A PRIORI ESTIMATES FOR SOLUTIONS OF A NONLINEAR DISPERSIVE EQUATION

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Abstract. In this work we obtain some a priori estimates for a higher order Schrödinger equation and in particular we obtain some a priori estimates for the modified Korteweg-de Vries equation.

1. Introduction

In this paper we will describe some results on the growth of Sobolev norms for solutions of the initial value problem (IVP)

$$\begin{cases} \partial_t u + ia\,\partial_x^2 u + b\,\partial_x^3 u + ic\,|u|^2 u + d\,|u|^2\partial_x u + e\,u^2\partial_x \bar{u} = 0, & x, t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where u is a complex valued function and a, b, c, d and e are real parameters with $be \neq 0$.

This model was proposed by Hasegawa and Kodama in [11, 14] to describe the nonlinear propagation of pulses in optical fibers. In literature, this model is called as a higher order nonlinear Schrödinger equation or also Airy-Schrödinger equation.

We consider the following gauge transformation

$$v(x,t) = \exp\left(i\lambda\, x + i(a\,\lambda^2 - 2b\,\lambda^3)t\right) u(x + (2a\,\lambda - 3b\lambda^2)t, t), \tag{1.2}$$

then, u solves (1.1) if and only if v satisfies the IVP

$$\begin{cases} \partial_t v + i(a - 3\lambda\, b)\partial_x^2 v + b\,\partial_x^3 v + i(c - \lambda(d-e))\,|v|^2 v + d\,|v|^2\partial_x v + e\,v^2\partial_x \bar{v} = 0, \\ v(x,0) = \exp(i\lambda\, x)\, u(x,0). \end{cases} \tag{1.3}$$

Thus, if we take $\lambda = a/3b$ in (1.2) and $c = (d-e)a/3b$, then the function

$$v(x,t) = \exp\left(i\frac{a}{3b}\,x + i\frac{a^3}{27b^2}\,t\right) u\left(x + \frac{a^2}{3b}\,t, t\right), \tag{1.4}$$

satisfies the complex modified Korteweg-de Vries type equation (complex mKdV)

$$\begin{cases} \partial_t v + b\,\partial_x^3 v + d\,|v|^2\partial_x v + e\,v^2\partial_x \bar{v} = 0, \\ v(x,0) = \exp(ia\,x/3b)\,u(x,0). \end{cases} \tag{1.5}$$

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It was shown in [15] that the flow associated to the IVP (1.1) leaves the following quantity

$$I_1(u) = \int_{\mathbb{R}} |u|^2(x,t)\,dx, \tag{1.6}$$

conserved in time. Also, when $be \neq 0$ we have the following conserved quantity

$$I_2(u) = k_1 \int_{\mathbb{R}} |\partial_x u|^2(x,t)\,dx + k_2 \int_{\mathbb{R}} |u|^4(x,t)dx + k_3 \operatorname{Im} \int_{\mathbb{R}} u(x,t)\partial_x \overline{u(x,t)}dx, \tag{1.7}$$

where $k_1 = 3be$, $k_2 = -e(e+d)/2$ and $k_3 = (3bc - a(e+d))$. We may suppose $k_3 = 0$. In fact, when $k_3 \neq 0$ we can take in the gauge transformation (1.2) $\lambda = -\frac{k_3}{6be}$. Then, u solves (1.1) if and only if v satisfies (1.3) and in this new IVP we have the constant $k_3 = 0$. The conserved quantity (1.7) with $k_3 = 0$, gives

$$\int_{\mathbb{R}} |\partial_x u|^2(x,t)\,dx + c\int_{\mathbb{R}} |u|^4(x,t)dx = \int_{\mathbb{R}} |\partial_x u|^2(x,0)\,dx + c\int_{\mathbb{R}} |u|^4(x,0)dx. \tag{1.8}$$

The main result in this work is

Theorem 1.1. *Let $u \in \mathcal{C}(\mathbb{R}, H^1)$ be the solution of IVP (1.5) and $0 \leq \theta < 1/2$, then*

$$\|D_x^\theta u(t)\|_{L_x^2} \leq c\|D_x^\theta u(0)\|_{L_x^2} \exp\{ct(\|D_x^{1/4} u(0)\|_{L_x^2}^4 + \|u(0)\|_{L_x^2}^6 + 1)\}, \tag{1.9}$$

where c is a constant.

