# THE GERSTENHABER BRACKET AND CYCLES IN THE MODULE CATEGORY OF A MONOMIAL QUADRATIC ALGEBRA 

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#### Abstract

We establish a link between the Gerstenhaber bracket in the Hochshild cohomology and the behaviour of cycles in the module category of a monomial quadratic algebra $A$.


## Introduction

Given a finite-dimensional algebra $A$ over a field $\mathbb{k}$, its Hochschild cohomology groups, denoted by $\mathrm{HH}^{i}(A)$, are $\mathrm{HH}^{i}(A)=\operatorname{Ext}_{A^{e}}^{i}(A, A)$ where $A^{e}=A \otimes_{\mathfrak{k}} A^{o p}$ is the enveloping algebra of $A$. The natural equivalence between $A-A$-bimodules and right $A^{e}$-modules allows to see the groups $\mathrm{HH}^{i}(A)$ as extension spaces of the $A-A$-bimodule by itself. The groups corresponding to the lower degrees admit interpretations in terms of the center, the outer derivations, as well as infinitesimal deformations of the algebra $A$.

Besides this, the sum $\mathrm{HH}^{*}(A)=\oplus_{i} \geqslant 0 \mathrm{HH}^{i}(A)$ is endowed with a very rich structure, that of a Gerstenhaber algebra. That is, it is a graded commutative ring (for the cup-product) which is equipped with a Lie bracket(of degree -1 ) and the two structures are compatible, the bracket inducing graded derivations with respect to the cup-product. The two operations are defined using the standard Hochschild complex of $A$, see the Gerstenhaber's paper [6].

While the cup-product coincides with the Yoneda splicing of extensions, the Gerstenhaber bracket is harder to interpret. Schwede provided in [16] an interpretation based on homotopy classes in extension categories. Besides this rather abstract description, some concrete computations and results have been established in the last years in particular cases : string algebras [3, 13], radical-square zero algebras [14, or some group algebras [11, 15]. The main difficulty to study the Gerstenhaber bracket comes from the fact that the usual bar resolution is normally too big to carry out actual concrete computations. However, recent developments show that the bracket can be treated using alternate approaches or other resolutions [11, 19, 18, 10].

Our work in the present paper, however, points in a different direction, perhaps more "representation - theoretic". Indeed, we are mainly interested in the information about mod $-A$ that can be recovered from the Hochschild cohomology of $A$. There are number of results concerning the vanishing of the groups corresponding to the lower degrees (see [17]), but as far as we know, there are no known results concerning the additional structure of $\mathrm{HH}^{*}(A)$.

Our idea originates in section 5.4 of [9, where it is established that for representation-finite algebras the groups $\mathrm{HH}^{i}(A)$ for $i \geqslant 2$ detect cycles in the module category mod $-A$. Let us now explain the idea while we describe the structure and contents of the paper.

[^0]We focus on monomial quadratic algebras, a class for which a very convenient minimal projective resolution of ${ }_{A} A_{A}$ (due to Bardzell [2]) and an explicit comparison morphism 3, 12 with the standard bar resolution are known. This makes possible to carry the operations defined using the standard Hochschild complex to the Bardzell complex. The required details are given in section 1 where in addition we recall the notions concerning algebras and modules that will be needed in the sequel.

The minimal resolution is parametrized by the extension groups between the simple $A$-modules. These extensions are known to be, after Bardzell [2], completely described by the so-called "associated sequences of paths", that we call by $\Gamma$-paths. In section 2 we introduce combinatorial and representation theoretic material for describing the defining composition operations " $\circ_{i}$ " of the Gerstenhaber bracket. This includes the concepts of $\Gamma$-bypasses and their composition as well specific cycles in mod $-A$ - called admissible cycles - associated to $\Gamma$-bypasses together with a certain composition of these cycles. On one hand, the extension spaces between simple modules have bases parametrised by $\Gamma$-paths and the cup product of these extensions can be translated in terms of compositions of specific $\Gamma$-bypasses. On the other hand, $\Gamma$-bypasses parametrize cochains on Bardzell's resolution with values in $A$; we prove that the " $\circ_{i}$ " composition of these cochains is encoded by the composition of the associated admissible cycles.

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## 1. Preliminaries

1.1. Algebras and modules. While we briefly recall some particular concepts concerning bound quiver algebras, we refer the reader to [1], for instance, for unexplained notions.

Let $Q=\left(Q_{0}, Q_{1}, \mathfrak{s}, \mathfrak{t}\right)$ be a finite quiver and $\mathbb{k}$ a commutative field. We consider algebras of the form $A=\mathbb{k} Q / I$ with $I$ an admissible ideal of the path algebra $\mathbb{k} Q$. This includes, but does not restrict to, all basic connected algebras over algebraically closed fields. The composition of two arrows $\alpha: i \rightarrow j$ and $\beta: j \rightarrow l$ is denoted by $\alpha \beta$. A two-sided ideal $I$ of the path algebra $\mathbb{k} Q$ is said to be monomial if it can be generated by a set of paths. If there is a set of generators formed by linear combinations of paths of length 2 , then $I$ is said to be quadratic. We will mainly focus on monomial and quadratic algebras, and by abuse of notation will identify a path $w$ in $\mathbb{k} Q$ with its class $w+I$ in $A$.

