Persistence of solutions to higher order nonlinear Schrödinger equation

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Abstract

Applying an Abstract Interpolation Lemma, we can show persistence of solutions of the initial value problem to higher order nonlinear Schrödinger equation, also called Airy-Schrödinger equation, in weighted Sobolev spaces $\mathcal{X}^{2,\theta}$, for $0 \leq \theta \leq 1$.

1 Introduction

Motivated by the difficulty question of how to show persistent properties of solutions to dispersive equations in the weighted Sobolev spaces, we proved an Abstract Interpolation Lemma. Then, applying this lemma we were able to show persistence for the so called Airy-Schrödinger equation in weighted Sobolev spaces $\mathcal{X}^{s,\theta}$, see definition in equation (1.4), for $0 \leq \theta \leq 1$ and $s = 2$.

Here, we have focus on the exponent of the weighted, that is, we have been concentrated on $\mathcal{X}^{2,\theta}$ for $\theta < 1$. In this direction our result is new, moreover to higher order nonlinear Schrödinger equations. We address the reader the paper of Nahas and Ponce [18] for similar results on the persistent properties of solutions to semi-linear Schrödinger equation in weighted Sobolev spaces. Although, that paper used different technics from ours.

1.1 Purpose and some results

In this paper we describe how to obtain some new results on the persistent properties in weighted Sobolev spaces for solutions of the initial value problem (IVP)

$$\begin{cases} \partial_t u + i\,a\,\partial_x^2 u + b\,\partial_x^3 u + i\,c\,|u|^2 u + d\,|u|^2 \partial_x u + e\,u^2 \partial_x \bar{u} = 0, \quad (t,x) \in \mathbb{R}^2, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

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where u is a complex-valued function, a, b, c, d and e are real parameters and u_0 is a given initial-data. This model was proposed by Hasegawa and Kodama in [12, 16] to describe the nonlinear propagation of pulses in optical fibers. In literature, it is called as a higher order nonlinear Schrödinger equation or also Airy-Schrödinger equation. Moreover, as we are going to show below in this first section, depending on the values of the constants a, b, c, d and e, (1.1) describes many interesting known problems.

It was shown in [17] that the flow associated to the IVP (1.1) leaves the following quantity

$$I_1(v(t)) := \int_{\mathbb{R}} |v(x,t)|^2\, dx, \tag{1.2}$$

conserved in time. Also, when $be \neq 0$ we have the following conserved quantity

$$\begin{aligned} I_2(v(t)) := C_1 \int_{\mathbb{R}} |\partial_x v(x,t)|^2\, dx + C_2 \int_{\mathbb{R}} |v(x,t)|^4\, dx \\ + C_3 \operatorname{Im} \int_{\mathbb{R}} v(x,t) \partial_x \overline{v(t,x)} dx, \end{aligned} \tag{1.3}$$

where $C_1 = 3\, b\, e$, $C_2 = -e\,(e+d)/2$ and $C_3 = (3\, b\, c - a\,(e+d))$.

Regarding the IVP (1.1) with $b \neq 0$, Laurey in [17] showed that the IVP is locally well-posed in $H^s(\mathbb{R})$ with $s > 3/4$, and using the quantities (1.2) and (1.3) she proved the global well-posedness in $H^s(\mathbb{R})$ with $s \geq 1$. In [22] Staffilani established the local well-posedness in $H^s(\mathbb{R})$ with $s \geq 1/4$, for the IVP associated to (1.1), improving Laurey's result.

In the problem (1.1), when a, b are real functions of t, ($b \neq 0$), was proved in [4] the local well-posedness in $H^s(\mathbb{R})$, for $s \geq 1/4$. Moreover, in [6] when $c = (d-e)\, a/3\, b$, global well-posedness was established in $H^s(\mathbb{R})$ with $s > 1/4$. Also, in [5] one has the unique continuation property for the solution of (1.1).

One stress the importance of the weighted Sobolev spaces. This question goes back to work of Kato [13], where the space $\mathcal{X}^{2r,r}$ for ($r = 1, 2, \ldots$) was first introduced to prove well-posedness with weight for the KdV and generalized KdV equations. Following Kato, we observe that functions in the Sobolev spaces H^s do not necessarily decay fast as $|x| \to \infty$. Therefore, since we want to prove well-posedness in spaces of fast-decaying functions, a simple choice is a weighted Sobolev space $H^s(\mathbb{R}) \cap L^2\big(\omega(x)\, dx\big)$ for some appropriated weight function ω, see [13].

One of the purposes of the present paper is to show well-posedness of (1.1), with $b \neq 0$, in the weighted Sobolev space $\mathcal{X}^{2,\theta}$, for $\theta \in [0,1]$. The most difficult question in this work is to prove the persistence in the weighted Sobolev space to the solution $u(t)$ of the IVP (1.1) given by the global theory in H^2, which seems to be so regular, but it is not the point of the paper. To establish this, we approximate the solution by a sequence of smoothing solutions of (1.1) (see Lemma 3.4). We show that this sequence belongs to a family $\mathcal{A}$ of functions (see conditions (2.9)-(2.12)) where is possible apply the Abstract Interpolation Lemma (Lemma 2.2), which permits to obtain the persistence in $\mathcal{X}^{2,\theta}$ for this sequence of smoothing solutions. Then passing to the limit in this sequence, we get the persistence for the solution $u(t)$ in $\mathcal{X}^{2,\theta}$ as desired. Moreover, since $\mathcal{X}^{s,1} \subseteq \mathcal{X}^{s,\theta}$, for all $s \in \mathbb{R}$ and $\theta \in [0,1]$, see Remark 1.1, we have also extended the well-posedness results in [13] for the KdV and generalized KdV obtained in the weighted Sobolev space $\mathcal{X}^{2,1}$. The authors would like to observe that, as it is known by them, it is the first time in literature, where the weighted Sobolev space $\mathcal{X}^{s,\theta}$ for $\theta \in [0,1]$ appears.

An outline of this paper follows. In the rest of this section we fix the notation, give the definition of well-posedness and present some background concerning the theory of well-posedness for the Airy-Schrödinger equation. The Abstract Interpolation Lemma is given at Section 2. In Section 3, first we show some conserved quantities, and prove a nonlinear estimate. Then, we formulate the approximated problems associated to the IVP (1.1) and prove Lemma 3.4, which is important to show Theorem 3.7 at the end of this section.

1.2 Notation and background

At this point we fix some functional notation used in the paper. By dx we denote the Lebesgue measure on $\mathbb{R}$ and, for $\theta \geq 0$,

$$d\mu_\theta(x) := (1+|x|^2)^\theta \, dx,$$
$$d\dot{\mu}_\theta(x) := |x|^{2\theta} \, dx$$

denote the Lebesgue-Stieltjes measures on $\mathbb{R}$. Hence, given a set X, a measurable function $f \in L^2(X; d\mu_\theta)$ means that

$$\|f\|^2_{L^2(X;d\mu_\theta)} = \int_X |f(x)|^2 \, d\mu_\theta(x) < \infty.$$

When $X = \mathbb{R}$, we write: $L^2(d\mu_\theta) \equiv L^2(\mathbb{R}; d\mu_\theta)$, and for simplicity

$$L^2 \equiv L^2(d\mu_0), \quad L^2(d\mu) \equiv L^2(d\mu_1).$$

Analogously, for the measure $d\dot{\mu}_\theta$. We will use the Lebesgue space-time $L^p_x \mathcal{L}^q_\tau$ endowed with the norm

$$\|f\|_{L^p_x \mathcal{L}^q_\tau} = \big\| \|f\|_{\mathcal{L}^q_\tau} \big\|_{L^p_x} = \Big(\int_{\mathbb{R}} \Big(\int_0^\tau |f(x,t)|^q dt \Big)^{p/q} dx \Big)^{1/p} \quad (1 \leq p, q < \infty).$$

When the integration in the time variable is on the whole real line, we use the notation $\|f\|_{L^p_x L^q_t}$. The notation $\|u\|_{L^p}$ is used when there is no doubt about the variable of integration. Similar notations when p or q are ∞. As usual, $H^s \equiv H^s(\mathbb{R})$, $\dot{H}^s \equiv \dot{H}^s(\mathbb{R})$ are the classic Sobolev spaces in $\mathbb{R}$, endowed respectively with the norms

$$\|f\|_{H^s} := \|\widehat{f}\|_{L^2(d\mu_s)}, \quad \|f\|_{\dot{H}^s} := \|\widehat{f}\|_{L^2(d\dot{\mu}_s)}.$$

In this work, we study the solutions of (1.1) in the Sobolev spaces with weight $\mathcal{X}^{s,\theta}$, defined as

$$\mathcal{X}^{s,\theta} := H^s \cap L^2(d\mu_\theta), \tag{1.4}$$

with the norm

$$\|f\|_{\mathcal{X}^{s,\theta}} := \|f\|_{H^s} + \|f\|_{L^2(d\mu_\theta)}.$$

Remark 1.1. *We remark that, $\mathcal{X}^{s,1} \subseteq \mathcal{X}^{s,\theta}$, for all $s \in \mathbb{R}$ and $\theta \in [0,1]$. Indeed, using Hölder's inequality*

$$\|f\|_{L^2(d\dot{\mu}_\theta)} \leq \|f\|_{L^2}^{1-\theta} \, \|f\|_{L^2(d\dot{\mu})}^{\theta}.$$

The following definition tell us in which sense we consider the well-posedness for the IVP (1.1).

Definition 1.2. *Let X be a Banach space and $T > 0$. We say that the IVP (1.1) is locally well-posed in X, if the solution u uniquely exists in certain time interval $[-T,T]$ (unique existence), describes a continuous curve in X in the interval $[-T,T]$ whenever initial data belongs to X (persistence), and varies continuously depending upon the initial data (continuous dependence), that is, continuity of the application*

$$u_0 \mapsto u \quad \textit{from } X \textit{ to } \mathcal{C}([-T,T];X).$$

Moreover, we say that the IVP (1.1) is globally well-posed in X if the same properties hold for all time $T > 0$. If some hypotheses in the definition of local well-posed fail, we say that the IVP is ill-posed.

