$$

Note that, clearly $\|c_{ij}\left(m\right)\|_1 = \|m\|_1$ and $\|f_{i\theta}\left(m\right)\|_1 = \|m\|_1 + m_i \left(\sum_{k\geq 1} \theta_k - 1\right) \leq \|m\|_1$ for all $m \in \ell_\lambda$, since $\sum_{k\geq 1} \theta_k \leq 1$ β-a.e. Then,

$$\sup_{[0,t]} \|M(s)\|_1 \leq \|m\|_1, \; a.s. \; \forall t \geq 0.$$

This implies for all $s \in [0,t]$, $\sup_i M_i(s) \leq \sup_{[0,t]} \|M(s)\|_1 \leq \|m\|_1$ $a.s.$ We set

$$\overline{K}_m = \sup_{(x,y)\in[0,\|m\|_1]^2} K(x,y) \qquad \text{and} \qquad \overline{F}_m = \sup_{x\in[0,\|m\|_1]} F(x) \tag{4.7}$$

which are finite since K and F are bounded on every compact in $[0,\infty)^2$ and $[0,\infty)$ respectively.

In the same way, from (4.1) for $\lambda \in (0,1)$ we have for $k \geq 1$,

$$\begin{aligned}[M_k(t)]^\lambda &= [M_k(0)]^\lambda + \int_0^t \int_{i<j} \int_0^\infty \left[[c_{ij}(M(s-))]_k^\lambda - [M_k(s-)]^\lambda \right] \mathbb{1}_{\{z \leq K(M_i(s-), M_j(s-))\}} \\ &\qquad\qquad N(dt, d(i,j), dz) \\ &\quad + \int_0^t \int_i \int_\Theta \int_0^\infty \left[[f_{i\theta}(M(s-))]_k^\lambda - [M(s-)]_k^\lambda \right] \mathbb{1}_{\{z \leq F(M_i(s-))\}} \\ &\qquad\qquad M(dt, di, d\theta, dz),\end{aligned}$$

and summing on k, we deduce

$$\begin{aligned}\|M(t)\|_\lambda &= \|m\|_\lambda + \int_0^t \int_{i<j} \int_0^\infty \left[\|c_{ij}(M(s-))\|_\lambda - \|M(s-)\|_\lambda \right] \mathbb{1}_{\{z \leq K(M_i(s-), M_j(s-))\}} \\ &\qquad\qquad N(dt, d(i,j), dz) \\ &\quad + \int_0^t \int_i \int_0^\infty \left[\|f_{i\theta}(M(s-))\|_\lambda - \|M(s-)\|_\lambda \right] \mathbb{1}_{\{z \leq F(M_i(s-))\}} \\ &\qquad\qquad M(dt, di, d\theta, dz). \end{aligned} \tag{4.8}$$

We take the expectation, use (A.4) and (A.5) with (2.7) and (4.7), to obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{s \in [0,t]} \|M(s)\|_\lambda \right] &\leq \|m\|_\lambda + C_\beta^\lambda \int_0^t \mathbb{E}\left[\sum_{i \geq 1} F(M_i(s)) M_i^\lambda(s) \right] ds \\ &\leq \|m\|_\lambda + \overline{F}_m C_\beta^\lambda \int_0^t \mathbb{E}\left[\|M(s)\|_\lambda \right] ds.\end{aligned}$$

We conclude using the Gronwall Lemma.

Bound for δ_λ.- The aim of this paragraph is to prove *ii)*. For this, we consider for $m, \tilde{m} \in \ell_\lambda$ some solutions to $SDE(K, F, m, N, M)$ and $SDE(K, F, \tilde{m}, N, M)$ which will be written $M(t) := M(m,t)$ and $\tilde{M}(t) := M(\tilde{m}, t)$ for simplicity. Since M and $\tilde{M}$ solve (4.1) with the same Poisson measures N and M, and since the numbers of particles in the systems are $a.s.$ finite, we have

$$\delta_\lambda(M(t), \tilde{M}(t)) = \delta_\lambda(m, \tilde{m}) + A_t^c + B_t^c + C_t^c + A_t^f + B_t^f + C_t^f, \tag{4.9}$$

where

$$\begin{aligned}A_t^c &= \int_0^t \int_{i<j} \int_0^\infty \left\{ \delta_\lambda\left(c_{ij}(M(s-)), c_{ij}(\tilde{M}(s-)) \right) - \delta_\lambda\left(M(s-), \tilde{M}(s-) \right) \right\} \\ &\qquad\qquad \mathbb{1}_{\left\{ z \leq K(M_i(s-), M_j(s-)) \wedge K\left(\tilde{M}_i(s-), \tilde{M}_j(s-)\right) \right\}} N(ds, d(i,j), dz),\end{aligned}$$

$$\begin{aligned}B_t^c &= \int_0^t \int_{i<j} \int_0^\infty \left\{ \delta_\lambda\left(c_{ij}(M(s-)), \tilde{M}(s-) \right) - \delta_\lambda\left(M(s-), \tilde{M}(s-) \right) \right\} \\ &\qquad\qquad \mathbb{1}_{\left\{ K\left(\tilde{M}_i(s-), \tilde{M}_j(s-)\right) \leq z \leq K(M_i(s-), M_j(s-)) \right\}} N(ds, d(i,j), dz),\end{aligned}$$

$$\begin{aligned}C_t^c &= \int_0^t \int_{i<j} \int_0^\infty \left\{ \delta_\lambda\left(M(s-), c_{ij}(\tilde{M}(s-)) \right) - \delta_\lambda\left(M(s-), \tilde{M}(s-) \right) \right\} \\ &\qquad\qquad \mathbb{1}_{\left\{ K(M_i(s-), M_j(s-)) \leq z \leq K\left(\tilde{M}_i(s-), \tilde{M}_j(s-)\right) \right\}} N(ds, d(i,j), dz),\end{aligned}$$

$$\begin{aligned}
A_t^f &= \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda\left(f_{i\theta}(M(s-)), f_{i\theta}(\tilde{M}(s-))\right) - \delta_\lambda\left(M(s-), \tilde{M}(s-)\right)\right\} \\
&\qquad\qquad\qquad\qquad \mathbb{1}_{\left\{z\leq F(M_i(s-))\wedge F\left(\tilde{M}_i(s-)\right)\right\}} M(ds, di, d\theta, dz), \\
B_t^f &= \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda\left(f_{i\theta}(M(s-)), \tilde{M}(s-)\right) - \delta_\lambda\left(M(s-), \tilde{M}(s-)\right)\right\} \\
&\qquad\qquad\qquad\qquad \mathbb{1}_{\left\{F\left(\tilde{M}_i(s-)\right)\leq z\leq F(M_i(s-))\right\}} M(ds, di, d\theta, dz), \\
C_t^f &= \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda\left(M(s-), f_{i\theta}(\tilde{M}(s-))\right) - \delta_\lambda\left(M(s-), \tilde{M}(s-)\right)\right\} \\
&\qquad\qquad\qquad\qquad \mathbb{1}_{\left\{F(M_i(s-))\leq z\leq F\left(\tilde{M}_i(s-)\right)\right\}} M(ds, di, d\theta, dz).
\end{aligned}$$

Note also that

(4.10)
$$\left|\delta_\lambda\left(c_{ij}(M(s-)), \tilde{M}(s-)\right) - \delta_\lambda\left(M(s-), \tilde{M}(s-)\right)\right| \leq \delta_\lambda\left(c_{ij}(M(s-)), M(s-)\right)$$

(4.11)
$$\left|\delta_\lambda\left(f_{i\theta}(M(s-)), \tilde{M}(s-)\right) - \delta_\lambda\left(M(s-), \tilde{M}(s-)\right)\right| \leq \delta_\lambda\left(f_{i\theta}(M(s-)), M(s-)\right)$$

We now search for an upper bound to the expression in (4.9). We define, for all $x > 0$, the stopping time $\tau(m, x) := \inf\{t \geq 0; \|M(m, t)\|_\lambda \geq x\}$. We set $\tau_x = \tau(m, x) \wedge \tau(\tilde{m}, x)$.

Furthermore, since for all $s \in [0, t]$, $\sup_i M_i(s) \leq \sup_{[0,t]} \|M(s)\|_1 \leq \|m\|_1 := a_m$ *a.s*, equivalently for $\tilde{M}$, we put $a_{\tilde{m}} = \|\tilde{m}\|_1$. For $a := a_m \vee a_{\tilde{m}}$ we set κ_a and μ_a the constants for which the kernels K and F satisfy (2.2) and (2.3). Finally, we set $\overline{F}_m$ as in (4.7).

Term A_t^c: using (A.8) we deduce that this term is non-positive, we bound it by 0.

Term B_t^c: we take the expectation, use (4.10), (A.6) and (2.2), to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]} B_s^c\right] &\leq \mathbb{E}\left[\int_0^{t\wedge\tau_x} \sum_{i<j} 2M_j^\lambda(s)\left|K\left(M_i(s), M_j(s)\right) - K\left(\tilde{M}_i(s), \tilde{M}_j(s)\right)\right| ds\right] \\
&\leq 2\kappa_a \mathbb{E}\left[\int_0^{t\wedge\tau_x} \sum_{i<j} M_j^\lambda(s)\Big(\left|M_i^\lambda(s) - \tilde{M}_i^\lambda(s)\right| + \left|M_j^\lambda(s) - \tilde{M}_j^\lambda(s)\right|\Big) ds\right] \\
&\leq 2\kappa_a \mathbb{E}\left[\int_0^{t\wedge\tau_x} \sum_{i\geq 1} \left|M_i^\lambda(s) - \tilde{M}_i^\lambda(s)\right| \sum_{j\geq i+1} M_j^\lambda(s) ds\right] \\
&\quad +2\kappa_a \mathbb{E}\left[\int_0^{t\wedge\tau_x} \sum_{j\geq 2} \left|M_j^\lambda(s) - \tilde{M}_j^\lambda(s)\right| \sum_{i=1}^{j-1} M_i^\lambda(s) ds\right] \\
&\leq 4\kappa_a \mathbb{E}\left[\int_0^{t\wedge\tau_x} \|M(s)\|_\lambda \, \delta_\lambda\left(M(s), \tilde{M}(s)\right) ds\right] \\
&\leq 4\kappa_a\, x \int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_x]} \delta_\lambda\left(M(u), \tilde{M}(u)\right)\right] ds, \qquad (4.12)
\end{aligned}$$

we used that for $m \in \ell_\lambda$, $\sum_{i=1}^{j-1} m_j^\lambda \leq \sum_{i=1}^{j-1} m_i^\lambda \leq \|m\|_\lambda$.