Given an arrow $\alpha: x \rightarrow y$ we denote by $\alpha^{-1}: y \rightarrow x$ its formal inverse, and agree that $\left(\alpha^{-1}\right)^{-1}=\alpha$. A walk in $Q$ is a composition $w=\alpha_{1} \cdots \alpha_{n}$ of arrows or formal inverses of arrows (so the target of $\alpha_{i}$ is the source of $\alpha_{i+1}$ for $i \in\{1, \ldots, n-1\}$ ). A walk $w$ as above is reduced if $\alpha_{i-1} \neq \alpha_{i}^{-1}$ for $i \in\{2, \ldots, n\}$. Let $A=\mathbb{k} Q / I$ with $I$ monomial. A reduced walk $w$ in $Q$ is a string in $(Q, I)$ if neither $w$ nor $w^{-1}$ contain a subpath that is a path in $I$. A string is direct if it is composed entirely by arrows of $Q$ and inverse if it is composed only by formal inverses of arrows. The trivial walks are both inverse and direct.

To a string $w=\alpha_{1} \cdots \alpha_{n}$ is associated the so-called string module $M(w)$. If $w=e_{x}$ is the trivial walk at some vertex $x$, then $M(w)$ is simply the corresponding simple module $S_{x}$. Let $w=\alpha_{1} \ldots \alpha_{n}$ be a string with $n \geqslant 1$. For $i \in 1, \ldots, n+1$, define $K_{i}=\mathbb{k}$ and define $\varphi_{\alpha_{i}}$ as the identity map $K_{i} \rightarrow K_{i+1}$ if $\alpha_{i}$ is an arrow, and the identity map $K_{i+1} \rightarrow K_{i}$ in case $\alpha_{i}^{-1}$ is an arrow of $Q$.

The $A$-module $M(w)$ (equivalently the representation of $(Q, I))$ is defined as follows:

- For a vertex $a \in Q_{0}, M(w)_{a}$ is the sum of all the vector spaces $K_{i}$ such that the source of $\alpha_{i}$ is $a$, together with $K_{n+1}$ in case the target of $\alpha_{n}$ is $a$. In particular, if $w$ does not pass trough $a$, then $M(w)_{a}=0$.
- Given $\gamma \in Q_{1}$ and $\epsilon \in\{ \pm 1\}$, if $\gamma^{\epsilon}$ appears in $w$, then $M(w)_{\gamma}$ is the direct sum of the maps $\varphi_{\alpha_{i}}$ such that $\alpha_{i}=\gamma^{\epsilon}$. If $\gamma^{\epsilon}$ does not appear in $w$, them $M(w)_{\gamma}=0$.
Given a string $u=\alpha_{1} \cdots \alpha_{n}$ and a substring $v=\alpha_{i} \cdots \alpha_{j}$ (for some $i, j$ with $1 \leqslant i<j \leqslant n$ ) we also have that :
there is a monomorphism

$$
M(v) \rightarrow M(u) \text { if and only if }\left\{\begin{array}{lll}
\alpha_{j+1} \text { is inverse } & \text { or } & j=n \\
\text { and } & & \\
\alpha_{i-1} \text { is direct } & \text { or } i=1
\end{array}\right.
$$

And, dually, there is an epimorphism

$$
M(u) \rightarrow M(v) \text { if and only if }\left\{\begin{array}{lll}
\alpha_{j+1} \text { is direct } & \text { or } & j=n \\
\text { and } & & \\
\alpha_{i-1} \text { is inverse } & \text { or } & i=1
\end{array}\right.
$$

In particular, if $\alpha: i \rightarrow j$ is an arrow, we have a monomorphism $S_{j} \rightarrow M(\alpha)$ and an epimorphism $M(\alpha) \rightarrow S_{i}$.
1.2. Gerstenhaber bracket in Hochschild cohomology. In 6] Gerstenhaber defined a Lie bracket $[-,-]$ on $\mathrm{HH}^{*}(A)$. The original definition was given in terms of the complex obtained upon applying $\operatorname{Hom}_{A^{e}}(-, A)$ to the standard bar resolution of $A_{A} A_{A}$.