2. Growth of Sobolev norms

In the following proposition we obtain a sharp estimate for $|\ \|u_x(t)\|_{L^2} - \|u_x(0)\|_{L^2}|$, we will need the following elemental lemma

Lemma 2.1. *Let $x \geq 0$, $a \geq 0$ and $b \geq 0$, if $x^2 - ax - b \leq 0$ then*

$$0 \leq x \leq a + b^{1/2}. \tag{2.10}$$

Proposition 2.2. *Let $u_0 \in H^1$ and $u(t)$ the solution of IVP (1.1) with $k_3 = 0$, then*

$$|\ \|u_x(t)\|_{L^2} - \|u_x(0)\|_{L^2}| \leq c\|u(0)\|_{L^2}^3.$$

Proof. If $c < 0$ in (1.8), Gagliardo-Nirenberg's inequality yields

$$\begin{aligned}\int_{\mathbb{R}} |\partial_x u|^2(x,t)\,dx \leq& |c| \int_{\mathbb{R}} |u|^4(x,t)dx + \int_{\mathbb{R}} |\partial_x u|^2(x,0)\,dx \\ \leq& |c|\,\|u(0)\|_{L^2}^3 \|u_x(t)\|_{L^2} + \int_{\mathbb{R}} |\partial_x u|^2(x,0),\end{aligned}$$

therefore by (2.10)

$$\|u_x(t)\|_{L^2} \leq \|u_x(0)\|_{L^2} + |c|\,\|u(0)\|_{L^2}^3.$$

If $c>0$ in (1.8), applying Gagliardo-Nirenberg's inequality we have

$$\begin{aligned}\int_{\mathbb{R}} |\partial_x u|^2(x,t)\,dx &\le c\int_{\mathbb{R}} |u|^4(x,0)dx + \int_{\mathbb{R}} |\partial_x u|^2(x,0)\,dx\\ &\le c\,\|u(0)\|_{L^2}^3\|u_x(0)\|_{L^2} + \int_{\mathbb{R}} |\partial_x u|^2(x,0),\end{aligned}\tag{2.11}$$

similarly we get

$$\begin{aligned}\int_{\mathbb{R}} |\partial_x u|^2(x,0)\,dx &\le c\int_{\mathbb{R}} |u|^4(x,t)dx + \int_{\mathbb{R}} |\partial_x u|^2(x,t)\,dx\\ &\le c\,\|u(0)\|_{L^2}^3\|u_x(t)\|_{L^2} + \int_{\mathbb{R}} |\partial_x u|^2(x,t),\end{aligned}$$

this implies that

$$\begin{aligned}\|u_x(0)\|_{L^2} &\le c^{1/2}\,\|u(0)\|_{L^2}^{3/2}\|u_x(t)\|_{L^2}^{1/2} + \|u_x(t)\|_{L^2}\\ &\le \frac{c}{2}\|u(0)\|_{L^2}^3 + \frac{1}{2}\|u_x(t)\|_{L^2} + \|u_x(t)\|_{L^2},\end{aligned}\tag{2.12}$$

using (2.12) in (2.11) it follows that

$$\|u_x(t)\|_{L^2}^2 \le c\,\|u(0)\|_{L^2}^3\left(\frac{c}{2}\|u(0)\|_{L^2}^3 + \frac{3}{2}\|u_x(t)\|_{L^2}\right) + \|u_x(0)\|_{L^2}^2,\tag{2.13}$$

and again using (2.10) we obtain

$$\|u_x(t)\|_{L^2} \le \|u_x(0)\|_{L^2} + c\,\|u(0)\|_{L^2}^3.$$

Analogously when $c\in\mathbb{R}\setminus\{0\}$, by symmetry we get

$$\|u_x(0)\|_{L^2} \le \|u_x(t)\|_{L^2} + c\,\|u(0)\|_{L^2}^3.$$

This concludes the proof of the proposition. □

Lemma 2.3. *Let $u(x,t)=U(t)u_0(x)$ be the solution of the IVP (1.1) with $c=d=e=0$, we have the following local smoothing effect*

$$\|\partial_x U(t)u_0\|_{L_x^\infty L_T^2} \le c\|u_0\|_{L^2},\tag{2.14}$$

and the dual version

$$\left\|\partial_x\int_0^t U(t-t')f(t',x)dt'\right\|_{L_x^2} \le c\|f\|_{L_x^1L_t^2}.\tag{2.15}$$