Particular cases of (1.1) are the following:

- Cubic nonlinear Schrödinger equation (NLS), ($a = \pm 1$, $b = 0$, $c = -1$, $d = e = 0$).

  $$iu_t \pm u_{xx} + |u|^2 u = 0, \quad x, t \in \mathbb{R}. \tag{1.5}$$

  The best known local result for the IVP associated to (1.5) is in $H^s(\mathbb{R})$, $s \geq 0$, obtained by Tsutsumi [25]. Since the L^2 norm is preserved in (1.5), one has that (1.5) is globally well-posed in $H^s(\mathbb{R})$, $s \geq 0$.

- Nonlinear Schrödinger equation with derivative ($a = -1$, $b = 0$, $c = 0$, $d = 2e$).

  $$iu_t + u_{xx} + i\lambda(|u|^2 u)_x = 0, \quad x, t \in \mathbb{R}, \tag{1.6}$$

  where $\lambda \in \mathbb{R}$. The best known local result for the IVP associated to (1.6) is in $H^s(\mathbb{R})$, $s \geq 1/2$, obtained by Takaoka [24]. Colliander et al. [9] proved that (1.6) is globally well-posed in $H^s(\mathbb{R})$, $s > 1/2$.

- Complex modified Korteweg-de Vries (mKdV) equation ($a = 0$, $b = 1$, $c = 0$, $d = 1$, $e = 0$)

$$u_t + u_{xxx} + |u|^2 u_x = 0, \quad x, t \in \mathbb{R}. \tag{1.7}$$

  If u is real, (1.7) is the usual mKdV equation. Kenig et al. [15] proved that the IVP associated to it is locally well-posed in $H^s(\mathbb{R})$, $s \geq 1/4$ and Colliander et al. [10], proved that (1.7) is globally well-posed in $H^s(\mathbb{R})$, $s > 1/4$.

- When $a \neq 0$ and $b = 0$, we obtain a particular case of the well-known mixed nonlinear Schrödinger equation

$$u_t = iau_{xx} + \lambda(|u|^2)_x u + g(u), \quad x, t \in \mathbb{R}, \tag{1.8}$$

  where g satisfies some appropriated conditions and $\lambda \in \mathbb{R}$ is a constant. Ozawa and Tsutsumi in [19] proved that for any $\rho > 0$, there is a positive constant $T(\rho)$ depending only on ρ and g, such that the IVP (1.8) is locally well-posed in $H^{1/2}(\mathbb{R})$, whenever the initial data satisfies

$$\|u_0\|_{\mathrm{H}^{1/2}} \leq \rho.$$

There are other dispersive models similar to (1.1), see for instance [1, 8, 11, 20, 21, 23] and the references therein.

Remark 1.3. *1. We can suppose $C_3 = 0$ in (1.3). In fact, when $C_3 \neq 0$ we have the following gauge transformation*

$$v(x,t) = \exp\Big(ia\,x + i(a\,\alpha^2 + b\,\alpha^3)t\Big)\, u(x + (2a\,\alpha + 3b\alpha^2)t, t),$$

where

$$\alpha = \frac{3\,b\,c - a(d+e)}{6\,b\,e}.$$

Then, u solves (1.1) if and only if v satisfies the equation

$$\partial_t v + i(a + 3\alpha\, b)\partial_x^2 v + b\,\partial_x^3 v + i(c - \alpha(e-d))\,|v|^2 v + d\,|v|^2 \partial_x v + e\,v^2 \partial_x \bar{v} = 0,$$

and in this equation we have the factor $C_3 = 0$.

2. Let $c = (d-e)\,a/3\,b$ and $u(x,t)$ be a solution of (1.1). If we choose a new unknown function $v(x,t)$ related to u by the relation

$$v(x,t) = \exp\Big(i\frac{a}{3b}\,x + i\frac{a^3}{27b^2}\,t\Big)\, u(x + \frac{a^2}{3b}\,t, t).$$

Then, u solves (1.1) if and only if v satisfies the complex modified Korteweg-de Vries type equation

$$\partial_t v + b\,\partial_x^3 v + d\,|v|^2 \partial_x v + e\,v^2 \partial_x \bar{v} = 0.$$

2 The Abstract Interpolation Lemma

The aim of this section is to prove an interpolation lemma with weight concerning space time-value functions.

Let f be a function from $[-T,T]$ in $H^s(\mathbb{R})$ for each $T>0$ and $s>1/2$. We suppose that for all $t\in[-T,T]$, f satisfies the following conditions:

(i) For each $t\in[-T,T]$, $\mathcal{L}^1\big(\xi\in\mathbb{R}; f(t,\xi)\neq 0\big)>0$, where $\mathcal{L}^1\big(E\big)$ is the Lebesgue measure of measurable set $E\subset\mathbb{R}$.

(ii) There exist constants $C_0,\tilde{C_0}>0$, $\tilde{C_1}\geq 0$ (independent of f, t), such that

$$\|f(t)\|_{L^2}^2 \leq C_0\,\|f(0)\|_{L^2}^2, \tag{2.9}$$

$$\|f(t)\|_{L^2(d\dot{\mu})}^2 \leq \tilde{C_0}\,\|f(0)\|_{L^2(d\dot{\mu})}^2 + \tilde{C_1}. \tag{2.10}$$

(iii) For all $\theta\in[0,1]$, there exist $\Theta>0$ (independent of f, t) and $\gamma_1\in(0,1)$, such that

$$\int_{\{|f(t)|^2<\Theta\}} |f(t)|^2\,d\dot{\mu}_\theta \leq \gamma_1\int_{\mathbb{R}} |f(t)|^2\,d\dot{\mu}_\theta. \tag{2.11}$$

(iv) There exist $R>0$ and $\gamma_2\in(0,1)$ (both independent of f), such that

$$\int_{\{\mathbb{R}\setminus(-R,R)\}} |f(0)|^2\,d\dot{\mu} \leq \gamma_2\int_{\mathbb{R}} |f(0)|^2\,d\dot{\mu}. \tag{2.12}$$

We denote by $\mathcal{A}$ a set of functions that satisfies the above conditions. The following remark shows a non-enumerable number of non-empty sets $\mathcal{A}$.

Remark 2.1. *Let $R_0,T>0$ be constants and $b>0$, such that for each $\theta\in[0,1]$,*

$$\int_{\{R_0\leq|\xi|\leq R_0+b\}} \xi^{2\theta}d\xi \leq \frac{1}{2(T+1)^2}\int_{\{0\leq|\xi|\leq R_0\}} \xi^{2\theta}d\xi. \tag{2.13}$$

Let $\mathcal{A}_0^A$ the set of the continuous functions in $\mathbb{R}$ such that $f(\xi)=0$ if $|\xi|>R_0+b$, $f(\xi)=A$ if $|\xi|\leq R_0$ and $0\leq f(\xi)\leq A$, where A is any positive real number, fixed. Now, we set

$$\mathcal{A}_1^A := \big\{f(t,\xi)=f(\xi)(1+|t|);\ t\in[-T,T], f(\xi)\in\mathcal{A}_0^A\big\}.$$

Then, for each A, $\mathcal{A}_1^A$ is a like set $\mathcal{A}$. In fact, condition (i) is clearly satisfied. The condition (ii) is satisfied with $C_0=\tilde{C_0}=1+T$. The condition (iv) is satisfied with $R=R_0+b$ for all $\gamma_2\in(0,1)$, since the first integral in (2.12) is null. And the condition (iii) is satisfied with $\Theta=A^2$ and $\gamma_1=1/2$, since (2.13) *implies*

$$
\begin{aligned}
\int_{\{|f(t)|^2<A^2\}} \xi^{2\theta}|f(t,\xi)|^2 d\xi &\leq (1+T)^2A^2 \int_{\{R_0\leq|\xi|\leq R_0+b\}} \xi^{2\theta} d\xi \\
&\leq \frac{(1+T)^2A^2}{2(1+T)^2} \int_{\{0\leq|\xi|\leq R_0\}} \xi^{2\theta} d\xi \\
&\leq \frac{1}{2} \int_{\{0\leq|\xi|\leq R_0\}} \xi^{2\theta}|f(t,\xi)|^2 d\xi \\
&\leq \frac{1}{2} \int_{\mathbb{R}} \xi^{2\theta}|f(t,\xi)|^2 d\xi.
\end{aligned}
$$

Lemma 2.2. *For each $\theta \in (0,1)$, there exists a positive constant $\rho(\theta)$, such that, for each $t \in [-T,T]$*

$$
\|f(t)\|^2_{L^2(d\dot{\mu}_\theta)} \leq \|f(t)\|^{2\rho}_{H^s} \Big(K_0 \, \|f(0)\|^2_{L^2} + K_1 \, \|f(0)\|^2_{L^2(d\dot{\mu}_\theta)} + K_2 \Big) \tag{2.14}
$$

for all $f \in \mathcal{A}$, where

$$
K_0 = C_0 \, R^{2\theta} \left(\frac{4}{\Theta}\right)^{\rho+1}, \quad K_1 = \frac{\tilde{C}_0}{\rho(1-\gamma_2)} \left(\frac{4}{\Theta}\right)^{\rho}, \quad K_2 = \frac{\tilde{C}_1}{\rho R^{2\theta\rho}}.
$$

Proof. 1. For simplicity of notation, we sometimes write $f(t,\xi) \equiv f(\xi)$ and $f(0,\xi) \equiv f_0(\xi)$. Let $\theta_j > 0$, $(j = 0,1)$, constants independents of t, and for $\theta \in [0,1]$ set

$$
\begin{aligned}
I_1^{\theta_1} &:= \int_{\mathbb{R}} |\xi|^{2\theta} \, |f(\xi)|^2 \, \chi_{\{|f(\xi)|^2>\theta_1\}} \, d\xi \\
I_2^{\theta_1} &:= \, \theta_1 \int_{\mathbb{R}} |\xi|^{2\theta} \, \chi_{\{|f(\xi)|^2>\theta_1\}} \, d\xi, \\
I_3^{\theta_1} &:= \int_{\mathbb{R}} |\xi|^{2\theta} \, |f(\xi)|^2 \, \chi_{\{|f(\xi)|^2\leq\theta_1\}} \, d\xi,
\end{aligned}
$$

where χ_A is the characteristic function of the set A. Therefore, we have

$$
I := \int_{\mathbb{R}} |\xi|^{2\theta} \, |f(\xi)|^2 \, d\xi = I_1^{\theta_1} + I_3^{\theta_1} = I_1^{\theta_1} - \theta_0 I_2^{\theta_1} + I_3^{\theta_1} + \theta_0 I_2^{\theta_1}.
$$