In [14] Sánchez - Flores showed that the bracket can be defined using the so-called reduced resolution (or radical resolution, see [7, 4]). We now follow [14]. Let $A=\mathbb{k} Q / I$ and $E$ be the semi-simple subalgebra of $A$ generated by $Q_{0}$, the vertices if $Q$. As $E-E$-bimodules, we have that $A=E \oplus r$ (where $r$ is the Jacobson radical of $A$ ). In the remaining part of this section all tensor products are taken over $E$. Let $r^{\otimes_{k}^{n}}$ denote the $n^{\text {th }}$ tensor power of $r$ with itself. One then has a projective resolution of $A_{A}$

$$
\mathbf{R}_{\bullet}: \cdots \rightarrow A \otimes_{\mathbb{k}} r^{\otimes_{\mathfrak{k}} n} \otimes_{\mathbb{k}} A \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_{1}} A \otimes_{\mathbb{k}} r \otimes_{\mathbb{k}} A \xrightarrow{\delta_{0}} A \otimes_{\mathbb{k}} A \xrightarrow{\mu} A \rightarrow 0
$$

where $\mu$ is the multiplication of $A$ and

$$
\delta_{n-1}=\sum_{j=0}^{n}(-1)^{j} \mathbf{1}^{\otimes j} \otimes \mu \otimes \mathbf{1}^{\otimes(n-j)}
$$

In order to compute the Hochschild cohomology spaces we apply the functor $\operatorname{Hom}_{A^{e}}(-, A)$ to R., use the identification

$$
\begin{equation*}
\operatorname{Hom}_{A^{e}}\left(A \otimes_{\mathbb{k}} r^{\otimes_{\mathfrak{k}}^{n}} \otimes_{\mathbb{k}}, A\right) \simeq \operatorname{Hom}_{E^{e}}\left(r^{\otimes_{\mathfrak{k}}^{n}}, A\right) \tag{1.2.1}
\end{equation*}
$$

and denote the right-hand term by by $\left(r^{n}, A\right)$. The identification allows to carry de differential, still denoted by $\delta_{\bullet}$, giving a complex that we will denote by $\left(\mathbf{r}^{\bullet}, A\right)$.

Let $\pi: A \rightarrow r$ be the canonical projection of $A$ onto its radical. Note that it is an $E-E$ morphism.

For $f \in\left(r^{n}, A\right), g \in\left(r^{m}, A\right)$ and $i \in\{1, \ldots, n\}$ let $f \circ_{i} g \in\left(r^{n+m-1}, A\right)$ be defined by the formula

$$
f \circ_{i} g=f\left(\mathbf{1}^{\otimes i-1} \otimes \pi g \otimes \mathbf{1}^{n-i}\right) .
$$

Further, let

$$
f \circ g=\sum_{i=1}^{n}(-1)^{(i-1)(m-1)} f \circ_{i} g
$$

and finally

$$
[f, g]=f \circ g-(-1)^{(n-1)(m-1)} g \circ f
$$

This operation induces the so-called Gerstenhaber bracket in $\mathrm{HH}^{*}(A)$.
1.3. The minimal resolution. The reduced resolution is smaller than the standard one, but still too big to be used efficiently for our purpose.

From now on, we will be interested in monomial quadratic algebras, so we can use the minimal resolution of Bardzell [2] which has the following description. Let $\Gamma^{0}=Q_{0}, \Gamma^{1}=Q_{1}$ and for $n \geqslant 2, \Gamma^{n}=\left\{\alpha_{1} \cdots \alpha_{n} \mid \alpha_{i} \in Q_{1}, \alpha_{i} \alpha_{i+1} \in I\right.$ for $\left.1 \leqslant i<n\right\}$. The elements of some $\Gamma^{n}$ will be called $\Gamma$ - paths. If $x, y \in Q_{0}$ are fixed, we will write $\Gamma_{x, y}^{n}$ for the set of $\Gamma$-paths going from $x$ to $y$. We know from [8] that for every natural number $n$ and every fixed vertices $x, y \in Q_{0}$, the set $\Gamma_{x, y}^{n}$ is in bijection with a basis for $\operatorname{Ext}_{A}^{n}\left(S_{x}, S_{y}\right)$.

For $n \geqslant 0$, denote by $\mathbb{k} \Gamma^{n}$ the $\mathbb{k}$-vector space with basis $\Gamma^{n}$. With these notations we have a minimal projective resolution of ${ }_{A} A_{A}$.

$$
\mathbf{M}_{\bullet}: \cdots \rightarrow A \otimes_{\mathbb{k}} \mathbb{k} \Gamma^{n} \otimes_{\mathbb{k}} A \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_{1}} A \otimes_{\mathbb{k}} \mathbb{k} \Gamma^{1} \otimes_{\mathbb{k}} A \xrightarrow{\partial_{0}} A \otimes_{\mathbb{k}} \mathbb{k} \Gamma^{0} \otimes_{\mathbb{k}} A \xrightarrow{\epsilon} A \rightarrow 0
$$

where $\epsilon$ is the composition of the isomorphism $A \otimes_{\mathbb{k}} \mathbb{k} \Gamma^{0} \otimes_{\mathbb{k}} A \simeq A \otimes_{\mathbb{k}} A$ with the multiplication of $A$. The differentials are given by

$$
\partial_{n-1}\left(1 \otimes \alpha_{1} \cdots \alpha_{n} \otimes 1\right)=\alpha_{1} \otimes \alpha_{2} \cdots \alpha_{n} \otimes 1+(-1)^{n} \otimes \alpha_{1} \cdots \alpha_{n-1} \otimes \alpha_{n}
$$