We have also the maximal function estimate

$$\|U(t)u_0\|_{L_x^4L_T^\infty} \le c\|u_0\|_{H^{1/4}}.\tag{2.16}$$

And

$$\left\|\partial_x^2\int_0^t U(t-t')f(t',x)dt'\right\|_{L_x^\infty L_t^2} \le c\|f\|_{L_x^1L_t^2}.\tag{2.17}$$

Proof. For the proof of the inequalities (2.14)-(2.16) see [4] or [12] or [20]. For the proof of inequality (2.17) see [12]. □

The following lemmas were proved in [12].

Lemma 2.4. *Let be $\theta\in(0,1)$, $1<p_1,p_2,q_1,q_2<\infty$, such that $1/p_1+1/p_2=1$, $1/q_1+1/q_2=1/2$, then*

$$\|D_x^\theta(uv)-uD_x^\theta v-vD_x^\theta u\|_{L_x^1L_t^2}\le c\|u\|_{L_x^{p_1}L_t^{q_1}}\|D_x^\theta v\|_{L_x^{p_2}L_t^{q_2}}. \tag{2.18}$$

Lemma 2.5. *Let be $\theta\in(0,1)$, $1<p_1,p_2,p,r,r_2<\infty$, $r_1\in(1,\infty]$ such that $1/p_1+1/p_2=1/p$, $1/r_1+1/r_2=1/r$ then*

$$\|D_x^\theta F(f)\|_{L_x^pL_t^r}\le c\|F'(f)\|_{L_x^{p_1}L_t^{r_1}}\|D_x^\theta f\|_{L_x^{p_2}L_t^{r_2}}. \tag{2.19}$$

Without loss of generality we will restringed our attention, for the modified Korteweg-de Vries equation, mKdV ($\alpha=c=d=0$ and $u_0(x)\in\mathbb{R}$ in (1.1)).

Initially we considered the generalized KdV equation

$$\begin{cases}\partial_t u+\partial_x^3u+\partial_x u^k=0, & x,t\in\mathbb{R},\ k\ge 2,\ k\in\mathbb{Z}^+,\\ u(x,0)=u_0(x),\end{cases} \tag{2.20}$$

and the equivalent integral equality

$$u(t)=U(t)u_0-\int_0^t U(t-t')\partial_x u^k(t')dt'. \tag{2.21}$$

The IVP (2.20) have the following conserved quantities

$$I_1(u)=\|u(t)\|_{L_x^2}^2,\quad I_2(u)=\|u_x(t)\|_{L_x^2}^2+c_k\|u(t)\|_{L_x^{k+1}}^{k+1}. \tag{2.22}$$

Theorem 2.6. *Let $u\in\mathcal{C}(\mathbb{R},H^1)$ be the solution of IVP (2.20) and $0\le\theta<1/2$, then*

$$\|D_x^\theta u(t)\|_{L_x^2}^2\le c\|D_x^\theta u(0)\|_{L_x^2}^2+c\|u(t')\|_{L_x^{2(k-1)}L_t^\infty}^{2(k-1)}\int_0^t\|D_x^\theta u(t')\|_{L_x^2}^2dt'. \tag{2.23}$$