Moreover, it is clear that $I_2^{\theta_1} < I_1^{\theta_1}$, indeed

$$
I_1^{\theta_1} - I_2^{\theta_1} = \int_{\mathbb{R}} |\xi|^{2\theta} \left(|f(\xi)|^2 - \theta_1\right) \chi_{\{|f(\xi)|^2>\theta_1\}} \, d\xi > 0.
$$

Hence, $\theta_0 I_2^{\theta_1} < \theta_0 I_1^{\theta_1} < \theta_0 \, (I_1^{\theta_1} + I_3^{\theta_1}) = \theta_0 \, I$. Therefore, we have

$$
(1-\theta_0) \, I < I - \theta_0 I_2^{\theta_1} = I_1^{\theta_1} - \theta_0 I_2^{\theta_1} + I_3^{\theta_1}. \tag{2.15}
$$

2. Claim $\sharp 1$: There exist $\theta_1 > 0$ independent of f, $t \in [-T,T]$, and a positive constant $\beta < 1$, such that $I_3^{\theta_1} < \beta I_1^{\theta_1}$.

Proof of Claim $\sharp 1$: We must show that

$$\begin{aligned}\int_{\mathbb{R}} |\xi|^{2\theta} \, |f(t,\xi)|^2 \, \chi_{\{|f|^2 \leq \theta_1\}} \, d\xi &\leq \beta \int_{\mathbb{R}} |\xi|^{2\theta} \, |f(t,\xi)|^2 \, \chi_{\{|f|^2 > \theta_1\}} \, d\xi \\ &= \beta \int_{\mathbb{R}} |\xi|^{2\theta} \, |f(t,\xi)|^2 \, d\xi \\ &\quad - \beta \int_{\mathbb{R}} |\xi|^{2\theta} \, |f(t,\xi)|^2 \, \chi_{\{|f|^2 \leq \theta_1\}} \, d\xi.\end{aligned}$$

Therefore, it is enough to show that

$$\int_{\mathbb{R}} |\xi|^{2\theta} \, |f(t,\xi)|^2 \, \chi_{\{|f|^2 \leq \theta_1\}} \, d\xi \leq \frac{\beta}{1+\beta} \int_{\mathbb{R}} |\xi|^{2\theta} \, |f(t,\xi)|^2 \, d\xi,$$

which is satisfied since $f \in \mathcal{A}$. Consequently, we take $\theta_1 = \Theta$ of inequality (2.11).

3. Now from item 2, we are going to show the existence of a positive constant $\alpha < 1/2$, such that

$$I_3^{\theta_1} < \alpha (I_1^{\theta_1} + I_3^{\theta_1}) = \alpha I. \tag{2.16}$$

Indeed, we have

$$\begin{aligned}I_3^{\theta_1} < \alpha (I_1^{\theta_1} + I_3^{\theta_1}) &\Leftrightarrow (1-\alpha) I_3^{\theta_1} < \alpha \, I_1^{\theta_1} \\ &\Leftrightarrow I_3^{\theta_1} < \frac{\alpha}{1-\alpha} \, I_1^{\theta_1}.\end{aligned}$$

Therefore, it is enough to take $\beta < 1$ and, we have

$$0 < \alpha = \frac{\beta}{1+\beta} < \frac{1}{2}.$$

Now, we fix $\theta_0 = (3/4 - \alpha) > 1/4$ and, from (2.15), (2.16), we obtain

$$I < \frac{I_1^{\theta_1} - \theta_0 I_2^{\theta_1}}{1 - (\theta_0 + \alpha)} = 4 \left(I_1^{\theta_1} - \theta_0 I_2^{\theta_1} \right). \tag{2.17}$$

4. Claim $\sharp 2$: There exist $N_1 \in \mathbb{N}$ and a constant $C_1 > 0$ both independent of f and t, such that, for all $\eta \geq N_1$

$$\int_{\{|\xi| < \eta\}} |f(\xi)|^2 |\xi|^2 \, d\xi \leq C_1 \int_{\{|\xi| < \eta\}} |f_0(\xi)|^2 |\xi|^2 \, d\xi + \tilde{C}_1.$$

Proof of Claim $\sharp 2$: Equivalently, we have to show that

$$\begin{aligned}&\int_{\mathbb{R}} |f(\xi)|^2 |\xi|^2 \, d\xi - \int_{\{|\xi| \geq \eta\}} |f(\xi)|^2 |\xi|^2 \, d\xi \\ &\qquad \leq C_1 \int_{\mathbb{R}} |f_0(\xi)|^2 |\xi|^2 \, d\xi - C_1 \int_{\{|\xi| \geq \eta\}} |f_0(\xi)|^2 |\xi|^2 \, d\xi + \tilde{C}_1,\end{aligned}$$

for each $\eta \geq N_1$. Hence using (2.10) and supposing $C_1 > \tilde{C}_0$, it is sufficient to prove that

$$\tilde{C}_1 + \tilde{C}_0 \int_{\mathbb{R}} |f_0(\xi)|^2 |\xi|^2 \, d\xi - \int_{\{|\xi| \geq \eta\}} |f(\xi)|^2 |\xi|^2 \, d\xi \\ \leq C_1 \int_{\mathbb{R}} |f_0(\xi)|^2 |\xi|^2 \, d\xi - C_1 \int_{\{|\xi| \geq \eta\}} |f_0(\xi)|^2 |\xi|^2 \, d\xi + \tilde{C}_1.$$

By a simple algebraic manipulation, it is sufficient to show that

$$C_1 \int_{\{|\xi| \geq \eta\}} |f_0(\xi)|^2 |\xi|^2 \, d\xi \leq \big(C_1 - \tilde{C}_0\big) \int_{\mathbb{R}} |f_0(\xi)|^2 |\xi|^2 \, d\xi \\ + \int_{\{|\xi| \geq \eta\}} |f(\xi)|^2 |\xi|^2 \, d\xi.$$

Therefore, it is enough to show that

$$\int_{\{|\xi| \geq \eta\}} |f_0(\xi)|^2 |\xi|^2 \, d\xi \leq \frac{C_1 - \tilde{C}_0}{C_1} \int_{\mathbb{R}} |f_0(\xi)|^2 |\xi|^2 \, d\xi,$$

which it is true for $f \in \mathcal{A}$. Consequently, we take $N_1 = R$ of inequality (2.12).

5. Finally, we estimate $I_1^{\theta_1} - \theta_0 I_2^{\theta_1}$.
If $\theta = 0, 1$ by (2.9) and (2.10) is obvious that

$$I_1^{\theta_1} - \theta_0 I_2^{\theta_1} \leq C_0 \int_{\mathbb{R}} |\xi|^{2\theta} \, |f_0(\xi)|^{2\theta} \, d\xi.$$

Consequently, we consider in the following $\theta \in (0,1)$.

$$\begin{aligned} I_1^{\theta_1} - \theta_0 I_2^{\theta_1} &= \int_{\mathbb{R}} \big(|\xi|^{2\theta} \, |f(\xi)|^2 - \theta_0 \theta_1 |\xi|^{2\theta} \big) \, \chi_{\{|f(\xi)|^2 > \theta_1\}} \, d\xi \\ &= \int_{\mathbb{R}} \big((\, |\xi| \, |f(\xi)|^{1/\theta} \,)^{2\theta} - ((\theta_0\theta_1)^{1/2\theta} |\xi| \,)^{2\theta} \big) \, \chi_{\{|f(\xi)|^2 > \theta_1\}} \, d\xi. \end{aligned}$$

For each $\eta > 0$, let $\varphi(\eta) = \eta^{2\theta}$. Hence, $\varphi'(\eta) = 2\,\theta\,\eta^{2\theta-1} > 0$ and, we have

$$\begin{aligned} I_1^{\theta_1} - \theta_0 I_2^{\theta_1} &= \int_{\mathbb{R}} \int_{(\theta_0\theta_1)^{1/2\theta}|\xi|}^{|\xi|\,|f(\xi)|^{1/\theta}} \varphi'(\eta) \, d\eta \, d\xi \\ &= 2\theta \int_0^\infty \eta^{2\theta-1} \int_{\mathbb{R}} \chi_{E(\eta)}(\xi) \, d\xi \, d\eta \\ &= 2\theta \int_0^\infty \eta^{2\theta-1} \, \mathcal{L}^1\big((E(\eta)\big) \, d\eta, \end{aligned}$$

where

$$E(\eta) = \big\{ \xi \in \mathbb{R} \, / \, |f(\xi)|^{1/\theta} |\xi| > \eta \, \big\} \bigcap \big\{ \xi \in \mathbb{R} \, / \, \kappa \, |\xi| < \eta, \; \kappa = (\theta_0\theta_1)^{1/2\theta} \big\}.$$

We observe that,

$$\mathcal{L}^1\big((E(\eta)\big) \leq \int_{\{\kappa\,|\xi|<\eta\}} \frac{|f(\xi)|^{2/\theta}|\xi|^2}{\eta^2}\, d\xi.$$

Hence we obtain

$$\begin{aligned} I_1^{\theta_1} - \theta_0 I_2^{\theta_1} &\leq 2\theta \int_0^\infty \eta^{2\theta-1} \int_{\{\kappa\,|\xi|<\eta\}} \frac{|f(\xi)|^{2/\theta}\,|\xi|^2}{\eta^2}\, d\xi\, d\eta \\ &\leq 2\theta\, \|f\|_{H^s}^{(2/\theta)-2} \int_0^\infty \eta^{2\theta-3} \int_{\{\kappa\,|\xi|<\eta\}} |f(\xi)|^2\,|\xi|^2\, d\xi\, d\eta. \end{aligned}$$