We then apply $\operatorname{Hom}_{A^{e}}(-, A)$ to $\mathbf{M}_{\bullet}$ and make the identification

$$
\begin{equation*}
\operatorname{Hom}_{A^{e}}\left(A \otimes_{\mathbb{k}} \mathbb{k} \Gamma^{n} \otimes_{\mathbb{k}} A, A\right) \simeq \operatorname{Hom}_{E^{e}}\left(\mathbb{k} \Gamma^{n}, A\right) \tag{1.3.1}
\end{equation*}
$$

We denote the right-hand term by $\left(\Gamma^{n}, A\right)$, and the resulting complex by $\left(\boldsymbol{\Gamma}^{\bullet}, A\right)$.
In order to define the Gerstenhaber bracket using the complex $\left(\boldsymbol{\Gamma}^{\bullet}, A\right)$, an explicit comparison of complexes may be used. This approach was used in [3, 1.3 and 1.4], and in [13] for quadratic string algebras but can be generalized to all monomial quadratic algebras.

The comparison morphisms

$$
\mu_{\bullet}: A \otimes_{\mathbb{k}} \mathbb{k} \Gamma^{n} \otimes_{\mathbb{k}} A=\mathbf{M}_{\bullet} \rightleftarrows \mathbf{R}_{\bullet}=A \otimes_{\mathbb{k}} r^{\otimes_{k}^{n}} \otimes_{\mathbb{k}} A: \omega_{\bullet}
$$

are defined as follows (recall that $E=\mathbb{k} Q_{0}=\mathbb{k} \Gamma^{0}$ and keep in mind that all the tensor products involved are taken over $E$ ):

- The map $\mu_{\bullet}$ is defined by :

$$
\begin{aligned}
\mu_{0}\left(1 \otimes e_{i} \otimes 1\right) & =1 \otimes e_{i}=e_{i} \otimes 1 \\
\mu_{n}\left(1 \otimes \alpha_{1} \cdots \alpha_{n} \otimes 1\right) & =1 \otimes \alpha_{1} \otimes \alpha_{2} \otimes \cdots \otimes \alpha_{n} \otimes 1 \text { for } n \geqslant 1
\end{aligned}
$$

- The map $\omega_{\bullet}$ is defined by

$$
\begin{aligned}
& -\omega_{0}(1 \otimes 1)=1 \otimes \sum_{i \in Q_{0}} e_{o} \otimes 1 \\
& -\omega_{1}\left(1 \otimes \alpha_{1} \cdots \alpha_{n} \otimes 1\right)=\sum_{i=1}^{n} \alpha_{1} \cdots \alpha_{i-1} \otimes \alpha_{i} \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_{n} \text { for all paths } \alpha_{1} \cdots \alpha_{n} \\
& \quad\left(\alpha_{i} \in Q_{1}\right)
\end{aligned}
$$

- and $\omega_{n}(1 \otimes p \otimes 1)$ is equal to $u \otimes v \otimes w$ if the path $p$ decomposes as $u v w$ where $u, v, w$ are paths such that $v \in \Gamma^{n}, u$ being of minimal length and 0 otherwise.
The maps $\mu_{\bullet}$ and $\omega_{\bullet}$ are both morphisms of complexes and moreover $\omega_{\bullet} \mu_{\bullet}=\mathbf{1}$. Upon applying $\operatorname{Hom}_{A^{e}}(-, A)$ and making the identifications (1.3.1) and (1.2.1) mentioned above, we obtain two quasi-isomorphisms

$$
-\circ \mu^{\bullet}:\left(\mathbf{r}^{\bullet}, A\right) \rightleftarrows\left(\boldsymbol{\Gamma}^{\bullet}, A\right):-\circ \omega^{\bullet}
$$

This allows to define the operations in $\left(\Gamma^{\bullet}, A\right)$ which will be used to investigate the Gerstenhaber bracket. More precisely, given $f \in\left(\Gamma^{n}, A\right), g \in\left(\Gamma^{m}, A\right)$, and $i \in\{1, \ldots, n\}$, one defines

$$
f \circ_{i} g=\mu^{n+m-1}\left(\omega^{n} f \circ_{i} \omega^{m} g\right)
$$

where the symbol $\circ_{i}$ on the right-hand term denotes the operation defined using the reduced bar resolution. From this point, define a bracket, still denoted $[-,-]$ on $\left(\boldsymbol{\Gamma}^{\bullet}, A\right)$ on $\left(\boldsymbol{\Gamma}^{\bullet}, A\right)$ following the same steps as for $\left(\mathbf{r}^{\bullet}, A\right)$ (see section 1.2).