Term C_t^c: it is treated exactly as B_t^c.

Term A_t^f: We take the expectation, and use (A.9) together with (2.7), to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]} A_s^f\right] &\leq C_\beta^\lambda \mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{i\geq 1}\Big(F(M_i(s))\wedge F(\tilde{M}_i(s))\Big)\left|M_i^\lambda(s)-\tilde{M}_i^\lambda(s)\right|\right] ds\\
&\leq \overline{F}_m\, C_\beta^\lambda \mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{i\geq 1}\left|M_i^\lambda(s)-\tilde{M}_i^\lambda(s)\right|\right] ds\\
&\leq \overline{F}_m\, C_\beta^\lambda \int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_x]}\delta_\lambda\Big(M(u),\tilde{M}(u)\Big)\right] ds. \qquad (4.13)
\end{aligned}$$

Term B_t^f: we take the expectation and use (2.3) (recall $a := a_m \vee a_{\tilde{m}}$), (4.11), (A.7) together with (2.7), (A.3) and finally Proposition 4.4. *ii)*, to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]} B_s^f\right] &\leq C_\beta^\lambda \mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{i\geq 1}\left|F(M_i(s))-F(\tilde{M}_i(s))\right| M_i^\lambda(s)\right] ds\\
&\leq \mu_a\, C_\beta^\lambda \mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{i\geq 1}\left|M_i(s)^\alpha-\tilde{M}_i(s)^\alpha\right|\Big(M_i^\lambda(s)+\tilde{M}_i^\lambda(s)\Big)\right] ds\\
&\leq \mu_a\, C_\beta^\lambda\, C\, \mathbb{E}\left[\int_0^{t\wedge\tau_x}\Big(\|M(s)\|_1^\alpha+\|\tilde{M}(s)\|_1^\alpha\Big)\times\sum_{i\geq 1}\left|M_i^\lambda(s)-\tilde{M}_i^\lambda(s)\right|\right] ds\\
&\leq 2\mu_a\, C_\beta^\lambda\, C\,(\|m\|_1^\alpha\vee\|\tilde{m}\|_1^\alpha)\times\int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_x]}\delta_\lambda\Big(M(u),\tilde{M}(u)\Big)\right] ds. \qquad (4.14)
\end{aligned}$$

Term C_t^f: it is treated exactly as B_t^f.

Conclusion.- we take the expectation on (4.9) and gather (4.12), (4.13) and (4.14) to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]}\delta_\lambda\Big(M(s),\tilde{M}(s)\Big)\right] &\leq \delta_\lambda(m,\tilde{m})\\
&\quad + \left[8\kappa_a\, x + 4\mu_a\, C_\beta^\lambda\, C\,(\|m\|_1^\alpha\vee\|\tilde{m}\|_1^\alpha) + \overline{F}_m C_\beta^\lambda\right]\\
&\quad \times\int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_x]}\delta_\lambda\Big(M(u),\tilde{M}(u)\Big)\right] ds. \qquad (4.15)
\end{aligned}$$

We conclude using the Gronwall Lemma:

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]}\delta_\lambda\Big(M(s),\tilde{M}(s)\Big)\right] &\leq \delta_\lambda(m,\tilde{m})\times e^{C\,(x\vee 1\vee\|m\|_1^\alpha\vee\|\tilde{m}\|_1^\alpha)\,t}\\
&\leq \delta_\lambda(m,\tilde{m})\; e^{C\,(x+1)\,t}.
\end{aligned}$$

Where C is a positive constant depending on λ, α, κ_a, μ_a, K, F, C_β^λ, $\|m\|_1$ and $\|\tilde{m}\|_1$.

This ends the proof of Proposition 4.4. $\square$

4.2. **Existence for *SDE*: general case.** We may now prove existence for (SDE). For this, we will build a sequence of coupled finite Coalescence-Fragmentation process which will be proved to be a Cauchy sequence in $\mathbb{D}([0,\infty),\ell_\lambda)$.

Theorem 4.6. *Let $\lambda \in (0,1]$, $\alpha \geq 0$ and $m \in \ell_\lambda$. Consider the coagulation kernel K, the fragmentation kernel F, the measure β and the Poisson measures N and M as in Definition 4.1.*

Then, there exists a solution $(M(m,t))_{t\geq 0}$ to $SDE(K,F,m,N,M)$.

We point out that we do not provide a pathwise uniqueness result for such processes. This is because, under our assumptions, we cannot take advantage of Proposition (4.4) for this process since the expressions in (4.6), (4.8) and (4.9) are possibly not true in general.

Nevertheless, when adding the hypothesis $K(0,0)=0$ to the coagulation kernel we can prove that these expressions hold by considering finite sums and passing to the limit. We believe that this is due to a possible injection of *dust* (particles of mass 0) into the system which could produce an increasing in the total mass of the system; see [9].

For proving this theorem, we first need the following lemma.

Lemma 4.7. *Let $\lambda \in (0,1]$ and $\alpha \geq 0$ be fixed. Assume that the coagulation kernel K, the fragmentation kernel F and a measure β satisfy Hypotheses 2.2. Consider for all $k \geq 1$ the measure β_k defined by (3.3). Finally, consider also a subset $\mathcal{A}$ of ℓ_{0+} such that $\sup_{m\in\mathcal{A}} \|m\|_\lambda < \infty$ and $\lim_{i\to\infty} \sup_{m\in\mathcal{A}} \sum_{k\geq i} m_k^\lambda = 0$.*

For each $m \in \mathcal{A}$ and each $k \geq 1$, let $(M^k(m,t))_{t\geq 0}$ be the unique solution to $SDE(K,F,m,N,M_k)$ constructed in Lemma 3.2., define $\tau_k(m,x) = \inf\{t \geq 0 : \|M^k(m,t)\|_\lambda \geq x\}$. Then for each $t \geq 0$ we have $\lim\limits_{x\to\infty} \alpha(t,x) = 0$, where

$$\alpha(t,x) := \sup_{m\in\mathcal{A}} \sup_{k\geq 1} P\left[\sup_{s\in[0,t]} \|M^k(m,s)\|_\lambda \geq x\right].$$

Remark that this convergence does not depend on β_k since is based on a bound not depending in the number of fragments but only on C_β^λ.

4.2.1. *Proofs.*

Proof of Lemma 4.7. It suffices to remark that from Proposition 4.4. *i)*, we have

$$\begin{aligned}
\sup_{m\in\mathcal{A}} \sup_{k\geq 1} P\left[\sup_{[0,t]} \|M^k(m,s)\|_\lambda \geq x\right] &\leq \frac{1}{x} \sup_{m\in\mathcal{A}} \sup_{k\geq 1} \mathbb{E}\left[\sup_{[0,t]} \|M^k(m,s)\|_\lambda\right] \\
&\leq \frac{1}{x} \sup_{m\in\mathcal{A}} \|m\|_\lambda e^{\overline{F}_m C_\beta^\lambda t}.
\end{aligned}$$

We make x tend to infinity and the lemma follows. □

Proof of Theorem 4.6. First, recall ψ_n defined by (3.2) and the measure $\beta_n = \mathbb{1}_{\theta\in\Theta(n)} \beta \circ \psi_n^{-1}$. Consider the Poisson measure $M(dt,di,d\theta,dz)$ associated to the fragmentation, as in Definition 4.1.

We set $M_n = \mathbb{1}_{\Theta(n)} M \circ \psi_n^{-1}$. This means that writing M as $M = \sum_{k\geq 1} \delta_{(T_k,i_k,\theta_k,z_k)}$, we have $M_n = \sum_{k\geq 1} \delta_{(T_k,i_k,\psi_n(\theta_k),z_k)} \mathbb{1}_{\theta\in\Theta(n)}$. Defined in this way, M_n is a Poisson measure on $[0,\infty) \times \mathbb{N} \times \Theta \times [0,\infty)$ with intensity measure $dt \left(\sum_{k\geq 1} \delta_k(di)\right) \beta_n(d\theta)\, dz$. In this paragraph $\delta_{(\cdot)}$ holds for the Dirac measure on $(\cdot)$.

We define $m^n \in \ell_{0+}$ by $m^n = (m_1, m_2, \cdots, m_n, 0, \cdots)$ and denote $M^n(t) := M(m^n, t)$ the unique solution to $SDE(K, F, m^n, N, M_n)$ obtained in Proposition 4.3. Note that $M^n(t)$ satisfies the following equation

$$\begin{aligned} M^n(t) &= m^n + \int_0^t \int_{i<j} \int_0^\infty \left[c_{ij}\left(M^n(s-)\right) - M^n(s-)\right] \mathbb{1}_{\{z \le K(M_i^n(s-), M_j^n(s-))\}} \\ &\qquad\qquad N(dt, d(i,j), dz) \\ &\quad + \int_0^t \int_i \int_\Theta \int_0^\infty \left[f_{i\psi_n(\theta)}\left(M^n(s-)\right) - M^n(s-)\right] \mathbb{1}_{\{z \le F(M_i^n(s-))\}} \mathbb{1}_{\{\theta \in \Theta(n)\}} \\ &\qquad\qquad M(dt, di, d\theta, dz). \end{aligned} \tag{4.16}$$

This setting allows us to couple the processes since they are driven by the same Poisson measures.