From item 4 and applying (2.9), it follows that

$$\begin{aligned} I_1^{\theta_1} - \theta_0 I_2^{\theta_1} &\leq 2\theta\, \|f(t)\|_{H^s}^{(2/\theta)-2} \int_0^{N_1} \eta^{2\theta-3} \int_{\{\kappa\,|\xi|<\eta\}} |f(\xi)|^2\, \frac{\eta^2}{\kappa^2}\;\, d\xi\, d\eta \\ &+ 2\theta\, \|f(t)\|_{H^s}^{(2/\theta)-2} \int_{N_1}^\infty \eta^{2\theta-3} \int_{\{\kappa\,|\xi|<\eta\}} |f(\xi)|^2\,|\xi|^2\;\, d\xi\, d\eta \\ &\leq \frac{C_0\, N_1^{2\theta}}{\kappa^2}\, \|f(t)\|_{H^s}^{(2/\theta)-2} \int_{\mathbb{R}} |f_0(\xi)|^2\, d\xi \\ &+ 2\theta\, C_1\, \|f(t)\|_{H^s}^{(2/\theta)-2} \int_{\mathbb{R}} |f_0(\xi)|^2\,|\xi|^2 \int_{\{\eta>\kappa\,|\xi|\}} \eta^{2\theta-3}\;\, d\eta\, d\xi + \Xi \\ &= \frac{C_0\, N_1^{2\theta}}{\kappa^2}\, \|f(t)\|_{H^s}^{(2/\theta)-2} \int_{\mathbb{R}} |f_0(\xi)|^2\;\, d\xi \\ &+ \frac{\theta}{1-\theta}\, C_1\, \|f(t)\|_{H^s}^{(2/\theta)-2} \int_{\mathbb{R}} |f_0(\xi)|^2\,|\xi|^2\,|\xi|^{2\theta-2}\kappa^{2\theta-2}\, d\xi + \Xi \\ &= \left(\frac{4}{\theta_1}\right)^{1/\theta} C_0\, N_1^{2\theta}\, \|f(t)\|_{H^s}^{(2/\theta)-2} \int_{\mathbb{R}} |f_0(\xi)|^2\, d\xi + \\ &+ \left(\frac{4}{\theta_1}\right)^{(1-\theta)/\theta} \frac{C_1\,\theta}{1-\theta}\, \|f(t)\|_{H^s}^{(2/\theta)-2} \int_{\mathbb{R}} |f_0(\xi)|^2\,|\xi|^{2\theta}\, d\xi + \Xi, \end{aligned}$$

where

$$\Xi = \frac{\theta \tilde{C_1} \|f(t)\|_{H^s}^{(2/\theta)-2}}{(1-\theta) N_1^{2(1-\theta)}}.$$

$\square$

3 Statement of the well-posedness result

This is the section where the well-posedness of the Cauchy problem (1.1) in weighted Sobolev space $\mathcal{X}^{2,\theta}$, for $\theta \in [0,1]$ is proved.

3.1 A priori estimates

Lemma 3.1. *If $u(t)$ is a solution of the IVP* (1.1) *with $u(0)$ in H^2, then for each $T > 0$,*

$$\|u(t)\|_{L^2} = \|u(0)\|_{L^2}, \tag{3.18}$$

$$\|u_x(t)\|_{L^2} \leq 2\|u_x(0)\|_{L^2} + C\,|C_2/C_1|\,\|u(0)\|^3_{L^2} + 2|C_3/C_1|\,\|u(0)\|_{L^2}, \tag{3.19}$$

for all $t \in [-T, T]$. Moreover, we have

$$\|u_{xx}(t)\|^2_{L^2} \leq \left(\|u_{xx}(0)\|^2_{L^2} + \hbar\right)(1 + T)\, e^{\hbar T}, \tag{3.20}$$

where $\hbar = \hbar(\|u_0\|_{L^2}, \|u_{0x}\|_{L^2})$ is a continuous function with $\hbar(0, 0) = 0$.

Proof. The inequalities (3.18) and (3.19) are consequence of conserved laws (1.2), (1.3) and the Gagliardo- Nirenberg inequality $\|v\|_{L^4} \leq \|v\|^{3/4}_{L^2}\|v_x\|^{1/4}_{L^2}$, see [7] and [17]. For inequality (3.20), we address [17]. □

Proposition 3.2. *Let u be a solution of the IVP* (1.1)*. If $u(t) \in \mathcal{X}^{s,1}$ for each $t \in [-T, T]$, with $s \geq 3$, then*

$$\|u(t)\|^2_{L^2(d\dot{\mu})} \leq e^{a_0 T}\,(\|u_0\|^2_{L^2(d\dot{\mu})} + a_0\hbar_0 T),$$

for all $t \in [-T, T]$, where $a_0 := 2|a| + 3|b| + (|d + e|)/2$ and

$$\hbar_0 = \hbar_0(\|u(0)\|_{L^2}, \|u_x(0)\|_{L^2}, \|u_{xx}(0)\|_{L^2})$$

is a continuous function with $\hbar_0(0, 0, 0) = 0$.

Proof. 1. First, let us consider a convenient function, i.e. $\varphi_n \in C^\infty(\mathbb{R})$, φ_n a non-negative even function, such that, for each $x \geq 0$, $0 \leq \varphi_n(x) \leq x^2$, $0 \leq \varphi_n'(x) \leq 2x$ and $|\varphi_n^{(j)}(x)| \leq 2$, $(j = 2, 3)$. Moreover, for $0 \leq x \leq n$, $\varphi_n(x) = x^2$, and for $x > 10\,n$, $\varphi_n(x) = 10\,n^2$.

2. Now, multiplying the equation (1.1) by $\varphi_n \bar{u}$ and taking the real part, we get after integration by parts,

$$\begin{aligned} &\frac{\partial}{\partial t}\int_{\mathbb{R}}\varphi_n|u|^2dx + 2a\,\Re\left(i\int_{\mathbb{R}}\varphi_n\bar{u}u_{xx}dx\right) \\ &\qquad + 2b\,\Re\left(\int_{\mathbb{R}}\varphi_n\bar{u}u_{xxx}dx\right) + 2c\,\Re\left(i\int_{\mathbb{R}}\varphi_n|u|^4dx\right) \\ &\qquad + 2d\,\Re\left(\int_{\mathbb{R}}\varphi_n\bar{u}|u|^2u_xdx\right) + 2e\,\Re\left(\int_{\mathbb{R}}\varphi_n\bar{u}u^2\bar{u}_xdx\right) = 0. \end{aligned} \tag{3.21}$$

The term with coefficient c is zero. Integrating by parts two times the integral with coefficient b, we obtain

$$\int_{\mathbb{R}}\varphi_n\bar{u}u_{xxx}dx = \int_{\mathbb{R}}\varphi_n\bar{u}_{xx}u_xdx + 2\int_{\mathbb{R}}\varphi_n'\bar{u}_xu_xdx + \int_{\mathbb{R}}\varphi_n''u_x\bar{u}dx. \tag{3.22}$$

Integrating by parts the first term in the right-hand side of (3.22)

$$\int_{\mathbb{R}} \varphi_n \bar{u}_{xx} u_x dx = -\int_{\mathbb{R}} \varphi_n u \bar{u}_{xxx} dx - \int_{\mathbb{R}} \varphi_n' u \bar{u}_{xx} dx, \tag{3.23}$$

and integrating by parts the second term in the right-hand side of (3.22), we have

$$2\int_{\mathbb{R}} \varphi_n' \bar{u}_x u_x dx = -2\int_{\mathbb{R}} \varphi_n'' \bar{u}_x u dx - 2\int_{\mathbb{R}} \varphi_n' \bar{u}_{xx} u dx. \tag{3.24}$$

Now, combining the equations (3.22)-(3.24), we get

$$\begin{aligned}\int_{\mathbb{R}} \varphi_n \bar{u} u_{xxx} dx = -\int_{\mathbb{R}} \varphi_n u \bar{u}_{xxx} dx - 3\int_{\mathbb{R}} \varphi_n' \bar{u}_{xx} u dx \\ - 2\int_{\mathbb{R}} \varphi_n'' \bar{u}_x u dx + \int_{\mathbb{R}} \varphi_n'' u_x \bar{u} dx,\end{aligned}$$

and thus

$$2\Re\left(\int_{\mathbb{R}} \varphi_n \bar{u} u_{xxx} dx\right) = -3\Re\int_{\mathbb{R}} \varphi_n' \bar{u}_{xx} u dx - \Re\int_{\mathbb{R}} \varphi_n'' \bar{u}_x u dx. \tag{3.25}$$

Integrating by parts the integral with coefficient a in (3.21)

$$\int_{\mathbb{R}} \varphi_n u_{xx} \bar{u} dx = -\int_{\mathbb{R}} \varphi_n \bar{u}_x u_x dx - \int_{\mathbb{R}} \varphi_n' \bar{u} u_x dx.$$

Therefore,

$$\Re\left(i\int_{\mathbb{R}} \varphi_n \bar{u} u_{xx} dx\right) = \Im\left(\int_{\mathbb{R}} \varphi_n' \bar{u} u_x dx\right). \tag{3.26}$$

Now, we consider the integral with coefficient d in (3.21) and integrating by parts, we have

$$\int_{\mathbb{R}} \varphi_n \bar{u}|u|^2 u_x dx = -\int_{\mathbb{R}} \varphi_n u \bar{u}^2 u_x dx - 2\int_{\mathbb{R}} \varphi_n u^2 \bar{u} \bar{u}_x dx - \int_{\mathbb{R}} \varphi_n' u^2 \bar{u}^2 dx,$$

and this inequality implies

$$\Re\left(\int_{\mathbb{R}} \varphi_n \bar{u}|u|^2 u_x dx\right) = -\frac{1}{4}\int_{\mathbb{R}} \varphi_n' |u|^4 dx. \tag{3.27}$$