The crucial point is that given $f \in\left(\Gamma^{n}, A\right), g \in\left(\Gamma^{m}, A\right)$ and $w=\alpha_{1} \cdots \alpha_{n+m-1} \in \Gamma^{n+n-1}$ then

$$
f \circ_{i} g(w)=f\left(\alpha_{1} \cdots \alpha_{i-1} g\left(\alpha_{i} \cdots \alpha_{i+m-1}\right) \alpha_{i+m} \cdots \alpha_{n+m-1}\right)
$$

whenever $\alpha_{1} \cdots \alpha_{i-1} g\left(\alpha_{i} \cdots \alpha_{i+m-1}\right) \alpha_{i+m} \cdots \alpha_{n+m-1} \in \Gamma^{n}$, and $f \circ_{i} g(w)$ vanishes otherwise. Note that $\alpha_{i} \cdots \alpha_{i+m-1} \in \Gamma^{m}$.

## 2. Operations on paths, extensions and cycles

2.1. Compositions of paths. Let $(Q, I)$ be a bound quiver with $I$ monomial and quadratic.

Let $m, n$ be positive integers. Let $r, s$ be integers such that $1 \leqslant s \leqslant r \leqslant n$. We define an $(s, r)$ $\Gamma$-bypass to be a pair $(u, v)$ where $u=\alpha_{1} \cdots \alpha_{n} \in \Gamma^{n}\left(\alpha_{i} \in Q_{1}\right), v=\beta_{1} \cdots \beta_{m} \in \Gamma^{m}\left(\beta_{i} \in Q_{1}\right)$, such that $\alpha_{1} \cdots \alpha_{s-1} \beta_{1} \cdots \beta_{m} \alpha_{r} \cdots \alpha_{n} \in \Gamma^{s+m+n-r}$. For such a $(u, v)$ we then define the composition of $u$ and $v$ at positions $s$ and $r$ as

$$
u_{s} \circ_{r} v=\alpha_{1} \cdots \alpha_{s-1} \beta_{1} \cdots \beta_{m} \alpha_{r} \cdots \alpha_{n}
$$

Informally, we think as "start with $u$, then switch to $v$ for the $s{ }^{\text {th }}$ arrow, go along $v$, then switch back to $u$ when its $r^{\text {th }}$ arrow is hit".

We define a $\Gamma$-bypass as a couple $(u, v)$ which is an $(s, r) \Gamma$-bypass for some $s, r$.
A particularly interesting case in view of the study of the Gerstenhaber bracket is when $u_{s} \circ_{r} v \in$ $\Gamma^{n+m-1}$, that is precisely when $r=s+1$. On this situation we shall write $u \circ_{s} v$ instead of $u_{s} \circ_{s+1} v$.

Note also that since $u$ and $v$ are allowed to contain cycles, the endpoints of $v$ can appear more than once while running along $u$. Several compositions are thus possible in general.

Example. Let us consider the quiver

bound by all the monomial relations of length 2. Let $u=a b c d e c d \in \Gamma^{7}$ and $v=f g \in \Gamma^{3}$. We then have $u_{3} \circ_{5} v=a b f g e c d \in \Gamma^{7}, u_{3} \circ_{8} v=a b f g \in \Gamma^{4}, u_{6} \circ_{8} v=a b c d e f g \in \Gamma^{7}$.
2.2. Paths and extensions. Let $x, y$ be vertices. It is well known that an arrow of $Q$ (equivalently a path in $\Gamma_{x, y}^{1}$ ) $\alpha: x \rightarrow y$ defines an extension between the simple modules $S_{x}$ and $S_{y}$; this extension is as follows (the morphisms are the natural ones)

$$
\mathbf{E}(\alpha): 0 \rightarrow S_{y} \rightarrow M(\alpha) \rightarrow S_{x} \rightarrow 0
$$

This construction on paths in $\Gamma_{x, y}^{1}$ extends to a construction on paths in $\Gamma_{x, y}^{n}$, for all integers $n$. For all $u=\alpha_{1} \cdots \alpha_{n} \in \Gamma_{x, y}^{n}\left(\alpha_{i} \in Q_{1}\right)$ denote by $\mathbf{E}(u)$ the following exact sequence where all the arrows are the natural morphisms

$$
\mathbf{E}(u): 0 \rightarrow S_{y} \rightarrow M\left(\alpha_{n}\right) \rightarrow M\left(\alpha_{n-1}\right) \rightarrow \cdots \rightarrow M\left(\alpha_{1}\right) \rightarrow S_{x} \rightarrow 0 .
$$

Recall that the vector space $\operatorname{Ext}_{A}^{n}\left(S_{x}, S_{y}\right)$ has several equivalent definitions. One of them defines this space as the set of appropriate equivalence classes $\overline{\mathbf{E}}$ of $n$-fold extensions $\mathbf{E}: 0 \rightarrow S_{y} \rightarrow M_{n} \rightarrow$ $\cdots \rightarrow M_{1} \rightarrow S_{x} \rightarrow 0$. Another equivalent point of view is to consider it as the cohomology group $H^{n} \operatorname{Hom}_{A}\left(P, S_{y}\right)$, where $P$ is any projective resolution of $S_{x}$ in $\bmod -A$. In this text it will be convenient to deal with the former description. The following lemma is part of the folklore and follows from [8]. We provide a proof for convenience. In what follows, given a path $p$ in $Q$ we denote by $\mathfrak{s}(p)$ and $\mathfrak{t}(p)$ its source and its target.