Proof. A straightforward calculation yields

$$\int_0^t\int_{\mathbb{R}}D_x^\theta u[D_x^\theta u_t+D_x^\theta(u^k)_x]=\int_0^t\int_{\mathbb{R}}D_x^\theta uD_x^\theta[u_t+u_{xxx}+(u^k)_x]=0,$$

therefore

$$2\int_0^t\int_{\mathbb{R}}D_x^\theta uD_x^\theta u_t=\int D_x^\theta u(t)^2-\int D_x^\theta u(0)^2=-2\int_0^t\int_{\mathbb{R}}D_x^\theta uD_x^\theta(u^k)_x:=2J$$

and using (2.21) and Lemma 2.3 we obtain

$$\begin{aligned}J=&\int_0^t\int_{\mathbb{R}}\partial_xU(t)D_x^\theta u_0D_x^\theta u^k+\int_0^t\int_{\mathbb{R}}\partial_x^2\int_0^{t'}U(t'-\tau)D_x^\theta u^k(\tau)d\tau D_x^\theta u^k\\ \le& c\|\partial_xU(t)D_x^\theta u_0\|_{L_x^\infty L_t^2}\|D_x^\theta u^k\|_{L_x^1L_t^2}+\left\|\partial_x^2\int_0^{t'}U(t'-\tau)D_x^\theta u^k(\tau)d\tau\right\|_{L_x^\infty L_t^2}\|D_x^\theta u^k\|_{L_x^1L_t^2}\\ \le& c\|D_x^\theta u_0\|_{L_x^2}\|D_x^\theta u^k\|_{L_x^1L_t^2}+c\|D_x^\theta u^k\|_{L_x^1L_t^2}\|D_x^\theta u^k\|_{L_x^1L_t^2}\\ \le& c\|D_x^\theta u_0\|_{L_x^2}^2+c\|D_x^\theta u^k\|_{L_x^1L_t^2}^2\\ \le& c\|D_x^\theta u_0\|_{L_x^2}^2+c\|u(t')\|_{L_x^{2(k-1)}L_t^\infty}^{2(k-1)}\int_0^t\|D_x^\theta u\|_{L_x^2}^2,\end{aligned}$$

where in the last inequality we used (2.18) and (2.19) (see Remark 2.7), thus

$$\|D_x^\theta u(t)\|_{L_x^2}^2 \le c\|D_x^\theta u(0)\|_{L_x^2}^2 + c\|u(t')\|^{2(k-1)}_{L_x^{2(k-1)}L_t^\infty}\int_0^t \|D_x^\theta u(t')\|_{L_x^2}^2\,dt'.$$

□

If $k=3$ (mKdV) in (2.23) we have

$$\|D_x^\theta u(t)\|_{L_x^2}^2 \le c\|D_x^\theta u(0)\|_{L_x^2}^2 + c\|u(t')\|^4_{L_x^4L_t^\infty}\int_0^t \|D_x^\theta u(t')\|_{L_x^2}^2\,dt'. \tag{2.24}$$

But if $u \in \mathcal{C}(\mathbb{R}, H^1)$, then $u(t')$ is continuous in $[0,t]$, therefore $\|u(t')\|_{L_x^4L_t^\infty} = \|u(t_0)\|_{L_x^4}$ for some $t_0 \in [0,t]$, then we have

$$\|u(t')\|_{L_x^4L_t^\infty} = \|u(t_0)\|_{L_x^4} \le \|u(t')\|_{L_t^\infty L_x^4} \le \|u(t')\|_{L_x^4L_t^\infty}. \tag{2.25}$$

Hence by (2.24) and (2.25) we obtain

$$\|D_x^\theta u(t)\|_{L_x^2}^2 \le c\|D_x^\theta u(0)\|_{L_x^2}^2 + c\|u(t')\|^4_{L_t^\infty L_x^4}\int_0^t \|D_x^\theta u(t')\|_{L_x^2}^2\,dt'. \tag{2.26}$$