Convergence $M_t^n \to M_t$.– Consider $p, q \in \mathbb{N}$ with $1 \le p < q$, from (4.16) we obtain

$$\begin{aligned} \delta_\lambda\left(M^p(t), M^q(t)\right) &\le \delta_\lambda(m^p, m^q) + A_c^{p,q}(t) + B_c^{p,q}(t) + C_c^{p,q}(t) \\ &\quad + A_f^{p,q}(t) + B_f^{p,q}(t) + C_f^{p,q}(t) + D_f^{p,q}(t). \end{aligned} \tag{4.17}$$

We obtain this equality, exactly as in (4.9), by replacing M by M^p and $\tilde{M}$ by M^q. The terms concerning the coalescence are the same. The terms concerning the fragmentation are, equivalently:

$$\begin{aligned} A_f^{p,q}(t) &= \int_0^t \int_i \int_\Theta \int_0^\infty \Big\{ \delta_\lambda\left(f_{i\psi_p(\theta)}(M^p(s-)), f_{i\psi_p(\theta)}(M^q(s-))\right) \\ &\qquad - \delta_\lambda\left(M^p(s-), M^q(s-)\right) \Big\} \mathbb{1}_{\{\theta \in \Theta(p)\}} \mathbb{1}_{\left\{z \le F\left(M_i^p(s-)\right) \wedge F\left(M_i^q(s-)\right)\right\}} \\ &\qquad\qquad M(ds, di, d\theta, dz), \end{aligned}$$

$$\begin{aligned} B_f^{p,q}(t) &= \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda\left(f_{i\psi_p(\theta)}(M^p(s-)), M^q(s-)\right) - \delta_\lambda\left(M^p(s-), M^q(s-)\right) \right\} \\ &\qquad \mathbb{1}_{\{\theta \in \Theta(p)\}} \mathbb{1}_{\left\{F\left(M_i^q(s-)\right) \le z \le F\left(M_i^p(s-)\right)\right\}} M(ds, di, d\theta, dz), \end{aligned}$$

$$\begin{aligned} C_f^{p,q}(t) &= \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda\left(f_{i\psi_p(\theta)}(M^q(s-)), M^p(s-)\right) - \delta_\lambda\left(M^p(s-), M^q(s-)\right) \right\} \\ &\qquad \mathbb{1}_{\{\theta \in \Theta(p)\}} \mathbb{1}_{\left\{F\left(M_i^p(s-)\right) \le z \le F\left(M_i^q(s-)\right)\right\}} M(ds, di, d\theta, dz), \end{aligned}$$

Finally, the term $D_f^{p,q}(t)$ is the term that collects the errors.

$$\begin{aligned} D_f^{p,q}(t) &= \int_0^t \int_i \int_\Theta \int_0^\infty \delta_\lambda\left(f_{i\psi_p(\theta)}(M^q(s-)), f_{i\psi_q(\theta)}(M^q(s-))\right) \mathbb{1}_{\{\theta \in \Theta(p)\}} \\ &\qquad \mathbb{1}_{\left\{z \le F\left(M_i^q(s-)\right)\right\}} M(ds, di, d\theta, dz) \\ &\quad + \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda\left(f_{i\psi_q(\theta)}(M^q(s-)), M^p(s-)\right) - \delta_\lambda\left(M^p(s-), M^q(s-)\right) \right\} \\ &\qquad \mathbb{1}_{\left\{z \le F\left(M_i^q(s-)\right)\right\}} \mathbb{1}_{\{\theta \in \Theta(q) \setminus \Theta(p)\}} M(ds, di, d\theta, dz). \end{aligned}$$

The first term of $D_f^{p,q}(t)$ results from the utilization of the triangle inequality that gives $A_f^{p,q}(t)$ and $C_f^{p,q}(t)$. The second term is issued from fragmentation of M^q when θ belongs to $\Theta(q) \setminus \Theta(p)$. This induces a fictitious jump to M^p which does not undergo fragmentation.

We proceed to bound each term. We define, for all $x > 0$ and $n \geq 1$, the stopping time $\tau_n^x = \inf\{t \geq 0 : \|M^n(t)\|_\lambda \geq x\}$.

From Proposition 4.4. we have for all $s \in [0,t]$,

$$\sup_{n\geq 1}\sup_{i\geq 1} M_i^n(s) \leq \sup_{n\geq 1}\sup_{i\geq 1}\sup_{[0,t]} \|M^n(s)\|_1 \leq \|m\|_1 := a_m \, a.s.$$

We set κ_{a_m} and μ_{a_m} the constants for which the kernels K and F satisfy (2.2) and (2.3). Finally, we set $\overline{F}_m = \sup_{[0,a_m]} F(x)$.

The terms concerning coalescence are upper bounded on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ with $t \geq 0$, exactly as in (4.9).

Term $A_f^{p,q}(t)$: we take the sup on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ and then the expectation. We use (A.9) together with (2.7). We thus obtain exactly the same bound as for A_t^f.

Term $B_f^{p,q}(t)$: we take the sup on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ and then the expectation. We use (4.11), (A.7) with (2.7) and (2.3). We thus obtain exactly the same bound as for B_t^f.

Term $C_f^{p,q}(t)$: it is treated exactly as $B_f^{p,q}(t)$.

Term $D_f^{p,q}(t)$: we take the sup on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ and then the expectation. For the first term we use (A.10). For the second term we use (4.11) and (A.7) together with (2.7). Finally, we use Proposition 4.4. *i)*. and the notation $C(\theta) := \sum_{k\geq 2} \theta_k^\lambda + (1 - \theta_1^\lambda)$, to obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_q^x\wedge\tau_p^x]} D_f^{p,q}(t)\right] \\
&\quad\leq \mathbb{E}\left[\int_0^{t\wedge\tau_p^x\wedge\tau_q^x} \sum_{i\geq 1} F\left(M_i^q(s)\right) \int_\Theta \mathbb{1}_{\{\theta\in\Theta(p)\}} \sum_{k=p+1}^{q} \theta_k^\lambda [M_i^q(s)]^\lambda \beta(d\theta) ds\right] \\
&\qquad + \mathbb{E}\left[\int_0^{t\wedge\tau_p^x\wedge\tau_q^x} \sum_{i\geq 1} F\left(M_i^q(s)\right) [M_i^q(s)]^\lambda ds \int_\Theta C(\theta) \mathbb{1}_{\{\theta\in\Theta(q)\setminus\Theta(p)\}} \beta(d\theta)\right] \\
&\quad\leq \overline{F}_m \int_\Theta \sum_{k>p} \theta_k^\lambda \beta(d\theta) \int_0^t \mathbb{E}\left[\sup_{u\in[0,t]} \|M^q(u)\|_\lambda\right] ds \\
&\qquad + \overline{F}_m \int_\Theta C(\theta) \mathbb{1}_{\{\theta\in\Theta\setminus\Theta(p)\}} \beta(d\theta) \int_0^t \mathbb{E}\left[\sup_{u\in[0,t]} \|M^q(u)\|_\lambda\right] ds \\
&\quad\leq \overline{F}_m \, t \, \|m\|_\lambda \, e^{\overline{F}_m C_\beta^\lambda t} (A(p) + B(p)),
\end{aligned}$$

where $A(p) := \int_\Theta \sum_{k>p} \theta_k^\lambda \beta(d\theta)$ and $B(p) := \int_\Theta C(\theta) \mathbb{1}_{\{\theta\in\Theta\setminus\Theta(p)\}} \beta(d\theta)$. Note that by (2.5) and since $\Theta \setminus \Theta(p)$ tends to the empty set, $A(p)$ and $B(p)$ tend to 0 as p tends to infinity.

Thus, gathering the terms as for the bound (4.15), we get

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_q^x\wedge\tau_p^x]}\delta_\lambda\left(M^p(s),M^q(s)\right)\right]&\\ \leq\ \delta_\lambda\left(m^p,m^q\right)+D_1t[A(p)+B(p)]&\\ +\left(8\kappa_1\,x+CC_\beta^\lambda\|m\|_1^\alpha\right)\int_0^t\mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_q^x\wedge\tau_p^x]}\delta_\lambda\left(M^p(u),M^q(u)\right)\right]ds,&\qquad(4.18)\end{aligned}$$

where $D_1=\overline{F}_m\,\|m\|_\lambda\,e^{\overline{F}_m\,C_\beta^\lambda\,t}$. The Gronwall Lemma allows us to obtain

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_q^x\wedge\tau_p^x]}\delta_\lambda\left(M^p(s),M^q(s)\right)\right]\leq\left\{\delta_\lambda\left(m^p,m^q\right)+D_1[A(p)+B(p)]t\right\}\times e^{D_2\,x\,t},\qquad(4.19)$$

where D_2 is a positive constants depending on λ, α, κ_{a_m}, μ_{a_m}, K, F, C_β^λ and $\|m\|_1$.

Since $\lim_{n\to\infty}\delta_\lambda(m^n,m)=0$, we deduce from Lemma 4.7. that for all $t\geq 0$,

$$\lim_{x\to\infty}\alpha(t,x)=0\ \text{ where }\ \alpha(t,x):=\sup_{n\geq1}P[\tau(m^n,x)\leq t].\qquad(4.20)$$

This means that the stopping times τ_n^x tend to infinity as $x\to\infty$, uniformly in n.