Lemma. The family $\{\overline{\mathbf{E}(u)}\}_{u \in \Gamma_{x, y}^{n}}$ is a basis of $\operatorname{Ext}_{A}^{n}\left(S_{x}, S_{y}\right)$.
Proof. Using a projective resolution $P$ of $S_{x}$ due to [8], we will make explicit the cohomology class in $H^{n} \operatorname{Hom}_{A}\left(P, S_{y}\right)$ corresponding to a given $u=\alpha_{1} \cdots \alpha_{n} \in \Gamma_{x, y}^{n}\left(\alpha_{i} \in Q_{1}\right)$. The conclusion will follow from this description. Recall that $S_{x}$ has of projective resolution of the following form

$$
\cdots \rightarrow P^{-\ell} \xrightarrow{d^{-\ell}} P^{-\ell+1} \rightarrow \cdots \rightarrow P^{-1} \rightarrow P^{0} \xrightarrow{\pi} S_{x}
$$

where

$$
P^{-\ell}=\bigoplus_{p \in \Gamma_{x,-}^{\ell}} e_{\mathfrak{t}(p)} A \text { for all } \ell \geqslant 0
$$

- $\pi$ is the natural projection,
- for all $\ell \geqslant 1$, the differential $d^{-\ell}$ is such that $d\left(e_{\mathfrak{t}\left(\beta_{1} \cdots \beta_{\ell}\right)}\right)=\beta_{\ell}\left(\in e_{\mathfrak{t}\left(\beta_{1} \cdots \beta_{\ell-1}\right)} A\right)$ for all $\beta_{1} \cdots \beta_{\ell} \in \Gamma_{x,-}^{\ell}\left(\beta_{i} \in Q_{1}\right)$.
In terms of Bardzell's resolution, this is $S_{x} \otimes_{A} \mathbf{M}$. Now, here is a natural morphism of complexes determined by $\mathbf{E}(u)$

where
- $f^{-n}: P^{-n} \rightarrow S_{y}$ is the following composition of natural morphisms

$$
P^{-n}=\bigoplus_{p \in \Gamma_{x,-}^{n}} e_{\mathfrak{t}}(p) A \rightarrow e_{\mathfrak{t}\left(\alpha_{1} \cdots \alpha_{n}\right)} A \rightarrow S_{\mathfrak{t}\left(\alpha_{n}\right)}
$$

- for all $\ell \in\{0,1, \ldots, n-1\}$, the morphism $f^{-\ell}: P^{-\ell} \rightarrow M\left(\alpha_{\ell+1}\right)$ is the following composition of natural morphisms

$$
P^{-\ell}=\bigoplus_{p \in \Gamma_{x,-}^{\ell}} e_{\mathfrak{t}(p)} A \rightarrow e_{\mathfrak{t}\left(\alpha_{1} \cdots \alpha_{\ell}\right)} A=e_{\mathfrak{s}\left(\alpha_{\ell+1}\right)} A \rightarrow M\left(\alpha_{\ell+1}\right)
$$

Under the identification of equivalence classes of $n$-fold extensions $0 \rightarrow S_{y} \rightarrow \cdots \rightarrow S_{x} \rightarrow 0$ and cohomology classes in $H^{n} \operatorname{Hom}_{A}\left(P, S_{y}\right)$, the equivalence class $\overline{\mathbf{E}(u)}$ corresponds to the cohomology class of $f^{-n}$.

In view of this reminder, $\{\overline{\mathbf{E}(u)}\}_{u \in \Gamma_{x, y}^{n}}$ is a basis of $\operatorname{Ext}_{A}^{n}\left(S_{x}, S_{y}\right)$ because $S_{y}$ is simple.
In addition to $n$ and $x, y$, let $m$ be an integer and $x^{\prime}, y^{\prime}$ be vertices of $Q$. We now introduce an operation, denoted by ${ }_{s} \circ_{r}$,

$$
{ }_{s} \circ_{r}: \operatorname{Ext}_{A}^{n}\left(S_{x}, S_{y}\right) \otimes_{\mathbb{k}} \operatorname{Ext}_{A}^{m}\left(S_{x^{\prime}}, S_{y^{\prime}}\right) \rightarrow \operatorname{Ext}_{A}^{n-r+s-m}\left(S_{x}, S_{y}\right)
$$

for all integers $s, r$ such that $1 \leqslant s \leqslant r \leqslant n$. It is defined by its behaviour on tensors of the shape $\mathbf{E}(u) \otimes_{\mathbb{k}} \mathbf{E}(v)\left(u \in \Gamma_{x^{\prime}, y^{\prime}}^{n}\right.$ and $\left.v \in \Gamma_{x, y}^{m}\right)$ as follows

$$
\mathbf{E}(u) \otimes_{\mathbb{k}} \mathbf{E}(v) \longmapsto \begin{cases}\mathbf{E}\left(u_{s} \circ_{r} v\right) & \text { if }(u, v) \text { is an }(s, r) \Gamma \text {-bypass }, \\ 0 & \text { otherwise }\end{cases}
$$