Remark 2.7. *1) The inequality (2.18) is valid with $2<q_1,q_2<\infty$, $1/q_1+1/q_2=1/2$, therefore for all $q_1>2$, $k\ge 2$ and $0\le\theta<1/2$ we get*

$$\begin{aligned}
\|D_x^\theta u^k\|_{L_x^1L_t^2} &\le c\|u^{k-1}\|_{L_x^2L_t^{q_1}}\|D_x^\theta u\|_{L_x^2L_t^{\frac{2}{(1-2/q_1)}}} + \|u\|_{L_x^{2(k-1)}L_t^\infty}\|D_x^\theta u^{k-1}\|_{L_x^{\frac{2(k-1)}{(2k-3)}}L_t^2}\\
&\le ct^{1/q_1}\|u^{k-1}\|_{L_x^2L_t^\infty}\|D_x^\theta u\|_{L_x^2L_t^{\frac{2}{(1-2/q_1)}}} + c\|u\|_{L_x^{2(k-1)}L_t^\infty}\|u^{k-2}\|_{L_x^{\frac{2(k-1)}{(k-2)}}L_t^2}\|D_x^\theta u\|_{L_x^2L_t^2}\\
&\le ct^{1/q_1}\|u\|^{k-1}_{L_x^{2(k-1)}L_t^\infty}\|D_x^\theta u\|_{L_x^2L_t^{\frac{2}{(1-2/q_1)}}} + c\|u\|^{k-1}_{L_x^{2(k-1)}L_t^\infty}\|D_x^\theta u\|_{L_x^2L_t^2}.
\end{aligned} \tag{2.27}$$

Now if $\theta\in[0,1/2)$, $u\in\mathcal{C}(\mathbb{R},H^1)$ and $\tau_n\to\tau$, then

$$\left|D_x^\theta u(\tau_n,\xi) - D_x^\theta u(\tau,\xi)\right| \le c\|u(\tau_n)-u(\tau)\|_{H^1}\to 0,$$

thus $D_x^\theta u(\tau,\xi)$ is continuous in the variable τ, and in similar way as in (2.25) we have $\|D_x^\theta u(t')\|_{L_x^2L_t^\infty} = \|D_x^\theta u(t')\|_{L_t^\infty L_x^2}$. Hence taking the limit when $q_1\to\infty$ in (2.27), Lebesgue's dominated convergence theorem yields

$$\|D_x^\theta u^k\|_{L_x^1L_t^2} \le c\|u\|^{k-1}_{L_x^{2(k-1)}L_t^\infty}\|D_x^\theta u\|_{L_x^2L_t^2}.$$

2) Seeking (2.22) and (2.23) we observe that $2(k-1)=k+1$ if and only if $k=3$ (mKdV).

Theorem 2.8. *Let $u\in\mathcal{C}(\mathbb{R},H^1)$ be the solution of IVP (2.20) with $k=3$ and $0\le\theta<1/2$, then*

$$\|D_x^\theta u(t)\|_{L_x^2} \le c\|D_x^\theta u(0)\|_{L_x^2}\exp\Big\{ct(\|D_x^{1/4}u(0)\|^4_{L_x^2} + \|u(0)\|^6_{L_x^2}+1)\Big\}. \tag{2.28}$$

Proof. We consider two cases:

Case 1) : if $\|D_x u(0)\|_{L^2_x} \leq 1$.

From (1.8), Proposition 2.2 and the Sobolev embedding $\|v\|_{L^4_x} \leq c\|v\|_{H^{1/4}}$, it follows that

$$\begin{aligned}
\int u^4(x,t) \leq & \int u^4(x,0) + c\|D_x u(t)\|^2_{L^2_x} + c\|D_x u(0)\|^2_{L^2_x} \\
\leq & \int u^4(x,0) + c\left(2\|D_x u(0)\|^2_{L^2_x} + c\|u(0)\|^6_{L^2_x}\right) + c\|D_x u(0)\|^2_{L^2_x} \\
\leq & c\left(\|D_x^{1/4} u(0)\|^4_{L^2_x} + \|u(0)\|^6_{L^2_x} + 1\right). \qquad (2.29)
\end{aligned}$$

The inequality (2.28) is a direct consequence of (2.26), (2.29) and Gronwall's inequality.

Case 2) : if $\|D_x u(0)\|_{L^2_x} > 1$.