Finally, we consider the last term, that is, with coefficient e

$$\Re\left(\int_{\mathbb{R}} \varphi_n u^2 \bar{u} \bar{u}_x dx\right) = \Re\left(\int_{\mathbb{R}} \varphi_n \bar{u}|u|^2 u_x dx\right) = -\frac{1}{4}\int_{\mathbb{R}} \varphi_n' |u|^4 dx. \tag{3.28}$$

3. From (3.21) and (3.25)-(3.28), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}} \varphi_n |u|^2 dx = &- 2a\,\Im\left(\int_{\mathbb{R}} \varphi_n' \bar{u} u_x dx\right) + 3b\,\Re\int_{\mathbb{R}} \varphi_n' \bar{u}_{xx} u dx \\ &+ b\,\Re\int_{\mathbb{R}} \varphi_n'' \bar{u}_x u dx + \frac{(d+e)}{2}\int_{\mathbb{R}} \varphi_n' |u|^4 dx,\end{aligned}$$

and using the properties of φ_n, we get

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}}\varphi_n|u|^2dx \leq & 2|a|\int_{\mathbb{R}}x^2|u|^2dx+2|a|\int_{\mathbb{R}}|u_x|^2dx+3|b|\int_{\mathbb{R}}x^2|u|^2dx\\ &+3|b|\int_{\mathbb{R}}|u_{xx}|^2dx+|b|\int_{\mathbb{R}}|u|^2dx+|b|\int_{\mathbb{R}}|u_x|^2dx\\ &+\frac{|d+e|}{2}\int_{\mathbb{R}}x^2|u|^2dx+\frac{|d+e|}{2}\int_{\mathbb{R}}|u|^6dx.\end{aligned}$$

Now, passing to the limit as $n\to\infty$ and applying the Dominated Convergence Theorem

$$\frac{\partial}{\partial t}\|u(t)\|^2_{L^2(d\dot{\mu})}\leq a_0\,\big(\|u(t)\|^2_{L^2(d\dot{\mu})}+A\big),$$

where $A=\|u(0)\|^2_{L^2}+(1+\|u(0)\|^4_{L^2})\sup_{t\in\mathbb{R}}\|u_x(t)\|^2_{L^2}+\sup_{t\in\mathbb{R}}\|u_{xx}(t)\|^2_{L^2}$. Observe that by (3.19) and (3.20)

$$A\leq\hbar_0(\|u(0)\|_{L^2},\|u_x(0)\|_{L^2},\|u_{xx}(0)\|_{L^2})<\infty,$$

where $\hbar_0$ is a continuous function with $\hbar_0(0,0,0)=0$. Now, applying Gronwall's inequality, we have for all $t\in[0,T]$

$$\|u(t)\|^2_{L^2(d\dot{\mu})}\leq e^{a_0T}\Big(\|u(0)\|^2_{L^2(d\dot{\mu})}+a_0\,T\,A\Big). \tag{3.29}$$

4. Let k_0 be a non-zero real parameter and set $\tilde{u}(x,t):=u(x,k_0\,t)$. Since u is a solution of (1.1) for every $t\in\mathbb{R}$, then $\tilde{u}$ is a global solution of the following Airy-Schrödinger equation

$$\partial_t\tilde{u}+\ i\,\tilde{a}\,\partial_x^2\tilde{u}+\tilde{b}\,\partial_x^3\tilde{u}+i\,\tilde{c}\,|\tilde{u}|^2\tilde{u}+\tilde{d}\,|\tilde{u}|^2\partial_x\tilde{u}+\tilde{e}\,\tilde{u}^2\partial_x\bar{\tilde{u}}=0,$$

where $\tilde{a}=k_0\,a,\ldots,\tilde{e}=k_0\,e$. Therefore, we have an analogously inequality for $\tilde{u}$, that is

$$\|\tilde{u}(t)\|^2_{L^2(d\dot{\mu})}\leq e^{\tilde{a}_0\tilde{T}}\Big(\|u(0)\|^2_{L^2(d\dot{\mu})}+\tilde{a}_0\,\tilde{T}\,A\Big),$$

for all $t\in[0,\tilde{T}]$, where $\tilde{a_0}=|k_0|\,a_0$. Now, taking $\tilde{T}=T$ and $k_0=-1$, we obtain that the inequality (3.29) is valid for all $t\in[-T,T]$.

$\square$

3.2 Unitary group and non-linear estimate

We begin defining the unitary group $U(t)$ as the solution of the linear initial value problem associated to (1.1),

$$\begin{cases}\partial_t u+ia\,\partial_x^2\,u+b\,\partial_x^3u=0,\\ u(x,0)=\ u_0(x).\end{cases}$$

Hence, we have

$$\widehat{U(t)u_0}(\xi) = \exp\big(it(a\xi^2 + b\xi^3)\big)\,\widehat{u_0}(\xi). \tag{3.30}$$

For convenience, we define the non-linear part of equation in (1.1) as

$$F(u) := ic\,|u|^2u + d\,|u|^2\partial_x u + e\,u^2\partial_x\bar{u}. \tag{3.31}$$

Next we recall a well known result, see for instance [4, 15].

If $f \in L^1_x\mathcal{L}^2_t$, $u_0 \in \dot{H}^{1/4}$ and $U(t)$ is the unitary group as in (3.30). Then, there exists a constant $C > 0$, such that

$$\left\|\partial_x \int_0^t U(t-t')f(x,t')dt'\right\|_{L^2_x} \leq C\,\|f\|_{L^1_x\mathcal{L}^2_t} \tag{3.32}$$

and

$$\|U(t')u_0\|_{L^4_xL^\infty_t} \leq C\,\|u_0\|_{\dot{H}^{1/4}}. \tag{3.33}$$

This result enable us to prove the following

Proposition 3.3. *Let $u \in \mathcal{C}(\mathbb{R}, H^2)$ be the solution of IVP (1.1), then*

$$\begin{aligned}\|u\|_{L^4_x\mathcal{L}^\infty_t} \leq & C\,\|u(0)\|_{\dot{H}^{1/4}} + C\int_0^t (\|u(t')\|_{H^{1/2+}}\|u(t')\|^2_{H^2} \\ & + \|u(t')\|^2_{H^{1/2+}}\|u(t')\|_{H^2})dt',\end{aligned} \tag{3.34}$$

where C is a positive constant.

Proof. In order to prove this inequality we rely on the integral equation form

$$u(t) = U(t)u_0 - \int_0^t U(t-\tau)F(u)(\tau)d\tau,$$

where $F(u)$ is given by (3.31). The linear estimate (3.33) shows that if $u(0) \in H^2$ then for any $t > 0$

$$\|u\|_{L^4_x\mathcal{L}^\infty_t} \leq C\|u(0)\|_{\dot{H}^{1/4}} + C\int_0^t (\|F(u)\|_{L^2_x} + \|\,\partial_x F(u)\|_{L^2_x})dt'. \tag{3.35}$$

First, we estimate $\|F(u)\|_{L^2_x}$. By the immersions $\|u(t)\|_{L^\infty_x} \leq C\|u(t)\|_{H^{1/2+}}$ and $\|u(t)\|_{L^4_x} \leq C\|u(t)\|_{\dot{H}^{1/4}}$, it follows that

$$\begin{aligned}\|\,|u|^2u(t')\,\|_{L^2_x} \leq & \|u(t')\,\|_{L^\infty_x}\|u^2(t')\,\|_{L^2_x} \leq C\|u(t')\,\|_{H^{1/2+}}\|u(t')\,\|^2_{L^4_x} \\ \leq & C\|u(t')\,\|_{H^{1/2+}}\|u(t')\,\|^2_{\dot{H}^{1/4}},\end{aligned} \tag{3.36}$$

and

$$\|\,|u|^2u_x(t')\,\|_{L^2_x} \leq \|u(t')\,\|^2_{L^\infty_x}\|u_x(t')\,\|_{L^2_x} \leq C\|u(t')\,\|^2_{H^{1/2+}}\|u(t')\,\|_{\dot{H}^1}. \tag{3.37}$$

Analogously, we treat the term $u^2\partial_x\bar{u}$.

Now consider $\|\partial_x F(u)\|_{L^2_x}$, we estimate the term $|u|^2 u_x$. The estimates for the other terms are similar. Using Leibniz rule, it is easy to see that

$$\begin{aligned}\|\partial_x(|u|^2u_x)(t')\|_{L^2_x} \leq & \|\bar{u}u_x^2(t')\|_{L^2_x} + \|\,|u_x|^2u(t')\,\|_{L^2_x} + \|\,|u|^2u_{xx}(t')\,\|_{L^2_x}\\ \leq & C\|u(t')\|_{H^{1/2+}}\|u(t')\|^2_{H^2} + \|u(t')\|^2_{H^{1/2+}}\|u(t')\|_{H^2}.\end{aligned} \tag{3.38}$$

Hence combining (3.35)-(3.38) we conclude (3.34). □

3.3 Approximated problem

Let X be a Banach space, $u_0 \in X$ and $(u_0^\lambda)_{\lambda>0}$ a family of regular functions, such that

$$u_0^\lambda \to u_0 \quad \text{in } X,$$

when $\lambda \to \infty$. For each $\lambda > 0$, we consider the following family of approximated problems obtained from (1.1)

$$\begin{cases}\partial_t u^\lambda + ia\,\partial_x^2 u^\lambda + b\,\partial_x^3 u^\lambda + ic\,|u^\lambda|^2 u^\lambda + d\,|u^\lambda|^2\partial_x u + e\,{u^\lambda}^2\partial_x\bar{u}^\lambda = 0, \quad x, t \in \mathbb{R},\\ u^\lambda(x,0) = u_0^\lambda(x).\end{cases} \tag{3.39}$$

As mentioned in the Introduction, we know that (1.1) is global well-posedness in H^2. In order to prove the well-posedness with weight result, Theorem 3.7, we initially proving the following