Here is an alternative description when $(u, v)$ is an $(s, r) \Gamma$-bypass. We use the following notation

- $u=\alpha_{1} \cdots \alpha_{n}\left(\alpha_{i} \in Q_{1}\right), v=\beta_{1} \cdots \beta_{m}\left(\beta_{i} \in Q_{1}\right)$,
- $u^{\prime}=\alpha_{1} \cdots \alpha_{s-1}\left(\in \Gamma^{s-1}\right), \widehat{u}=\alpha_{s} \cdots \alpha_{r-1}\left(\in \Gamma^{r-s}\right)$, and $u^{\prime \prime}=\alpha_{r} \cdots \alpha_{n}\left(\in \Gamma^{n-r+1}\right)$.

Hence $u=u^{\prime} \widehat{u} u^{\prime \prime}$ and $u_{s} \circ_{r} v=u^{\prime} v u^{\prime \prime}$. In terms of long exact sequences, $\mathbf{E}(u)$ is obtained by splicing $\mathbf{E}\left(u^{\prime}\right), \mathbf{E}(\widehat{u})$, and $\mathbf{E}\left(u^{\prime \prime}\right)$ :


In terms of cup-products, this means that the following equalities hold true

$$
\begin{aligned}
& \mathbf{E}(u)=\mathbf{E}\left(u^{\prime \prime}\right) \cup \mathbf{E}(\hat{u}) \cup \mathbf{E}\left(u^{\prime}\right){\text { in } \operatorname{Ext}_{A}^{n}\left(S_{x}, S_{y}\right)}^{\mathbf{E}(u)_{s} \circ r} \mathbf{E}(v):=\mathbf{E}\left(u^{\prime \prime}\right) \cup \mathbf{E}(v) \cup \mathbf{E}\left(u^{\prime}\right) \\
& \text { in } \operatorname{Ext}_{A}^{n-r+s-m}\left(S_{x}, S_{y}\right) .
\end{aligned}
$$

2.3. Cochains and cycles. In the previous sections we considered $\Gamma$-paths and extensions. We now turn our attention to the Bardzell complex, through which the Hochschild cohomology spaces are computed.

A non-zero cochain $f: \mathbb{k} \Gamma_{x, y}^{n} \rightarrow A$ sends and $\Gamma$-path $u$ to a linear combination of paths from $x$ to $y$. We will see that with some mild additional hypotheses, this context allows to obtain what we call a cycle in mod $-A$, that is, a diagram in mod $-A$ of the shape $X_{0} \rightarrow X_{1} \rightarrow \cdots \rightarrow X_{N}$ where each morphism is non-zero and non invertible, each $X_{i}$ is indecomposable, and $X_{N} \simeq X_{0}$.

Let $u=\alpha_{1} \cdots \alpha_{n} \in \Gamma_{x, y}^{n}$ and assume there is a non-zero path $p \in e_{x} A e_{y}$. We thus have a non-zero cochain, in the Bardzell complex : it sends $u$ to $p$ and every other $\Gamma$-path to zero. We denote it by $\chi_{u}^{p}$. Since the set of these cochains is a $\mathbb{k}$-basis for $\operatorname{Hom}_{E^{e}}\left(\mathbb{k} \Gamma_{x, y}^{n}, A\right)$, we call them basic cochains, and their degree is $n$. A basic cochain $\chi_{u}^{p}$ of degree $n$ is said to be reduced if $p$ and $u$ do not start with the same arrow $\alpha_{1}$ and do not end with the same arrow $\alpha_{n}$.


Notice that if $p$ is a stationary path, then $x=y$ and $u$ is an oriented cycle in $Q$. The oriented cycle $u$ then yields the following cycle in $\bmod -A$ which happens to be exact

$$
M\left(\alpha_{n}\right) \longrightarrow M\left(\alpha_{n-1}\right) \longrightarrow \cdots \longrightarrow M\left(\alpha_{1}\right) \longrightarrow M\left(\alpha_{n}\right)
$$

Moreover, the kernels and cokernels of the morphisms appearing in this cycle are precisely the simple modules corresponding to the sources and targets of the arrows $\alpha_{i}$ for $i \in\{1, \ldots, n\}$ so that $\mathbf{E}(u)$ can be easily recovered from the given oriented cycle. The same holds for the extensions $\mathbf{E}\left(\alpha_{i} \cdots \alpha_{n} \alpha_{1} \cdots \alpha_{i-1}\right)$ corresponding to the cycles obtained from $u$ by cyclic permutation.

Note also that this holds regardless if $A$ is of finite representation type or not. The cycle mentioned above is a cycle in $\bmod -A$, not necessarily a cycle in the Auslander - Reiten quiver of $A$.