Next, from (4.19), (4.20) and since $(m^n)_{n\geq1}$ is a Cauchy sequence for δ_λ and $(A(n))_{n\geq1}$ and $(B(n))_{n\geq1}$ converge to 0, we deduce that for all $\varepsilon>0$, $T>0$ we may find $n_\varepsilon>0$ such that for $p,q\geq n_\varepsilon$ we have

$$P\left[\sup_{[0,T]}\delta_\lambda\left(M^p(t),M^q(t)\right)\geq\varepsilon\right]\leq\varepsilon.\qquad(4.21)$$

Indeed, for all $x>0$,

$$\begin{aligned}P\left[\sup_{[0,T]}\delta_\lambda\left(M^p(t),M^q(t)\right)\geq\varepsilon\right]&\leq P[\tau_p^x\leq T]+P[\tau_q^x\leq T]+\frac{1}{\varepsilon}\mathbb{E}\left[\sup_{[0,T\wedge\tau_p^x\wedge\tau_q^x]}\delta_\lambda\left(M^p(t),M^q(t)\right)\right]\\&\leq 2\alpha(T,x)+\frac{1}{\varepsilon}[\delta_\lambda\left(m^p,m^q\right)+D_1T(A(p)+B(p))]\times e^{D_2\,x\,T}.\end{aligned}$$

Choosing x large enough so that $\alpha(T,x)\leq\varepsilon/8$ and n_ε large enough to have both $A(p)$ and $B(p)\leq(\varepsilon^2/4D_1\,T)e^{-D_2xT}$ and in a such a way that for all $p,q\geq n_\varepsilon$, $\delta_\lambda\left(m^p,m^q\right)\leq(\varepsilon^2/4)e^{-D_2xT}$, we conclude that (4.21) holds.

We deduce from (4.21) that the sequence of processes $(M_t^n)_{t\geq0}$ is Cauchy in probability in $\mathbb{D}([0,\infty),\ell_\lambda)$, endowed with the uniform norm in time on compact intervals. We are thus able to find a subsequence (not relabelled) and a $(\mathcal{H}_t)$-adapted process $(M(t))_{t\geq0}$ belonging *a.s.* to $\mathbb{D}([0,\infty),\ell_\lambda)$ such that for all $T>0$,

$$\lim_{n\to\infty}\sup_{[0,T]}\delta_\lambda\left(M^n(t),M(t)\right)=0.\ \ a.s.\qquad(4.22)$$

Setting now $\tau^x:=\inf\{t\geq0:\|M(t)\|_\lambda\geq x\}$, due to Lebesgue Theorem,

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{[0,T\wedge\tau_n^x\wedge\tau^x]}\delta_\lambda\left(M^n(t),M(t)\right)\right]=0.\qquad(4.23)$$

We have to show now that the limit process $(M(t))_{t\geq0}$ defined by (4.22) solves the equation $SDE(K,F,m,N,M)$ defined in (4.1).

We want to pass to the limit in (4.16), it suffices to show that $\lim_{n\to\infty} \Delta_n(t) = 0$, where

$$\begin{aligned}\Delta_n(t) \;=\; & \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\int_{i<j}\int_0^\infty \sum_{k\geq 1} 2^{-k}\Big| \left([c_{ij}\left(M(s-)\right)]_k - M_k(s-)\right]) \mathbb{1}_{\{z\leq K(M_i(s-),M_j(s-))\}} \\ & - \left([c_{ij}\left(M^n(s-)\right)]_k - M_k^n(s-)\right) \mathbb{1}_{\{z\leq K(M_i^n(s-),M_j^n(s-))\}}\Big| N(dt, d(i,j), dz) \\ & + \int_0^{t\wedge\tau_n^x\wedge\tau^x}\int_i\int_\Theta\int_0^\infty \sum_{k\geq 1} 2^{-k}\Big| \left([f_{i\theta}\left(M(s-)\right)]_k - [M(s-)]_k\right) \mathbb{1}_{\{z\leq F(M_i(s-))\}} \\ & - \left([f_{i\psi_n(\theta)}\left(M^n(s-)\right)]_k - M_k^n(s-)\right) \mathbb{1}_{\{z\leq F(M_i^n(s-))\}}\mathbb{1}_{\{\theta\in\Theta(n)\}}\Big| M(dt, di, d\theta, dz)\Bigg].\end{aligned}$$

Indeed, due to (4.22), for all $x > 0$ and for n large enough, $a.s.$ $\tau_n^x \geq \tau^{x/2}$. Thus M will solve $SDE(K, F, M(0), N, M)$ on the time interval $[0, \tau^{x/2})$ for all $x > 0$, and thus on $[0, \infty)$ since $a.s.$ $\lim_{x\to\infty} \tau^x = \infty$, because $M \in \mathbb{D}([0,\infty), \ell_\lambda)$.

Note that

$$\begin{aligned}&\Big| \left([c_{ij}\left(M(s)\right)]_k - M_k(s)\right]) \mathbb{1}_{\{z\leq K(M_i(s),M_j(s))\}} - \left([c_{ij}\left(M^n(s)\right)]_k - M_k^n(s)\right) \mathbb{1}_{\{z\leq K(M_i^n(s),M_j^n(s))\}}\Big| \\ &\quad\leq \Big| \left([c_{ij}\left(M(s)\right)]_k - M_k(s)\right]) - \left([c_{ij}\left(M^n(s)\right)]_k - M_k^n(s)\right) \Big| \mathbb{1}_{\{z\leq K(M_i(s),M_j(s))\}} \\ &\qquad + \Big|[c_{ij}\left(M^n(s)\right)]_k - M_k^n(s)\Big| \Big| \mathbb{1}_{\{z\leq K(M_i(s),M_j(s))\}} - \mathbb{1}_{\{z\leq K(M_i^n(s),M_j^n(s))\}}\Big|\end{aligned}$$

and

$$\begin{aligned}&\Big| \left([f_{i\theta}\left(M(s)\right)]_k - M_k(s)\right) \mathbb{1}_{\{z\leq F(M_i(s))\}} - \left([f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k - M_k^n(s)\right) \mathbb{1}_{\{z\leq F(M_i^n(s))\}}\mathbb{1}_{\{\theta\in\Theta(n)\}}\Big| \\ &\quad\leq \Big| \left([f_{i\theta}\left(M(s)\right)]_k - M_k(s)\right) - \left([f_{i\theta}\left(M^n(s)\right)]_k - M_k^n(s)\right) \Big| \mathbb{1}_{\{z\leq F(M_i(s))\}} \\ &\qquad + \Big| \left([f_{i\theta}\left(M^n(s)\right)]_k - [f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k\right) \Big| \mathbb{1}_{\{z\leq F(M_i(s))\}} \\ &\qquad + \Big|[f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k - M_k^n(s)\Big| \Big| \mathbb{1}_{\{z\leq F(M_i(s))\}} - \mathbb{1}_{\{z\leq F(M_i^n(s))\}}\Big| \\ &\qquad + \Big|[f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k - M_k^n(s)\Big| \mathbb{1}_{\{z\leq F(M_i^n(s))\}}\mathbb{1}_{\{\theta\in\Theta(n)^c\}},\end{aligned}$$

where $\Theta(n)^c = \Theta \setminus \Theta(n)$. We thus obtain the following bound

$$\Delta_n(t) \leq A_n^c(t) + B_n^c(t) + A_n^f(t) + B_n^f(t) + C_n^f(t) + D_n^f(t).$$

First, $A_n^c(t) = \sum_{i<j} A_n^{ij}(t)$ with

$$\begin{aligned}A_n^{ij}(t) \;=\; & \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} K\left(M_i(s), M_j(s)\right) \sum_{k\geq 1} 2^{-k} \\ & \qquad \left|\left([c_{ij}\left(M(s)\right)]_k - M_k(s)\right]) - \left([c_{ij}\left(M^n(s)\right)]_k - M_k^n(s)\right)\right| ds\Bigg],\end{aligned}$$

and using

$$\begin{aligned}&\Big|\mathbb{1}_{\{z\leq K(M_i(s),M_j(s))\}} - \mathbb{1}_{\{z\leq K(M_i^n(s),M_j^n(s))\}}\Big| \\ &\qquad = \mathbb{1}_{\{K(M_i(s),M_j(s))\wedge K(M_i^n(s),M_j^n(s))\leq z\leq K(M_i(s),M_j(s))\vee K(M_i^n(s),M_j^n(s))\}},\end{aligned}$$

$$B_n^c(t) = \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \sum_{i<j} \left|K\left(M_i(s), M_j(s)\right) - K\left(M_i^n(s), M_j^n(s)\right)\right| \sum_{k\geq 1} 2^{-k} \left|[c_{ij}\left(M^n(s)\right)]_k - M_k^n(s)\right| ds\Bigg].$$

For the fragmentation terms we have

$$A_n^f(t) = \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \int_\Theta \sum_{i\geq 1} F\left(M_i(s)\right) \sum_{k\geq 1} 2^{-k} \left|\left([f_{i\theta}\left(M(s)\right)]_k - M_k(s)\right) - \left([f_{i\theta}\left(M^n(s)\right)]_k - M_k^n(s)\right)\right| \beta(d\theta)ds\Bigg],$$

$$B_n^f(t) = \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \int_\Theta \sum_{i\geq 1} F\left(M_i(s)\right) \sum_{k\geq 1} 2^{-k} \left|\left([f_{i\theta}\left(M^n(s)\right)]_k - [f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k\right)\right| \beta(d\theta)ds\Bigg],$$

using

$$\left|\mathbb{1}_{\{z\leq F(M_i(s))\}} - \mathbb{1}_{\{z\leq F(M_i^n(s))\}}\right| = \mathbb{1}_{\{F(M_i(s))\wedge F(M_i^n(s))\leq z\leq F(M_i(s))\vee F(M_i^n(s))\}},$$

$$C_n^f(t) = \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \int_\Theta \mathbb{1}_{\{\theta\in\Theta(n)\}} \sum_{i\geq 1} \left|F\left(M_i(s)\right) - F\left(M_i^n(s)\right)\right| \sum_{k\geq 1} 2^{-k} \left|[f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k - M_k^n(s)\right| \beta(d\theta)ds\Bigg],$$

and finally,

$$D_n^f(t) = \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \int_\Theta \mathbb{1}_{\{\theta\in\Theta(n)^c\}} \sum_{i\geq 1} F\left(M_i^n(s)\right) \sum_{k\geq 1} 2^{-k} \left|[f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k - M_k^n(s)\right| \beta(d\theta)ds\Bigg],$$

We will show that each term converges to 0 as n tends to infinity.