By scaling we consider the solution

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^3 t), \quad \lambda = \frac{1}{\|D_x u(0)\|^{2/3}_{L^2_x}}.$$

We have

$$\|D_x^s u_\lambda(t)\|_{L^2_x} = \lambda^{1/2+s}\|D_x^s u(\lambda^3 t)\|_{L^2_x}, \qquad (2.30)$$

therefore

$$\|D_x u_\lambda(0)\|_{L^2_x} = \lambda^{3/2}\|D_x u(0)\|_{L^2_x} = 1,$$

hence by Case 1) we get

$$\|D_x^\theta u_\lambda(t)\|_{L^2_x} \leq c\|D_x^\theta u_\lambda(0)\|_{L^2_x} \exp\left\{ct\left(\|D_x^{1/4} u_\lambda(0)\|^4_{L^2_x} + \|u_\lambda(0)\|^6_{L^2_x} + 1\right)\right\}. \qquad (2.31)$$

And using (2.30) we obtain for all $t \in \mathbb{R}$:

$$\begin{aligned}
\|D_x^\theta u(\lambda^3 t)\|_{L^2_x} \leq & c\|D_x^\theta u(0)\|_{L^2_x} \exp\left\{ct\left(\lambda^3\|D_x^{1/4} u(0)\|^4_{L^2_x} + \lambda^3\|u(0)\|^6_{L^2_x} + 1\right)\right\} \\
\leq & c\|D_x^\theta u(0)\|_{L^2_x} \exp\left\{ct\left(\|D_x^{1/4} u(0)\|^4_{L^2_x} + \|u(0)\|^6_{L^2_x} + 1\right)\right\}.
\end{aligned}$$

This completes the proof of the lemma. □

In the next section we prove a refined a priori estimate.

3. Growth of Sobolev norms of low frequency solutions

In this section we will prove the following theorem

Theorem 3.1. *Let $u_0 \in H^s$, $s \geq 1/4$, $v_0(x) = \left(\chi_{\{|\xi|<N\}}\widehat{u}_0\right)^\vee(x)$ and $v(t)$ a solution of IVP (1.1) with initial date v_0 and $k_3 = 0$, then there exist a $\theta_0 = \theta_0(u_0, v(t)) \in (0,1)$ such that for all $\theta \in [0,1]$*

$$\|v(t)\|_{H^\theta} \leq \|v(0)\|_{H^\theta} + \frac{\langle N\rangle^{\theta_0}}{\sqrt{e\theta_0 \log\langle N^2\rangle}} P(\|v(0)\|_{L^2}), \qquad (3.32)$$

where P(x) is a polinomio of grau three.

To prove Theorem 3.1 we need the following result

Lemma 3.2. *Let $v \in H^1$, then for $\theta \in [0,1)$ we have*

$$\|v\|_{H^\theta}^2 \leq \|v\|_{L^2}^2 \exp\left\{\frac{\theta \int (1+\xi^2)^\theta \log(1+\xi^2)|\widehat{v}(\xi)|^2 d\xi}{\|v\|_{H^\theta}^2}\right\}. \tag{3.33}$$

Proof. Let $F(\theta) = \|v\|_{H^\theta}^2 \neq 0$ for $\theta \in [0,1)$ and $G(\theta) = F'(\theta)/F(\theta)$, then

$$F'(\theta) = \int (1+\xi^2)^\theta \log(1+\xi^2)|\widehat{v}(\xi)|^2 d\xi, \;\; F''(\theta) = \int (1+\xi^2)^\theta \log^2(1+\xi^2)|\widehat{v}(\xi)|^2 d\xi$$

and using Cauchy-Schwartz

$$F'(\theta) \leq F(\theta)^{1/2} F''(\theta)^{1/2}.$$

Thus

$$G'(\theta) = \frac{F(\theta)F''(\theta) - F'(\theta)^2}{F(\theta)^2} \geq 0.$$

Hence $G(\theta)$ is increasing. Now we define $H(\theta) = \log F(\theta)$, using the mean value theorem for $\theta \in (0,1)$ we get

$$\log F(\theta) - \log F(0) \leq \theta G(\gamma\theta) \leq \theta G(\theta) = \theta \frac{F'(\theta)}{F(\theta)},$$