Lemma 3.4. *Let $T > 0$, $u_0 \in H^2$, $u_0^\lambda \to u_0$ in H^2, and any $s \in [0,2)$ fixed. Then, for each $t \in [-T,T]$, the family $(u^\lambda)(t)$, solutions of the approximated problems* (3.39)*, converges to $u(t)$ in H^s, uniformly with respect to t, where $u(t) \in H^2$ is the global solution of the IVP* (1.1)*.*

Proof. 1. We begin proving that (u^λ) is a Cauchy sequence in L^2. Let $\mu := u^\lambda$, $v := u^{\lambda'}$ and $w = \mu - v$, we rely on the integral equation form

$$\mu(t) = U(t)u_0^\lambda - \int_0^t U(t-\tau)F(\mu)(\tau)d\tau, \tag{3.40}$$

and

$$v(t) = U(t)u_0^{\lambda'} - \int_0^t U(t-\tau)F(v)(\tau)d\tau, \tag{3.41}$$

where $F(u)$ is given by (3.31). Thus (3.40) and (3.41) imply

$$w(t) = U(t)(u_0^\lambda - u_0^{\lambda'}) - \int_0^t U(t-\tau)(F(\mu) - F(v))(\tau)d\tau.$$

Hence we have

$$\|w(t)\|_{L^2_x} \leq \|u_0^\lambda - u_0^{\lambda'}\|_{L^2} + \left\| \int_0^t U(t-\tau)(F(\mu)-F(v))(\tau)d\tau \right\|_{L^2_x}.$$

We can suppose a, b, c, d positive numbers, and using the definition of $F(u)$

$$\begin{aligned}
\| \int_0^t U(t-\tau)(F(\mu)-F(v))(\tau)d\tau\|_{L^2_x} &\leq c \int_0^t \left\| |\mu|^2 w + \mu v \overline{w} + |v|^2 w \right\|_{L^2_x} d\tau \\
&+ d \int_0^t \|\mu v_x \overline{w} + w \overline{v} v_x\|_{L^2_x} d\tau + e \int_0^t \|w \mu \overline{\mu}_x + w v \overline{\mu}_x\|_{L^2_x} d\tau \\
&+ \| \int_0^t U(t-\tau)(d|\mu|^2 w_x + e v^2 \overline{w}_x)(\tau) d\tau \|_{L^2_x} \\
&=: I_c + I_d + I_e + I_{de},
\end{aligned} \tag{3.42}$$

with obvious notation. Applying the Hölder inequality, the Sobolev immersion $\|u(t)\|_{L^\infty_x} \leq C \, \|u(t)\|_{H^{1/2+}}$ and conservation law in $\dot{H}^1$, it follows that

$$\begin{aligned}
I_c \leq & c \int_0^t (\|\mu\|_{L^\infty_x} + \|v\|_{L^\infty_x})^2 \|w\|_{L^2_x} d\tau \leq 2c \int_0^t (\|\mu\|^2_{H^1} + \|v\|^2_{H^1}) \|w\|_{L^2_x} d\tau \\
\leq & Cc(\|u_0^\lambda\|^2_{H^1} + \|u_0^\lambda\|^6_{L^2} + \|u_0^{\lambda'}\|^2_{H^1} + \|u_0^{\lambda'}\|^6_{L^2}) \int_0^t \|w\|_{L^2_x} d\tau \\
\leq & 2Cc(\|u_0\|^2_{H^1} + \|u_0\|^6_{L^2}) \int_0^t \|w\|_{L^2_x} d\tau.
\end{aligned}$$

Analogously, using Hölder inequality, the immersion $\|u(t)\|_{L^\infty_x} \leq C \|u(t)\|_{H^{1/2+}}$ and conservation laws in $\dot{H}^1$ and $\dot{H}^2$, it follows that

$$\begin{aligned}
I_d \leq & d \int_0^t (\|\mu\|_{L^\infty_x} + \|v\|_{L^\infty_x}) \|v_x\|_{L^\infty_x} \|w\|_{L^2_x} d\tau \\
\leq & d \int_0^t (\|\mu\|_{H^1} + \|v\|_{H^1}) \|v\|_{H^2} \|w\|_{L^2_x} d\tau \\
\leq & Cd(\|u_0^\lambda\|_{H^1} + \|u_0^\lambda\|^3_{L^2} + \|u_0^{\lambda'}\|_{H^1} + \|u_0^{\lambda'}\|^3_{L^2}) \Omega_0 \int_0^t \|w\|_{L^2_x} d\tau \\
\leq & 2C \, d(\|u_0\|_{H^1} + \|u_0\|^3_{L^2}) \Omega_0 \int_0^t \|w\|_{L^2_x} d\tau,
\end{aligned}$$

where $\Omega_0 = \Omega_0(\|u_0\|_{L^2}, \|u_0\|_{\dot{H}^1}, \|u_0\|_{\dot{H}^2}, T)$. Similarly we obtain

$$I_e \leq C \, e(\|u_0\|_{H^1} + \|u_0\|^3_{L^2}) \Omega_0 \int_0^t \|w\|_{L^2_x} d\tau.$$

Now, we estimate I_{de}. From (3.32), we have

$$
\begin{aligned}
I_{de} =& \| \int_0^t U(t-\tau)\left(d(|\mu|^2 w)_x - d(|\mu|^2)_x w + e(v^2\overline{w})_x - 2evv_x\overline{w}\right)(\tau)d\tau\|_{L^2_x} \\
\leq& \|\partial_x \int_0^t U(t-\tau)(d|\mu|^2 w + ev^2\overline{w})d\tau\|_{L^2_x} + \int_0^t \|d(|\mu|^2)_x w - 2evv_x\overline{w}\|_{L^2_x} d\tau \\
\leq& C\,\|d|\mu|^2 w + ev^2\overline{w}\|_{L^1_x\mathcal{L}^2_t} + c\,(d+e)(\|u_0\|_{H^1} + \|u_0\|^3_{L^2})\Omega_0 \int_0^t \|w\|_{L^2_x} d\tau \\
\leq& C\,|d|\|\mu\|^2_{L^4_x\mathcal{L}^\infty_t}\|w\|_{L^2_x\mathcal{L}^2_t} + c|e|\|v\|^2_{L^4_x\mathcal{L}^\infty_t}\|w\|_{L^2_x\mathcal{L}^2_t} \\
&+ C\,(d+e)(\|u_0\|_{H^1} + \|u_0\|^3_{L^2})\Omega_0 \int_0^t \|w\|_{L^2_x} d\tau \\
\leq& C\,(d+e)\Big((\|\mu\|^2_{L^4_x\mathcal{L}^\infty_t} + \|v\|^2_{L^4_x\mathcal{L}^\infty_t})\|w\|_{L^2_x\mathcal{L}^2_t} \\
&+ (\|u_0\|_{H^1} + \|u_0\|^3_{L^2})\Omega_0 \int_0^t \|w\|_{L^2_x} d\tau\Big).
\end{aligned}
$$

Applying Proposition 3.3, we conclude

$$
I_{de} \leq C\,(d+e)\left(\Omega_1\|w\|_{L^2_x\mathcal{L}^2_t} + (\|u_0\|_{H^1} + \|u_0\|^3_{L^2})\Omega_0 \int_0^t \|w\|_{L^2_x} d\tau\right),
$$

where $\Omega_1 = \Omega_1(\|u_0\|_{L^2}, \|u_0\|_{\dot{H}^1}, \|u_0\|_{\dot{H}^2}, T)$. Finally, we have

$$
\|w(t)\|_{L^2_x} \leq \|u_0^\lambda - u_0^{\lambda'}\|_{L^2} + C\Gamma_1 \int_0^t \|w(\tau)\|_{L^2} d\tau + C(d+e)\Omega_1\|w\|_{L^2_x\mathcal{L}^2_t}, \tag{3.43}
$$

where $\Gamma_1 = 2Cc\Gamma^2 + C(d+e)\Gamma\Omega_0$ and $\Gamma = \|u_0\|_{H^1} + \|u_0\|^3_{L^2}$. Moreover, since

$$
\int_0^t \|w(\tau)\|_{L^2} d\tau \leq T^{1/2}\|w\|_{L^2_x\mathcal{L}^2_t},
$$

it follows from inequality (3.43) that

$$
\|w(t)\|^2_{L^2_x} \leq C\|u_0^\lambda - u_0^{\lambda'}\|^2_{L^2} + C\Gamma_1^2 T \int_0^t \|w(\tau)\|^2_{L^2} d\tau + C\Omega_1^2 \int_0^t \|w(\tau)\|^2_{L^2} d\tau,
$$

and using Gronwall's inequality

$$
\|w(t)\|^2_{L^2_x} \leq C\|u_0^\lambda - u_0^{\lambda'}\|^2_{L^2} e^{Ct(\Gamma_1^2 T + \Omega_1^2)}.
$$

Consequently, (u^λ) is a Cauchy sequence in L^2, and hence $u^\lambda \to u \in L^2$.