Note first that from (4.22) we have, *a.s.* $\sup_{[0,t]} \|M(s)\|_1 \leq \limsup_{n\to\infty} \sup_{[0,t]} \|M^n(s)\|_1$ and *a.s.* $\sup_{[0,t]} \|M(s)\|_\lambda \leq \limsup_{n\to\infty} \sup_{[0,t]} \|M^n(s)\|_\lambda$ and from Proposition 4.4 *i)*, we get $\sup_{n\geq 1} \sup_{[0,t]} \|M^n(s)\|_1 \leq \|m\|_1$, implying for all $t\geq 0$

$$\sup_{s\in[0,t]} \|M(s)\|_1 \leq \|m\|_1 := a_m < \infty,\ a.s., \tag{4.24}$$

equivalently for M^n, we have $a_{m^n} = \|m^n\|_1 \leq \|m\|_1$. We set κ_{a_m} and μ_{a_m} the constants for which the kernels K and F satisfy (2.2) and (2.3). Finally, we set $\overline{K}_m = \sup_{[0,a_m]^2} K(x,y)$ and $\overline{F}_m = \sup_{[0,a_m]} F(x)$.

We prove that $A_n^c(t)$ tends to 0 using the Lebesgue dominated convergence Theorem. It suffices to show that:

a) for each $1\leq i<j$, $A_n^{ij}(t)$ tends to 0 as n tends to infinity,

b) $\lim_{k\to\infty}\limsup_{n\to\infty}\sum_{i+j\geq k} A_n^{ij}(t)=0$.

Now, for $A_n^{ij}(t)$ using (A.16), (A.14), (4.24) and Proposition 4.4. *i)*, we have

$$\begin{aligned}
A_n^{ij}(t) &\leq \overline{K}_m\mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x} d\left(c_{ij}\left(M(s)\right), c_{ij}\left(M^n(s)\right)\right)+d\left(M(s),M^n(s)\right)\,ds\right]\\
&\leq \overline{K}_m\mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\left(2^i+2^j+1\right)d\left(M(s),M^n(s)\right)\,ds\right]\\
&\leq C\,\overline{K}_m\left(2^i+2^j+1\right)\mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\left(\|M(s)\|_1^{1-\lambda}\vee\|M^n(s)\|_1^{1-\lambda}\right)\\
&\qquad\qquad\qquad\qquad\times\delta_\lambda\left(M(s),M^n(s)\right)\,ds\Bigg]\\
&\leq C\,\overline{K}_m\left(2^i+2^j+1\right)t\|m\|_1^{1-\lambda}\mathbb{E}\left[\sup_{[0,t\wedge\tau_n^x\wedge\tau^x]}\delta_\lambda\left(M(s),M^n(s)\right)\right].
\end{aligned}$$

which tends to 0 as $n\to\infty$ due to (4.23). On the other hand, using (A.15) we have

$$\begin{aligned}
A_n^{ij}(t) &\leq \overline{K}_m\mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x} d\left(c_{ij}\left(M(s)\right),M(s)\right)+d\left(c_{ij}\left(M^n(s)\right),M^n(s)\right)\,ds\right]\\
&\leq \frac{3\overline{K}_m}{2}2^{-i}\int_0^t\mathbb{E}\left[M_j(s)+M_j^n(s)\right]ds.
\end{aligned}$$

Since $\sum_{i\geq 1}2^{-i}=1$ and $\sum_{j\geq 1}\int_0^t\mathbb{E}[M_j(s)]ds\leq\|m\|_1 t$, b) reduces to

$$\lim_{k\to\infty}\limsup_{n\to\infty}\sum_{j\geq k}\int_0^t\mathbb{E}[M_j^n(s)]ds=0.$$

For each $k\geq 1$, since $M^n(s)$ and $M(s)$ belong to ℓ_1 for all $s\geq 0$ *a.s* and since the map $m\mapsto\sum_{j=1}^{k-1}m_j$ is continuous for the pointwise convergence topology,

$$\begin{aligned}
\limsup_{n\to\infty}\int_0^t\mathbb{E}\left[\sum_{j\geq k}M_j^n(s)\right] &= \int_0^t ds\left\{\lim_{n\to\infty}\|M^n(s)\|_1-\lim_{n\to\infty}\mathbb{E}\left[\sum_{j=1}^{k-1}M_j^n(s)\right]\right\}ds\\
&= \int_0^t\left\{\|M(s)\|_1-\mathbb{E}\left[\sum_{j=1}^{k-1}M_j(s)\right]\right\}ds\\
&= \int_0^t\mathbb{E}\left[\sum_{j=k}^{\infty}M_j(s)\right]ds.
\end{aligned}$$

We easily conclude using that *a.s.* $\|M(s)\|_1<\|m\|_1$ for all $s\geq 0$.

Using (2.2), (A.15) and Proposition 4.4. *i)*, we obtain

$$
\begin{aligned}
B_n^c(t) &\le \kappa_{a_m}\mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\sum_{i<j}\left[\left|M_i^n(s)^\lambda - M_i(s)^\lambda\right| + \left|M_j^n(s)^\lambda - M_j(s)^\lambda\right|\,ds\right] \\
&\qquad\qquad \times d\left(c_{ij}\left(M^n(s)\right), M^n(s)\right)\Bigg] \\
&\le \frac{3}{2}\kappa_{a_m}\mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\sum_{i<j}\Big[\left|M_i^n(s)^\lambda - M_i(s)^\lambda\right| + \left|M_j^n(s)^\lambda - M_j(s)^\lambda\right|\Big]2^{-i}M_j^n(s)\,ds\Bigg] \\
&\le 3t\,\kappa_{a_m}\|m\|_1\mathbb{E}\left[\sup_{[0,[t\wedge\tau_n^x\wedge\tau^x]}\delta_\lambda\left(M(s), M^n(s)\right)\right],
\end{aligned}
$$

which tends to 0 as $n\to\infty$ due to (4.23).

We use (A.18) and (A.14) both with (4.24) and Proposition 4.4. *i)* and (A.17) to obtain

$$
\begin{aligned}
A_n^f(t) &\le \overline{F}_m\mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\sum_{i\ge1}\int_\Theta\Bigg[\Bigg(d\left(f_{i\theta}\left(M(s)\right), f_{i\theta}\left(M^n(s)\right)\right) + d\left(M(s), M^n(s)\right)\Bigg) \\
&\qquad \wedge\Bigg(d\left(f_{i\theta}\left(M(s)\right), M(s)\right) + d\left(f_{i\theta}\left(M^n(s)\right), M^n(s)\right)\Bigg)\Bigg]\beta(d\theta)ds\Bigg] \\
&\le \overline{F}_m\mathbb{E}\Bigg\{\int_0^{t\wedge\tau_n^x\wedge\tau^x}\int_\Theta\sum_{i\ge1}\Bigg[\Bigg(2C\|m\|_\lambda^{1-\lambda}\,\delta_\lambda\left(M(s), M^n(s)\right)\Bigg)\wedge \\
&\qquad \Bigg(2^{-i}(1-\theta_1)\left(M_i(s)+M_i^n(s)\right)\Bigg)\Bigg]\beta(d\theta)ds\Bigg\}.
\end{aligned}
$$

We split the integral on Θ and the sum on i into two parts. Consider $\Theta_\varepsilon = \{\theta\in\Theta : \theta_1 \le 1-\varepsilon\}$ and $N\in\mathbb{N}$. Using (4.24) and Proposition 4.4. *i)* and relabelling the constant C, we deduce

$$
\begin{aligned}
&\int_\Theta\sum_{i\ge1}\Bigg[\Bigg(C\|m\|_\lambda^{1-\lambda}\,\delta_\lambda\left(M(s), M^n(s)\right)\Bigg)\wedge\Bigg(2^{-i}(1-\theta_1)\left(M_i(s)+M_i^n(s)\right)\Bigg)\Bigg]\beta(d\theta) \\
&\quad\le C\|m\|_\lambda^{1-\lambda}\int_{\Theta_\varepsilon}\sum_{i=1}^N\delta_\lambda\left(M(s), M^n(s)\right)\beta(d\theta) + \int_{\Theta_\varepsilon^c}(1-\theta_1)\beta(d\theta)\sum_{i\ge1}\left(M_i(s)+M_i^n(s)\right) \\
&\qquad + \int_\Theta\sum_{i>N}2^{-i}(1-\theta_1)\left(M_i(s)+M_i^n(s)\right)\beta(d\theta) \\
&\quad\le C\|m\|_1^{1-\lambda}N\beta(\Theta_\varepsilon)\delta_\lambda\left(M(s), M^n(s)\right) + 2\|m\|_1\int_{\Theta_\varepsilon^c}(1-\theta_1)\beta(d\theta) \\
&\qquad + 2\|m\|_1\int_\Theta(1-\theta_1)\beta(d\theta)\sum_{i>N}2^{-i}.
\end{aligned}
$$

Note that $\beta(\Theta_\varepsilon) = \int_\Theta \mathbb{1}_{\{1-\theta_1 \geq \varepsilon\}}\, \beta(d\theta) \leq \frac{1}{\varepsilon} \int_\Theta (1-\theta_1)\, \beta(d\theta) \leq \frac{1}{\varepsilon} C_\beta^\lambda < \infty$. Thus, we get

$$\begin{aligned} A_n^f(t) \quad &\leq \quad \frac{t}{\varepsilon} C_\beta^\lambda N \overline{F}_m C \|m\|_1^{1-\lambda} \mathbb{E}\left[\sup_{[0,[t\wedge\tau_n^x\wedge\tau^x]} \delta_\lambda\left(M(s), M^n(s)\right)\right] \\ &\quad\quad + 2t\overline{F}_m \|m\|_1 \int_{\Theta_\varepsilon^c} (1-\theta_1)\beta(d\theta) + 4t\overline{F}_m \|m\|_1 C_\beta^\lambda\, 2^{-N}. \end{aligned}$$

Thus, due to (4.23) we have for all $\varepsilon > 0$ and $N \geq 1$,

$$\limsup_{n\to\infty} A_n^f(t) \leq 2t\overline{F}_m \|m\|_1 \int_{\Theta_\varepsilon^c} (1-\theta_1)\beta(d\theta) + 4t\overline{F}_m \|m\|_1 C_\beta^\lambda\, 2^{-N}.$$

Since Θ_ε^c tends to the empty set as $\varepsilon \to 0$ we conclude using (2.7) with (2.5) and making $\varepsilon \to 0$ and $N \to \infty$.