where $\gamma \in (0,1]$. Therefore

$$F(\theta) \leq F(0) \exp\left\{\frac{\theta F'(\theta)}{F(\theta)}\right\}.$$

□

Proof of Theorem 3.1. If $\theta = 0$ or $\theta = 1$ is a obvious consequence of (1.6) and Proposition 2.2. Now for t fixed, we define the function

$$f_t(\delta) := \|v(t)\|_{H^\delta}^2 - \|v(0)\|_{H^\delta}^2, \quad \delta \in [0,1],$$

and we consider

$$f_t(\delta_0) = \max_{\delta \in [0,1]} f_t(\delta).$$

If $\delta_0 = 0$ or $\delta_0 = 1$, we have (3.32). If $\delta_0 \in (0,1)$ then $f_t'(\delta_0) = 0$, therefore

$$\int (1+\xi^2)^{\delta_0} \log(1+\xi^2)|\widehat{v}(t,\xi)|^2 d\xi = \int (1+\xi^2)^{\delta_0} \log(1+\xi^2)|\widehat{v}(0,\xi)|^2 d\xi. \tag{3.34}$$

From (3.33) and (3.34) we have

$$\begin{aligned}\|v(t)\|^2_{H^{\delta_0}} \leq& \|v(0)\|^2_{L^2}\,\exp\Big\{\frac{\delta_0\int(1+\xi^2)^{\delta_0}\log(1+\xi^2)|\widehat{v}(t,\xi)|^2d\xi}{\|v(t)\|^2_{H^{\delta_0}}}\Big\}\\ =&\|v(0)\|^2_{L^2}\,\exp\Big\{\frac{\delta_0\int(1+\xi^2)^{\delta_0}\log(1+\xi^2)|\widehat{v}(0,\xi)|^2d\xi}{\|v(t)\|^2_{H^{\delta_0}}}\Big\}\\ \leq&\|v(0)\|^2_{L^2}\,\exp\Big\{\frac{\delta_0\log(1+N^2)\int(1+\xi^2)^{\delta_0}|\widehat{v}(0,\xi)|^2d\xi}{\|v(t)\|^2_{H^{\delta_0}}}\Big\}\\ =&\|v(0)\|^2_{L^2}\,(1+N^2)^{\frac{\delta_0\|v(0)\|^2_{H^{\delta_0}}}{\|v(t)\|^2_{H^{\delta_0}}}}.\end{aligned}$$

Let $\chi(t)=\|v(t)\|^2_{H^{\delta_0}}$, $L_0=\|v(0)\|^2_{L^2}$ and $q=\langle N^2\rangle^{\delta_0}$, then

$$\frac{\chi(t)}{L_0}\leq q^{\frac{\chi(0)}{\chi(t)}},$$

taking logaritmo in the above inequality we have

$$\begin{aligned}\frac{\chi(t)}{L_0}\log\frac{\chi(t)}{L_0}\leq&\frac{\chi(0)}{L_0}\log q\\ =&\frac{\frac{\chi(0)}{L_0}\log q}{\big(\frac{\chi(0)}{L_0}+k_0\big)\log\{\frac{\chi(0)}{L_0}+k_0\}}\Big(\frac{\chi(0)}{L_0}+k_0\Big)\log\Big\{\frac{\chi(0)}{L_0}+k_0\Big\},\end{aligned}\tag{3.35}$$

where we consider $k_0=q/(e\log q)$, let us now the function

$$f(x)=(x+k_0)\log\{x+k_0\}-x\log q,\quad x\geq 0,$$

this function have a minimum $x_{min}=q/e-k_0>0$ in $[0,\infty)$, equal to $f(x_{min})=-q/e+k_0\log q=0$, and by inequality (3.35) this implies

$$\frac{\chi(t)}{L_0}\log\frac{\chi(t)}{L_0}\leq\Big(\frac{\chi(0)}{L_0}+k_0\Big)\log\Big\{\frac{\chi(0)}{L_0}+k_0\Big\}.$$

As the function $g(x)=x\log x$, $x>0$ is no decreasing, by the definition of δ_0 we obtain (3.32). We concludes the proof of the theorem. □

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