2. Let $s \in (0,2)$, the interpolation in Sobolev spaces shows that

$$
\begin{aligned}
\|u^\lambda - u^{\lambda'}\|_{H^s} \leq& \|u^\lambda - u^{\lambda'}\|^{1-s/2}_{L^2}\|u^\lambda - u^{\lambda'}\|^{s/2}_{H^2} \\
\leq& \|u^\lambda - u^{\lambda'}\|^{1-s/2}_{L^2}\Omega_2,
\end{aligned}
$$

where $\Omega_2 = \Omega_2(\|u_0\|_{L^2}, \|u_0\|_{\dot{H}^1}, \|u_0\|_{\dot{H}^2}, T)$. Hence we have

$$u^\lambda \to u \quad \text{in } H^s \quad \text{for all } s \in [0,2). \tag{3.44}$$

Observe that the conservations laws in L^2 and $\dot{H}^1$ for u^λ implies that the limit u also satisfies:

$$\|u(t)\|_{L^2} = \|u(0)\|_{L^2} \tag{3.45}$$

and

$$\|u(t)\|_{\dot{H}^1} \leq \|u(0)\|_{\dot{H}^1} + C\|u(0)\|_{L^2}^3. \tag{3.46}$$

Moreover, the conserved quantity (3.20) gives

$$\|u^\lambda_{xx}(t)\|_{L^2}^2 \leq \left(\|u^\lambda_{xx}(0)\|_{L^2}^2 + C_4\right)(1+T)\, e^{C_4 T},$$

where the positive constants $C_4 = C_4(\|u_0^\lambda\|_{L^2}, \|u_{0x}^\lambda\|_{L^2})$. Thus

$$\|u^\lambda_{xx}(t)\|_{L^2}^2 \leq C\big(\|u_{xx}(0)\|_{L^2}, \|u_x(0)\|_{L^2}, \|u(0)\|_{L^2}, T). \tag{3.47}$$

Applying the Banach-Alaoglu Theorem, there exist a subsequence already denoted by u^λ, and a function $\tilde{u} \in H^2$, such that

$$u^\lambda \rightharpoonup \tilde{u}, \quad \text{in } H^2. \tag{3.48}$$

Therefore, $u^\lambda \rightharpoonup \tilde{u}$ in H^1. On the other hand, by (3.44) we have $u^\lambda \to u$ in H^1 and thus $u^\lambda \rightharpoonup u$ in H^1. Consequently, the uniqueness of the limit gives $u = \tilde{u} \in H^2$. The inequality (3.47) and the limit (3.48) implies

$$\begin{aligned}\|u_{xx}(t)\|_{L^2} =& \|\tilde{u}_{xx}(t)\|_{L^2} \leq \liminf \|u^\lambda_{xx}(t)\|_{L^2}^2 \\ &\leq C\big(\|u_{xx}(0)\|_{L^2}, \|u_x(0)\|_{L^2}, \|u(0)\|_{L^2}, T).\end{aligned} \tag{3.49}$$

3. Now we will prove that u is a solution of (1.1). Using Duhamel principle, we will show that, for each $t \in [-T,T]$, $u(t) = \mathcal{L}(u)(t)$, where

$$\mathcal{L}(u)(t) := U(t)u(0,x) - \int_0^t U(t-\tau)F(u)(\tau)d\tau.$$

Let $\varpi := \mathcal{L}(u^\lambda) - \mathcal{L}(u)$, then

$$\|\varpi\|_{L^2} \leq \|u^\lambda(0,x) - u(0,x)\|_{L^2} + \|\int_0^t U(t-t')(F(u^\lambda) - F(u))(t')dt'\|_{L^2}. \tag{3.50}$$

In the same way as in (3.42), we get

$$\begin{aligned}\|\int_0^t U(t-\tau)(F(u^\lambda) - F(u))(\tau)d\tau\|_{L^2_x} \leq& \, c\int_0^t \left\||u|^2 w + uv\overline{w} + |v|^2 w\right\|_{L^2_x} d\tau \\ &+ d\int_0^t \left\|uv_x\overline{w} + w\overline{v}v_x + |u|^2 w_x\right\|_{L^2_x} d\tau + e\int_0^t \left\|wu\overline{u}_x + wv\overline{u}_x + v^2\overline{w}_x\right\|_{L^2_x} d\tau \\ &=: c\, I_c + dJ_d + eJ_e,\end{aligned} \tag{3.51}$$

where $v = u^\lambda$ and $w = u^\lambda - u$. We begin estimating

$$\begin{aligned} I_c \leq & C(\|u\|_{L_t^\infty H^1} + \|v\|_{L_t^\infty H^1})^2 \int_0^t \|w\|_{L_x^2} d\tau \\ & \leq (\|u(0)\|_{H^1} + \|u_0^\lambda\|_{H^1} + \|u(0)\|_{L^2}^3 + \|u_0^\lambda\|_{L^2}^3)^2 \int_0^t \|w\|_{L_x^2} d\tau. \end{aligned} \tag{3.52}$$

Therefore, $I_c \to 0$ as $\lambda \to \infty$. By (3.18)-(3.20) and (3.44)-(3.46)

$$\begin{aligned} J_d \leq & C(\|u\|_{L_t^\infty H^1} + \|v\|_{L_t^\infty H^1})\|v\|_{L_t^\infty H^2} \int_0^t \|w\|_{L_x^2} d\tau \\ & + \|u\|_{L_t^\infty H^1}^2 \int_0^t \|w_x\|_{L_x^2} d\tau. \end{aligned} \tag{3.53}$$

Hence $J_d \to 0$ as $\lambda \to \infty$. Similarly by (3.49), we have

$$\begin{aligned} J_e \leq & C(\|u\|_{L_t^\infty H^1} + \|v\|_{L_t^\infty H^1})\|u\|_{L_t^\infty H^2} \int_0^t \|w\|_{L_x^2} d\tau \\ & + \|v\|_{L_t^\infty H^1}^2 \int_0^t \|w_x\|_{L_x^2} d\tau \to 0. \end{aligned} \tag{3.54}$$

Then, combining (3.44) and (3.50)-(3.54) and passing to the limit as $\lambda \to \infty$

$$\varpi = \mathfrak{L}(u^\lambda) - \mathfrak{L}(u) = u^\lambda - \mathfrak{L}(u) \to 0 \quad \text{in } L^2.$$

The uniqueness of limit implies that $u = \mathfrak{L}(u)$. $\square$

Remark 3.5. 1 *Observe that the proof of Lemma 3.4 gives another way to prove the global well-posedness of* (1.1) *in* H^2. *For instance, in order to show the persistence we proceed as follow:*

We claim that $u \in C([0,T], H^s)$, *for* $s \in [0,2]$. *Indeed, let* $t_n \to t$ *in* $[0,T]$, *and using the Duhamel's formula we have*

$$\begin{aligned} \|u(t_n) - u(t)\|_{H^2} \leq & \|U(t_n)u_0 - U(t)u_0\|_{H^2} \\ & + \| \int_0^{t_n} U(t_n - t')F(u)(t')dt'_0 - \int_0^t U(t-t')F(u)(t')dt'\|_{H^2} \\ & =: L_1 + L_2, \end{aligned}$$

with the obvious notation. In L_1, *by the Dominated Convergence Theorem, passing to the limit as* $n \to \infty$,

$$L_1^2 = \int_{\mathbb{R}} (1+\xi^2)2|e^{it_n\phi(\xi)} - e^{it\phi(\xi)}|^2 |\widehat{u}_0(\xi)|^2 d\xi \to 0,$$

where $\phi(\xi) = a\xi^2 + b\xi^3$. *In* L_2 *also by Dominated Convergence Theorem, we have*

$$\begin{aligned} L_2 \leq & \| \int_{t_n}^t U(t_n - t')F(u)(t')dt'_0\|_{H^2} + \| \int_0^t (U(t_n)\psi(t')dt' - U(t)\psi(t'))\, dt'\|_{H^2} \\ \leq & \| \int_{t_n}^t U(-t')F(u)(t')dt'_0\|_{H^2} + \| \int_0^t (U(t_n)\psi(t')dt' - U(t)\psi(t'))\, dt'\|_{H^2} \to 0, \end{aligned}$$

where $\psi(t') = U(-t')F(u)(t')$ and this proves that $u \in C([0,T], H^2)$.

2 *In Lemma 3.4 we can consider $u_0^\lambda(x) := \mathcal{F}^{-1}\left(\chi_{\{|\xi|\le\lambda\}}\widehat{u_0}(\xi)\right)(x)$ $\big($or $u_0^\lambda(x) = \mathcal{F}^{-1}\left(\psi_\lambda(\xi)\widehat{u_0}(\xi)\right)(x)$ where ψ_λ is a continuous function with support in $[-2\lambda, 2\lambda]$ and such that $\psi_\lambda = 1$ in $[-\lambda,\lambda]\big)$. Therefore, if $u_0 \in H^s$ then*

$$\int |\widehat{u_0^\lambda} - \widehat{u_0}|^2(\xi)d\mu_\theta = \int |\chi_{\{|\xi|>\lambda\}}\widehat{u_0}|^2(\xi)d\mu_\theta = \int_{|\xi|>\lambda} |\widehat{u_0}|^2(\xi)d\xi d\mu_\theta \to 0,$$

when $\lambda \to \infty$. In this case, Paley-Wiener Theorem implies that the initial data u_0^λ in (3.39) has an analytic continuation to an entire analytic (in x) function. On the other hand, by [14] the solution u^λ of the IVP (3.39) also is an entire analytic function. Hence we have the following :

Corollary 3.6. *If $u_0 \in H^2$ and $u(t)$ is the global solution of the IVP (3.39) associated with the initial data u_0, then there exists a sequence of entire analytic functions u^λ such that $u^\lambda(t) \to u(t)$ in H^s, with $s \in [0,2)$.*

3 *If $u_0 \in L^2(d\dot{\mu}_\theta)$, $\theta \in [0,1]$, $\lambda > 0$ and $u_0^\lambda(x) = \mathcal{F}^{-1}\left(\chi_{\{|\xi|<\lambda\}}\widehat{u_0}\right)(x)$, then*

$$\|u_0^\lambda\|_{L^2(d\dot{\mu}_\theta)} \le \|u_0\|_{L^2(d\dot{\mu}_\theta)}. \tag{3.55}$$

In fact, if $\theta = 0$, (3.55) is a direct consequence of Plancherel's theorem and definition of u_0^λ. If $\theta = 1$, using properties of Fourier transform we obtain

$$|\widehat{xu_0^\lambda}(\xi)| = |\partial_\xi \widehat{u_0^\lambda}(\xi)| = |\chi_{\{|\xi|<\lambda\}}\partial_\xi\widehat{u_0}(\xi)| = \chi_{\{|\xi|<\lambda\}}|\widehat{xu_0}(\xi)|.$$

Thus by Plancherel's equality

$$\int_{\mathbb{R}} x^2|u_0^\lambda(x)|^2dx = \int_{\mathbb{R}} |\widehat{xu_0^\lambda}(\xi)|^2d\xi \le \int_{\mathbb{R}} |\widehat{xu_0}(\xi)|^2d\xi = \int_{\mathbb{R}} |xu_0(x)|^2dx.$$

When $\theta \in (0,1)$, we obtain (3.55) by interpolation between the cases $\theta = 0$ and $\theta = 1$, see [2].