Next, use (A.19) and Proposition 4.4. *i)* to obtain

$$B_n^f(t) \leq t\overline{F}_t \|m\|_1 \int_\Theta \sum_{k>n} \theta_k \beta(d\theta).$$

which tends to 0 as $n \to \infty$ due to (2.4).

Using (2.3), (A.17) with (2.6) and (2.7), (A.3), (A.14), (4.24) and Proposition 4.4. *i)*, we obtain

$$\begin{aligned} C_n^f(t) \quad &\leq \quad 2\mu_{a_m} \mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \int_{\Theta(n)} \sum_{i\geq 1} \left| [M_i(s)]^\alpha - [M_i^n(s)]^\alpha \right| 2^{-i} (1-\theta_1) M_i(s) \beta(d\theta) ds \right] \\ &\leq \quad 2\mu_{a_m}\, C_\beta^\lambda \mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \sum_{i\geq 1} 2^{-i} \left| M_i(s) - M_i^n(s) \right| \left([M_i^n(s)]^\alpha + [M_i(s)]^\alpha \right) ds \right] \\ &\leq \quad 2\mu_{a_m}\, C\, C_\beta^\lambda t \|m\|_1^{1-\lambda+\alpha} \mathbb{E}\left[\sup_{[0,t\wedge\tau_n^x\wedge\tau^x]} \delta_\lambda\left(M(s), M^n(s)\right)\right], \end{aligned}$$

which tends to 0 as $n \to \infty$ due to (4.23).

Finally, we use (A.17) with (2.6) and (2.7) and Proposition 4.4. *i)*, to obtain

$$D_n^f(t) \leq 2t\, \overline{F}_t \|m\|_1 \int_\Theta \mathbb{1}_{\{\theta \in \Theta(n)^c\}} (1-\theta_1)\beta(d\theta)$$

which tends to 0 as n tends to infinity since $\int_\Theta (1-\theta_1)\, \beta(d\theta) \leq C_\beta^\lambda$ and $\Theta(n)^c$ tends to the empty set.

This ends the proof of Theorem 4.6. □

4.3. **Conclusion.** It remains to conclude the proof of Theorem 3.3.

We start with some boundedness of the operator $\mathcal{L}_{K,F}^\beta$.

Lemma 4.8. *Let $\lambda \in (0,1]$, $\alpha \geq 0$, the coagulation kernel K, fragmentation kernel F and the measure β satisfying Hypotheses 2.2. Let $\Phi : \ell_\lambda \to \mathbb{R}$ satisfy, for all $m, \tilde{m} \in \ell_\lambda$, $|\Phi(m)| \leq a$ and $|\Phi(m) - \Phi(\tilde{m})| \leq a d(m, \tilde{m})$. Recall (2.11). Then $m \mapsto \mathcal{L}_{K,F}^\beta \Phi(m)$ is bounded on $\{m \in \ell_\lambda, \|m\|_\lambda \leq c\}$ for each $c > 0$.*

Proof. This Lemma is a straightforward consequence of the hypotheses on the kernels and Lemma A.3. Let $c > 0$ be fixed, and set $A := c^{1/\lambda}$. Notice that if $\|m\|_\lambda \le c$, then for all $k \ge 1$ $m_k \le A$.

Setting $\sup_{[0,A]^2} K(x,y) = \overline{K}$ and $\sup_{[0,A]} F(x) = \overline{F}$. We use (A.15) and (A.17) with (2.6) and (2.7), and deduce that for all $m \in \ell_\lambda$ such that $\|m\|_\lambda \le c$,

$$\begin{aligned}
|\mathcal{L}^\beta_{K,F}\Phi(m)| &\le \overline{K} \sum_{1\le i<j<\infty} [\Phi(c_{ij}(m)) - \Phi(m)] + \overline{F} \sum_{i\ge 1} \int_\Theta [\Phi(f_{i\theta}(m)) - \Phi(m)]\, \beta(d\theta) \\
&\le a\overline{K} \sum_{1\le i<j<\infty} d(c_{ij}(m), m) + a\overline{F} \int_\Theta \sum_{i\ge 1} d(f_{i\theta}(m), m)\beta(d\theta) \\
&\le \frac{3}{2} a\overline{K}\, \|m\|_1 + 2a\,\overline{F}\, C^\lambda_\beta \|m\|_1 \;\; \le \;\; \left(\frac{3}{2}\overline{K} + 2\overline{F}\, C^\lambda_\beta\right) ac^{1/\lambda}.
\end{aligned}$$

□

Finally, it remains to conclude the proof of Theorem 3.3.

Proof of Theorem 3.3. We consider the Poisson measures N and M as in Definition 4.1., and we fix $m \in \ell_\lambda$. We consider $M(t) := M(m,t)$ a solution to $SDE(K, F, M(0), N, M)$ built in Section 4.2. M is a strong Markov Process, since it solves a time-homogeneous Poisson-driven $S.D.E.$ We now check the points *i)* and *ii)*.

Consider any sequence $m^n \in \ell_{0+}$ such that $\lim_{n\to\infty} \delta_\lambda(m^n, m) = 0$ and $M^n(t) := M(m^n, t)$ the unique solution to $SDE(K, F, m^n, N, M_n)$ obtained in Proposition 4.3. Denote by $\tau^x = \inf\{t \ge 0, \|M(m,t)\|_\lambda \ge x\}$ and by τ^x_n the stopping time concerning M^n. We will prove that for all $T \ge 0$ and $\varepsilon > 0$

$$\lim_{n\to\infty} P\left[\sup_{[0,T]} \delta_\lambda\left(M(t), M^n(t)\right) > \varepsilon\right] = 0. \tag{4.25}$$

For this, consider the sequence $m^{(n)} \in \ell_{0+}$ defined by $m^{(n)} = (m_1, \cdots, m_n, 0, \dots)$ and $M^{(n)}(t) := M(m^{(n)}, t)$ the solution to $SDE(K, F, m^{(n)}, N, M_n)$ obtained in Proposition 4.3. and denote by $\tau^x_{(n)}$ the stopping time concerning $M^{(n)}$.

First, note that since $\lim_{n\to\infty} \delta_\lambda(m^{(n)}, m) = \lim_{n\to\infty} \delta_\lambda(m^n, m) = 0$, we deduce that $\sup_{n\ge1} \|m^{(n)}\|_\lambda < \infty$ and from Lemma 4.7. that for all $t \ge 0$,

$$\lim_{x\to\infty} \alpha_1(t,x) = 0 \text{ where } \alpha_1(t,x) := \sup_{n\ge1} P[\tau^x_{(n)} \le t], \tag{4.26}$$

$$\lim_{x\to\infty} \alpha_2(t,x) = 0 \text{ where } \alpha_2(t,x) := \sup_{n\ge1} P[\tau^x_n \le t]. \tag{4.27}$$

Thus, using Proposition 4.4. *ii)* we get for all $x > 0$

$$\begin{aligned}
&P\left[\sup_{[0,T]} \delta_\lambda \left(M(t), M^n(t)\right) > \varepsilon\right] \\
&\quad \leq \; P\left[\sup_{[0,T]} \delta_\lambda \left(M(t), M^{(n)}(t)\right) > \frac{\varepsilon}{2}\right] + P\left[\sup_{[0,T]} \delta_\lambda \left(M^{(n)}(t), M^n(t)\right) > \frac{\varepsilon}{2}\right] \\
&\quad \leq \; P[\tau^x \leq T] + \alpha_1(T,x) + \frac{2}{\varepsilon}\mathbb{E}\left[\sup_{[0,T\wedge\tau^x_{(n)}\wedge\tau^x]} \delta_\lambda \left(M(t), M^{(n)}(t)\right)\right] \\
&\qquad +\alpha_1(T,x) + \alpha_2(T,x) + \frac{2}{\varepsilon}e^{C(x+1)T}\delta_\lambda\left(m^{(n)}, m^n\right).
\end{aligned}$$

We first make n tend to infinity and use (4.23), then x to infinity and use (4.26) and (4.27). We thus conclude that (4.25) holds.

We may prove point *ii)* using a similar computation that for *i)*. The proof is easier since we do not need to use a triangle inequality.

Finally, consider $(M(m,t))_{t\geq 0}$ solution to $SDE(K,F,m,N,M)$ and the sequence of stopping times $(\tau^{x_n})_{n\geq 1}$ where $\tau^{x_n} = \inf\{t \geq 0, \|M(m,t)\|_\lambda \geq x_n\}$, with $x_n = n$. Since $M \in \mathbb{D}([0,\infty),\ell_\lambda)$, we have that $(\tau^{x_n})_{n\geq 1}$ is non-decreasing and $\tau^{x_n} \underset{n\to\infty}{\longrightarrow} \infty$ and from Lemma 4.8. we deduce that $(\mathcal{L}^\beta_{K,F}\Phi(M(m,s)))_{s\in[0,\tau^{x_n})}$ is uniformly bounded.

We thus apply Itô's Formula to $\Phi(M(m,t))$ on the interval $[0, t\wedge\tau^{x_n})$ to obtain

$$\begin{aligned}
&\Phi(M(m,t\wedge\tau^{x_n})) - \Phi(m) \; = \\
&\int_0^{t\wedge\tau^{x_n}} \int_{i<j}\int_0^\infty \left[\Phi\left(c_{ij}\left(M(m,s-)\right)\right) - \Phi\left(M(m,s-)\right)\right] \mathbb{1}_{\{z\leq K(M_i(m,s-),M_j(m,s-))\}} \\
&\hspace{25em} \tilde{N}(dt, d(i,j), dz) \\
&+ \int_0^{t\wedge\tau^{x_n}} \int_i\int_\Theta\int_0^\infty \left[\Phi\left(f_{i\theta}\left(M(m,s-)\right)\right) - \Phi\left(M(m,s-)\right)\right] \mathbb{1}_{\{z\leq F(M_i(m,s-))\}} \\
&\hspace{25em} \tilde{M}(dt, di, d\theta, dz) \\
&+ \int_0^{t\wedge\tau^{x_n}} \mathcal{L}^\beta_{K,F}\left(M(m,s)\right)ds,
\end{aligned}$$

where $\tilde{N}$ and $\tilde{M}$ are two compensated Poisson measures and point *iii)* follows.

This ends the proof of Theorem 3.3.

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Appendix A. Estimates concerning c_{ij}, $f_{i\theta}$, d and δ_λ

Here we put all the auxiliary computations needed in Sections 4.1.2 and 4.2.

Lemma A.1. *Fix $\lambda \in (0,1]$. Consider any pair of finite permutations σ, $\tilde{\sigma}$ of $\mathbb{N}$. Then for all m and $\tilde{m} \in \ell_\lambda$,*

$$d(m,\tilde{m}) \leq \sum_{k\geq 1} 2^{-k}|m_k - \tilde{m}_{\tilde{\sigma}(k)}|, \tag{A.1}$$

$$\delta_\lambda(m,\tilde{m}) \leq \sum_{k\geq 1} |m^\lambda_{\sigma(k)} - \tilde{m}^\lambda_{\tilde{\sigma}(k)}|. \tag{A.2}$$

This lemma is a consequence of [8, Lemma 3.1].

We also have the following inequality: for all $\alpha, \beta > 0$, there exists a positive constant $C = C_{\alpha,\beta}$ such that for all $x, y \geq 0$,

$$(x^\alpha + y^\alpha)|x^\beta - y^\beta| \leq 2|x^{\alpha+\beta} - y^{\alpha+\beta}| \leq C(x^\alpha + y^\alpha)|x^\beta - y^\beta|. \tag{A.3}$$

We now give the inequalities concerning the action of c_{ij} and $f_{i\theta}$ on δ_λ and $\|\cdot\|_\lambda$.

Lemma A.2. *Let $\lambda \in (0,1]$ and $\theta \in \Theta$. Then for all m and $\tilde{m} \in \ell_\lambda$, all $1 \leq i < j < \infty$,*

$$\|c_{ij}(m)\|_\lambda = \|m\|_\lambda + (m_i + m_j)^\lambda - m_i^\lambda - m_j^\lambda \leq \|m\|_\lambda, \tag{A.4}$$

$$\|f_{i\theta}(m)\|_\lambda = \|m\|_\lambda + m_i^\lambda \left(\sum_{k\geq 1} \theta_k^\lambda - 1\right), \tag{A.5}$$

$$\delta_\lambda(c_{ij}(m), m) \leq 2m_j^\lambda, \tag{A.6}$$

$$\delta_\lambda(f_{i\theta}(m), m) \leq m_i^\lambda \left[\sum_{k\geq 2} \theta_k^\lambda + \left(1 - \theta_1^\lambda\right)\right], \tag{A.7}$$

$$\delta_\lambda(c_{ij}(m), c_{ij}(\tilde{m})) \leq \delta_\lambda(m,\tilde{m}), \tag{A.8}$$

$$\delta_\lambda(f_{i\theta}(m), f_{i\theta}(\tilde{m})) \leq \delta_\lambda(m,\tilde{m}) + |m_i^\lambda - \tilde{m}_i^\lambda| \left(\sum_{k\geq 1} \theta_k^\lambda - 1\right). \tag{A.9}$$

On the other hand, recall (3.2), we have, for $u, v \in \mathbb{N}$ with $1 \leq u < v$,

$$\delta_\lambda(f_{i\psi_u(\theta)}(m), f_{i\psi_v(\theta)}(m)) \leq \sum_{k=u+1}^{v} \theta_k^\lambda m_i^\lambda. \tag{A.10}$$

Note that in the case $\sum_{k\geq 1} \theta_k^\lambda - 1 < 0$, we have that $\|\cdot\|_\lambda$ and δ_λ are respectively, decreasing and contracting under the action of fragmentation and the calculations in precedent sections would be simpler.

Proof. First (A.4) and (A.5) are evident. Next, (A.6) and (A.8) are proved in [10, Lemma A.2].

To prove (A.7) let $\theta = (\theta_1, \cdots) \in \Theta$, $i \geq 1$ and $p \geq 2$ and set $l := l(m) = \min\{k \geq 1 : m_k \leq \theta_p\, m_i\}$, we consider the largest particle of the original system (before dislocation of m_i) that is smaller than the p-th fragment of m_i, this is m_l. Consider now σ, the finite permutation of $\mathbb{N}$ that achieves:

(A.11)
$$\begin{aligned}(f_k)_{k\geq 1} &:= \Big([f_{i\theta}(m)]_{\sigma(k)}\Big)_{k\geq 1}\\ &= (m_1,\cdots,m_{i-1},\theta_1 m_i, m_{i+1},\cdots,m_{l-1},m_l,\theta_2 m_i,\theta_3 m_i,\cdots,\theta_p m_i,[f_{i\theta}(m)]_{l+1},\cdots).\end{aligned}$$

It suffices to compute the δ_λ-distance of the sequences $(f_k)_k$ and $(m_k)_k$:
(A.12)
$$\begin{array}{ccccccccccccc} m_1 & \cdots & m_{i-1} & \theta_1 m_i & m_{i+1} & \cdots & m_{l-1} & m_l & \theta_2 m_i & \theta_3 m_i & \cdots & \theta_p m_i & f_{l+p} & \cdots\\ m_1 & \cdots & m_{i-1} & m_i & m_{i+1} & \cdots & m_{l-1} & m_l & m_{l+1} & m_{l+2} & \cdots & m_{l+p-1} & m_{l+p} & \cdots\end{array}$$

Thus, using (A.2), we have

$$\begin{aligned}\delta_\lambda(f_{i\theta}(m),m) &\leq \sum_{k\geq 1}\left|f_k^\lambda - m_k^\lambda\right| = \left(\sum_{k=1}^{l}+\sum_{k=l+1}^{l+p-1}+\sum_{k\geq l+p}\right)\left|f_k^\lambda - m_k^\lambda\right|\\ &\leq (1-\theta_1^\lambda)m_i^\lambda + \sum_{k=l+1}^{l+p-1}\left|\theta_{k-l+1}^\lambda m_i^\lambda - m_k^\lambda\right| + \sum_{k\geq l+p}\left|f_k^\lambda - m_k^\lambda\right|\\ &\leq (1-\theta_1^\lambda)m_i^\lambda + \left(\sum_{k=2}^{p}\theta_k^\lambda m_i^\lambda + \sum_{k=l+1}^{l+p-1} m_k^\lambda\right) + \sum_{k\geq l+p}\left(f_k^\lambda + m_k^\lambda\right)\\ &= (1-\theta_1^\lambda)m_i^\lambda + m_i^\lambda\sum_{k=2}^{\infty}\theta_k^\lambda + 2\sum_{k>l} m_k^\lambda.\end{aligned}$$

For the last equality it suffices to remark that $\sum_{k\geq l} f_k^\lambda$ contains all the remaining fragments of m_i^λ and all the particles m_k^λ with $k>l$.

Note that if $m\in\ell_{0+}$ the last sum consists of a finite number of terms and it suffices to take p large enough (implying l large) to cancel this term. On the other hand, if $m\in\ell_\lambda\setminus\ell_{0+}$ then the last sum is the tail of a convergent serie and since $l\to\infty$ whenever $p\to\infty$, we conclude by making p tend to infinity and (A.7) follows.