3.4 Main result

Now, we state our main theorem of global existence:

Theorem 3.7. *The IVP (1.1) is globally well-posed in $\mathcal{X}^{2,\theta}$ for any $0 \le \theta \le 1$ fixed. Moreover, the solution u of (1.1) satisfies, for each $t \in [-T,T]$*

$$\|u(t)\|^2_{L^2(d\dot{\mu}_\theta)} \le C\left(\|u_0\|^2_{L^2} + \|u_0\|^2_{L^2(d\dot{\mu}_\theta)} + 1\right),$$

where $C = C(\theta, \|u(t)\|_{H^s}, \|u(0)\|_{L^2}, \|u_x(0)\|_{L^2}, \|u_{xx}(0)\|_{L^2}, T)$, $s > 1/2$.

Proof. Let $T > 0$ and $u_0 \in \mathcal{X}^{2,\theta}$, $u_0 \neq 0$, $\theta \in [0,1]$, we know that that there exists an function $u \in C([-T,T], H^2)$ such that the IVP (1.1) is global well-posed in H^2. Is well know that $\mathbf{S}(\mathbb{R})$ is dense in $\mathcal{X}^{s,\theta}$. Then for $u_0 \in \mathcal{X}^{2,\theta}$ there exist a sequence (u_0^λ) in $\mathbf{S}(\mathbb{R})$ such that

$$u_0^\lambda \to u_0 \quad \text{in } \mathcal{X}^{2,\theta}. \tag{3.56}$$

By (3.56) and Lemma 3.4 the sequence of solutions $u^\lambda(t)$ associated to IVP (3.39) and with initial data u_0^λ satisfy

$$\sup_{t\in[-T,T]} \|u^\lambda(t) - u(t)\|_{H^s} \overset{\lambda\to\infty}{\to} 0 \quad s \in [0,2). \tag{3.57}$$

Suppose temporarily that the solutions u^λ of the IVP (3.39) satisfy the conditions (i)-(iv) of Section 2. Therefore Lemma 2.2 gives

$$\int_{\mathbb{R}} |\xi|^{2\theta} |u^\lambda(t,\xi)|^2 \, d\xi \leq C \, \Big(\int_{\mathbb{R}} |u^\lambda(0,\xi)|^2 \, d\xi + \int_{\mathbb{R}} |\xi|^{2\theta} |u^\lambda(0,\xi)|^2 \, d\xi + 1 \Big),$$

where $C = C(\theta, \|u^\lambda(t)\|_{H^s}, \|u^\lambda(0)\|_{L^2}, \|u_x^\lambda(0)\|_{L^2}, \|u_{xx}^\lambda(0)\|_{L^2}, T)$, $s \in (1/2, 2)$, taking the limit when $\lambda \to \infty$, (3.57) implies

$$\int_{\mathbb{R}} |\xi|^{2\theta} |u(t,\xi)|^2 \, d\xi \leq C \, \Big(\int_{\mathbb{R}} |u(0,\xi)|^2 \, d\xi + \int_{\mathbb{R}} |\xi|^{2\theta} |u(0,\xi)|^2 \, d\xi + 1 \Big),$$

where $C = C(\theta, \|u(t)\|_{H^s}, \|u(0)\|_{L^2}, \|u_x(0)\|_{L^2}, \|u_{xx}(0)\|_{L^2}, T)$. Thus $u(t) \in \mathcal{X}^{2,\theta}$, $\theta \in [0,1]$, $t \in [-T,T]$, which proves the persistence. The global well-posedness theory in H^2 implies the uniqueness and continuous dependence upon the initial data in H^2, therefore is sufficient prove continuous dependence in the norm $\|\cdot\|_{L^2(d\dot{\mu}_\theta)}$. Let $u(t)$ and $v(t)$ be two solutions in $\mathcal{X}^{2,\theta}$, $\theta \in [0,1]$ of the IVP (1.1) with initial dates u_0 and v_0 respectively, let $u^\lambda(t)$, $v^\lambda(t)$ be the solutions of the IVP (3.39) with initial dates u_0^λ and v_0^λ respectively such that $u_0^\lambda, v_0^\lambda \in \mathbf{S}(\mathbb{R})$, $u_0^\lambda \to u_0$, $v_0^\lambda \to v_0$ in $\mathcal{X}^{2,\theta}$ and with $\lambda >> 1$, we have

$$\begin{aligned} \|u(t) - v(t)\|_{L^2(d\dot{\mu}_\theta)} \leq & \|u(t) - u^\lambda(t)\|_{L^2(d\dot{\mu}_\theta)} + \|u^\lambda(t) - v^\lambda(t)\|_{L^2(d\dot{\mu}_\theta)} \\ & + \|v^\lambda(t) - v(t)\|_{L^2(d\dot{\mu}_\theta)}. \end{aligned}$$

Convergence in (3.57) implies for $\lambda >> 1$ that

$$|u(x,t) - u^\lambda(x,t)| \leq 2|u(x,t)| \quad \text{and} \qquad |v(x,t) - v^\lambda(x,t)| \leq 2|v(x,t)|,$$

and the Dominated Convergence Lebesgue's Theorem gives

$$\|u(t) - u^\lambda(t)\|_{L^2(d\dot{\mu}_\theta)} \to 0 \quad \text{and} \quad \|v^\lambda(t) - v(t)\|_{L^2(d\dot{\mu}_\theta)} \to 0.$$

Let $w^\lambda := u^\lambda - v^\lambda$, then w^λ satisfies the equation

$$\begin{aligned} & w_t^\lambda + iaw_{xx}^\lambda + bw_{xxx}^\lambda + c(|u^\lambda|^2 w^\lambda + u^\lambda v^\lambda \bar{w}^\lambda + |v|^2 w^\lambda) \\ & \quad + d(u^\lambda v_x^\lambda \bar{w}^\lambda + w^\lambda \bar{v}^\lambda v_x^\lambda + |u^\lambda|^2 w_x^\lambda) + e(w^\lambda u^\lambda \bar{u}_x^\lambda + w^\lambda v^\lambda \bar{u}_x^\lambda + (v^\lambda)^2 \bar{w}_x^\lambda) = 0. \end{aligned}$$

Then, we multiply the above equation by $\bar{w}^\lambda$, integrate on $\mathbb{R}$ and take two times the real part, to obtain

$$\partial_t \int_{\mathbb{R}} |w^\lambda(t,x)|^2\,dx \leq h(\|u_0\|_{H^2}, \|v_0\|_{H^2}) \int_{\mathbb{R}} |w^\lambda(t,x)|^2\,dx,$$

where we have used convergence (3.57), Lema 3.1 and h is a polynomial function with $h(0,0)=0$. Therefore, by Gronwall's Lema, we have

$$\|w^\lambda(t)\|_{L^2} \leq \exp\big(T\,h(\|u_0\|_{H^2}, \|v_0\|_{H^2})\big)\,\|w_0^\lambda\|_{L^2},$$

which gives the continuous dependence in case $\theta = 0$.

Now, when $\theta = 1$ a similar argument as used in the proof of Proposition 3.2 gives

$$\|w^\lambda(t)\|_{L^2(d\dot{\mu})} \leq \exp\big(T\,h_1(\|u_0\|_{H^2}, \|v_0\|_{H^2})\big)\Big(\|w_0^\lambda\|_{L^2(d\dot{\mu})} + h_1(\|u_0\|_{H^2}, \|v_0\|_{H^2})\Big),$$

where h_1 is a continuous function with $h_1(0,0)=0$.

Consequently, applying the Abstract Interpolation Lemma, we obtain the continuous dependence for $\theta \in (0,1)$, where we have assumed temporarily that the family (w^λ) satisfies the hypothesis of the Abstract Interpolation Lemma.

Finally we prove that the sequence of solutions $(u^{\lambda_n}(t))$ satisfy the conditions (i)-(iv). Similarly, we could obtain for the sequence $(w^{\lambda_n}(t))$.

Condition (i): There exists N_0 such that $\forall \lambda \geq N_0$ and for all $t \in [-T,T]$,

$$\mathcal{L}^1(\{x \in \mathbb{R}; u^\lambda(t,x) \neq 0\}) > 0.$$

In fact, by contradiction we suppose that there exist sequences $\lambda_n \to \infty$ and $t_0 \in [-T,T]$ such that, $u^{\lambda_n}(t_0,x) = 0$ almost everywhere. By convergence (3.57) we conclude that $u(t_0) = 0$, the uniqueness of the solution implies $u = 0$, in particular $u_0 = 0$, which is a contradiction.

In the following, we consider $\mathcal{A} = (u^\lambda)_{\lambda > N_0}$, see Section 2.

Condition (ii): Inequality (2.9) is a consequence of the conservation law in L^2 and (2.10) is a consequence of the Proposition 3.2.

Condition (iii): We prove (2.11) by contradiction. If there exists a $\tilde{\theta} \in [0,1]$, such that for all $\Theta > 0$, there exist $\lambda_n > N_0$, $t_0 \in [-T,T]$, $\gamma_0 \in (0,1)$, such that

$$\int_{\{|u^{\lambda_n}(t_0)|^2 < \Theta\}} |u^{\lambda_n}(t_0)|^2\,d\dot{\mu}_{\tilde{\theta}} > \gamma_0 \int_{\mathbb{R}} |u^{\lambda_n}(t_0)|^2\,d\dot{\mu}_{\tilde{\theta}}.$$

Then, taking the limit as $\Theta \to 0^+$ in the above inequality, we arrive to contradiction.

Condition (iv): We prove (2.12) also by contradiction. If for all $R > 0$ and each $\gamma_2 \in (0,1)$, there exists $\lambda_n > N_0$, such that

$$\int_{\{|\xi| \geq R\}} |u^{\lambda_n}(0)|^2\,d\dot{\mu} > \gamma_2 \int_{\mathbb{R}} |u^{\lambda_n}(0)|^2\,d\dot{\mu}, \quad \gamma_2 \in (0,1),$$

similarly passing to the limit as $R \to +\infty$, carry to a contradiction, which proves the condition (iv). $\square$

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