To prove (A.9) consider $\tilde m$, $l := l(m)\vee l(\tilde m)$ and the permutations σ and $\tilde\sigma$ associated to this l, exactly as in (A.11). Let f and $\tilde f$ be the corresponding objects concerning m and $\tilde m$:

$$\begin{array}{ccccccccccccc} m_1 & \cdots & m_{i-1} & \theta_1 m_i & m_{i+1} & \cdots & m_{l-1} & m_l & \theta_2 m_i & \theta_3 m_i & \cdots & \theta_p m_i & f_{l+p} & \cdots\\ \tilde m_1 & \cdots & \tilde m_{i-1} & \theta_1 \tilde m_i & \tilde m_{i+1} & \cdots & \tilde m_{l-1} & \tilde m_l & \theta_2 \tilde m_i & \theta_3 \tilde m_i & \cdots & \theta_p \tilde m_i & \tilde f_{l+p} & \cdots\end{array}\tag{A.13}$$

Using again (A.2) for $(f_k)_k$ and $(\tilde f_k)_k$, we have

$$\begin{aligned}&\delta_\lambda(f_{i\theta}(m),f_{i\theta}(\tilde m))\\ &\quad\leq \sum_{k\geq 1}\left|f_k^\lambda - \tilde f_k^\lambda\right| = \left(\sum_{k=1}^{l}+\sum_{k=l+1}^{l+p-1}+\sum_{k\geq l+p}\right)\left|f_k^\lambda - \tilde f_k^\lambda\right|\\ &\quad= \sum_{k=1}^{l}\left|m_k^\lambda - \tilde m_k^\lambda\right| - \left|m_i^\lambda - \tilde m_i^\lambda\right| + \sum_{k=1}^{p}\theta_k^\lambda\left|m_i^\lambda - \tilde m_i^\lambda\right| + \sum_{k\geq l+p}\left(f_k^\lambda + \tilde f_k^\lambda\right)\end{aligned}$$

$$= \sum_{k=1}^{l} \left|m_k^\lambda - \tilde{m}_k^\lambda\right| - \left|m_i^\lambda - \tilde{m}_i^\lambda\right| + \sum_{k=1}^{p} \theta_k^\lambda \left|m_i^\lambda - \tilde{m}_i^\lambda\right| + \sum_{k>p} \theta_k^\lambda \left(m_i^\lambda + \tilde{m}_i^\lambda\right) + \sum_{k>l} \left(m_k^\lambda + \tilde{m}_k^\lambda\right)$$

$$= \sum_{k=1}^{l} \left|m_k^\lambda - \tilde{m}_k^\lambda\right| + |m_i^\lambda - \tilde{m}_i^\lambda| \left(\sum_{k=1}^{p} \theta_k^\lambda - 1\right) + \left(m_i^\lambda + \tilde{m}_i^\lambda\right) \sum_{k>p} \theta_k^\lambda + \sum_{k>l} \left(m_k^\lambda + \tilde{m}_k^\lambda\right).$$

Notice that the last two sums are the tails of convergent series, note also that $l \to \infty$ whenever $p \to \infty$. We thus conclude making p tend to infinity.

Finally, to prove (A.10) we consider the permutation σ as in (A.11) with $p = v$ and $l := l(m)$. Recall (A.13), we have

$$\begin{aligned} \delta_\lambda(f_{i\psi_u(\theta)}(m), f_{i\psi_v(\theta)}(m)) &= \delta_\lambda(f_{i\psi_v(\psi_u(\theta))}(m), f_{i\psi_v(\theta)}(m)) \\ &\leq \sum_{k\geq 1} \left| [f_{i\psi_v(\psi_u(\theta))}(m)]_{\sigma(k)}^\lambda - [f_{i\psi_v(\theta)}(m)]_{\sigma(k)}^\lambda \right| \\ &\leq \sum_{k=u+1}^{v} \theta_k^\lambda m_i^\lambda + 2 \sum_{k>l} m_k^\lambda. \end{aligned}$$

We used that $[\psi_v(\psi_u(\theta))]_k = 0$ for $k = u+1, \cdots, v$. Since $m \in \ell_\lambda$, we conclude making l tend to infinity.

□

Lemma A.3. *Consider m, $\tilde{m} \in S^\downarrow$ and $1 \leq i < j < \infty$. Recall the definition of d (2.9), δ_λ (2.10), $c_{ij}(m)$ and $f_{i\theta}(m)$ (2.8) and $\psi_n(\theta)$ (3.2). For $\lambda \in (0,1)$ and for all $m, \tilde{m} \in \ell_\lambda$ there exists a positive constant C depending on λ such that*

$$d(m, \tilde{m}) \leq \delta_1(m, \tilde{m}) \leq C(\|m\|_1^{1-\lambda} \vee \|\tilde{m}\|_1^{1-\lambda})\, \delta_\lambda(m, \tilde{m}). \tag{A.14}$$

Next,

$$d(c_{ij}(m), m) \leq \frac{3}{2} 2^{-i} m_j, \qquad \sum_{1\leq k<l<\infty} d(c_{kl}(m), m) \leq \frac{3}{2} \|m\|_1, \tag{A.15}$$

$$d(c_{ij}(m), c_{ij}(\tilde{m})) \leq (2^i + 2^j) d(m, \tilde{m}). \tag{A.16}$$

$$d(f_{i\theta}(m), m) \leq 2(1-\theta_1)2^{-i} m_i, \tag{A.17}$$

$$d(f_{i\theta}(m), f_{i\theta}(\tilde{m})) \leq C(\|m\|_1^{1-\lambda} \vee \|\tilde{m}\|_1^{1-\lambda})\, \delta_\lambda(m, \tilde{m}), \tag{A.18}$$

$$d(f_{i\theta}(m), f_{i\psi_n(\theta)}(m)) \leq m_i \sum_{k>n} \theta_k. \tag{A.19}$$

Proof. The first inequality in (A.14) follows readily from the definition of d and the second one comes from (A.3), with $\alpha = 1 - \lambda$ and $\beta = \lambda$. The inequalities (A.15) and (A.16) involving d are proved in [8, Corollary 3.2.].

We prove (A.17) exactly as (A.7). Consider p, l and the permutation σ defined by (A.11), from (A.1) and since $i \leq l + 1 \leq l + p$, we obtain

$$\begin{aligned} d(f_{i\theta}(m), m) &\leq \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+p-1} + \sum_{k\geq l+p}\right) 2^{-k} |f_k - m_k| \\ &\leq (1-\theta_1)2^{-i} m_i + \sum_{k=l+1}^{l+p-1} 2^{-k} |\theta_{k-l+1} m_i - m_k| + \sum_{k\geq l+p} 2^{-k} |f_k - m_k| \end{aligned}$$

$$\begin{aligned} &\leq (1-\theta_1)2^{-i}m_i + \left(\sum_{k=2}^{p} 2^{-i}\theta_k m_i + \sum_{k=l+1}^{l+p-1} m_k\right) + \sum_{k\geq l+p} 2^{-i}\left(f_k + m_k\right) \\ &\leq (1-\theta_1)2^{-i}m_i + 2^{-i}m_i \sum_{k=2}^{\infty}\theta_k + 2\sum_{k>l} m_k. \end{aligned}$$

Since $m \in \ell_1$, we conclude using (2.4) and making l tend to infinity.

Next, we prove (A.18) as (A.9) using δ_1. Consider p, l and the permutations σ and $\tilde{\sigma}$ defined by (A.11). Recall (A.13), using (A.14) then (A.2) (applied to δ_1) and since, $i \leq l+1 \leq l+p$ we obtain

$$\begin{aligned} &d(f_{i\theta}(m), f_{i\theta}(\tilde{m})) \\ &\quad\leq \delta_1(f_{i\theta}(m), f_{i\theta}(\tilde{m})) \;\leq\; \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+p-1} + \sum_{k\geq l+p}\right)\left|f_k - \tilde{f}_k\right| \\ &\quad\leq \sum_{k=1}^{l}|m_k - \tilde{m}_k| + (\theta_1 - 1)\,|m_i - \tilde{m}_i| + \sum_{k=l+1}^{l+p-1}\theta_{k-l+1}\,|m_i - \tilde{m}_i| + \sum_{k\geq l+p}\left(f_k + \tilde{f}_k\right) \\ &\quad\leq \sum_{k=1}^{l}|m_k - \tilde{m}_k| + |m_i - \tilde{m}_i|\left(\sum_{k=1}^{p}\theta_k - 1\right) + (m_i + \tilde{m}_i)\sum_{k>p}\theta_k + \sum_{k>l}(m_k + \tilde{m}_k) \\ &\quad\leq \sum_{k=1}^{l}|m_k - \tilde{m}_k| + (m_i + \tilde{m}_i)\sum_{k>p}\theta_k + \sum_{k>l}(m_k + \tilde{m}_k)\,. \end{aligned}$$

We used that for $k \geq l+p$, f_k contains all the remaining fragments of m_i and the particles m_j with $j > l$ and (2.4). Since $m, \tilde{m} \in \ell_1$ we conclude making p tend to infinity and using (A.14).

Finally, for inequality (A.19), let $i \geq 1$, $p \geq 1$ and $l := l_p(m) = \min\{k \geq 1 : m_k \leq (\theta_n/p)m_i\}$ and consider σ, the finite permutation of $\mathbb{N}$ that achieves:

$$\begin{aligned} (f_k)_{k\geq 1} &:= \left([f_{i\theta}(m)]_{\sigma(k)}\right)_{k\geq 1} \\ &= (m_1, \cdots, m_{i-1}, \theta_1 m_i, \cdots, \theta_n m_i, m_{i+1}, \cdots, m_{l-1}, m_l, [f_{i\theta}(m)]_{l+n}, \cdots). \end{aligned} \tag{A.20}$$

Thus, from (A.14) and (A.2), and since $i \leq l+1 \leq l+n+1$, we deduce

$$\begin{aligned} d\left((f_{i\theta}(m), f_{i\psi_n(\theta)}(m)\right) &\leq \delta_1\left((f_{i\theta}(m), f_{i\psi_n(\theta)}(m)\right) \;=\; \sum_{k\geq 1}\left|[f_{i\theta}(m)]_k - [f_{i\psi_n(\theta)}(m)]_k\right| \\ &\leq \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+n-1} + \sum_{k\geq l+n}\right)\left|[f_{i\theta}(m)]_{\sigma(k)} - [f_{i\psi_n(\theta)}(m)]_{\sigma(k)}\right| \\ &\leq \sum_{k>n}\theta_k m_i + 2\sum_{k>l} m_k. \end{aligned}$$

The last sum being the tail of a convergent series we conclude making $l \to \infty$.

This concludes the proof of Lemma A.3. □

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Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050. Université Paris-Est. 61, avenue du Général de Gaulle, 94010 Créteil Cédex

E-mail address: eduardo.cepeda@math.cnrs.fr