In general it is still possible to associate a cycle in mod $-A$ to a basic and reduced cochain. The construction is detailed below. The resulting cycle has a specific shape and we give a name to it for later purposes. We define an admissible cycle of degree $n \mathrm{in} \bmod -A$ as a cycle of the following shape

$$
\begin{equation*}
M\left(\alpha_{n-1}\right) \rightarrow M\left(\alpha_{n-2}\right) \rightarrow \cdots \rightarrow M\left(\alpha_{2}\right) \rightarrow M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right) \rightarrow M\left(\alpha_{n-1}\right) \tag{2.3.2}
\end{equation*}
$$

where

- $\alpha_{1} \cdots \alpha_{n}$ is a path in $Q$ (with $\left.\alpha_{i} \in Q_{1}\right)$ with length $n$ at least 2 .
- $p$ is a path parallel to $\alpha_{1} \cdots \alpha_{n}$ such that $\alpha_{1}$ is not a prefix of $p$ and $\alpha_{n}$ is not a suffix of $p$ and such that $p$ is non-zero in $A$, in particular the module $M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)$ is well-defined. This ensures that the arrows $\alpha_{1}, \ldots, \alpha_{n}$ and the path $p$ hence form a diagram the one shown in (2.3.1), above, from which we adopt the numbering of the vertices.
- all the morphisms are the natural ones, note that
$-S_{1}$ is a direct summand of $\operatorname{soc}\left(M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)\right)$ and the map $M\left(\alpha_{2}\right) \rightarrow M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)$ is the composite morphism $M\left(\alpha_{2}\right) \rightarrow S_{1} \hookrightarrow M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)$,
$-S_{n-1}$ is a direct summand of $\operatorname{top}\left(M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)\right)$ and the map $M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right) \rightarrow M\left(\alpha_{n-1}\right)$ is the composite morphism $M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right) \rightarrow S_{n-1} \hookrightarrow M\left(\alpha_{n-1}\right)$.
Note that both the sequence of arrows $\alpha_{1}, \ldots, \alpha_{n}$ and the path $p$ may be recovered from the admissible cycle. First, the sequence of two-dimensional modules $M\left(\alpha_{n-1}\right), M\left(\alpha_{n-2}\right), \ldots, M\left(\alpha_{2}\right)$ determine the arrows. Next, $M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)$ is the only non uniserial module of the cycle and it determines the modules $M\left(\alpha_{1}\right), M\left(\alpha_{n}\right)$ and $M(p)$ (hence the arrows $\alpha_{1}, \alpha_{p}$ and the path $p$ ):
- $M\left(\alpha_{1}\right)$ is the unique quotient of $M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)$ of the shape $M\left(p^{\prime}\right)$ for some path $p^{\prime}$, and such that the socle and the top of $M\left(p^{\prime}\right)$ are direct summands of the socle and the top, respectively, of $M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)$,
- $M\left(\alpha_{n}\right)$ is the unique submodule of $M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)$ of the shape $M\left(p^{\prime}\right)$ for some path $p^{\prime}$, and such that the socle and the top of $M\left(p^{\prime}\right)$ are direct summands of the socle and the top, respectively, of $M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)$,
- finally $M(p)$ is the unique non zero cohomology group of (2.3.2).

Thus, the modules $M\left(\alpha_{1}\right), \ldots, M\left(\alpha_{n}\right), M(p)$, and hence the of arrows $\alpha_{1}, \ldots, \alpha_{n}$ and the path $p$ are determined by the admissible cycle.

Now let $\chi_{u}^{p}$ be a basic and reduced cochain ( $u=\alpha_{1} \cdots \alpha_{n}, \alpha_{i} \in Q_{1}$ ) of degree at least two, and keep in mind the labelling of the vertices and arrows of (2.3.1). It follows from the definition that the following diagram, where all the arrows are the natural morphisms and which we will denote by $\mathbf{C}(u, p)$, is an admissible cycle

$$
\mathbf{C}(u, p): M\left(\alpha_{n-1}\right) \rightarrow M\left(\alpha_{n-2}\right) \rightarrow \cdots \rightarrow M\left(\alpha_{2}\right) \rightarrow M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right) \rightarrow M\left(\alpha_{n-1}\right) .
$$

Note that this sequence is exact except at $M\left(\alpha_{1}^{-1} p \alpha_{n-1}^{-1}\right)$. Also, the kernel of the map with domain $M\left(\alpha_{j}\right)$ is the simple module $S_{j}$ for $j \in\{2, \ldots, n-1\}$ and $S_{1}$ is the image of $M\left(\alpha_{2}\right) \rightarrow M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)$. It follows from this remark that the admissible cycle $\mathbf{C}(u, p)$ alows to obtain cycles starting and ending at each of theses simple modules.

On the other hand, in case $n=2$ we obtain an admissible cycle that is reduced to one single module $M\left(\alpha_{1}^{-1} p \alpha_{2}^{-1}\right)$ with a non invertible non trivial endomorphism (see the example below).

The cycle $\mathbf{C}(u, p)$ may be recovered using homological algebra as follows. Denote $\alpha_{1} \cdots \alpha_{n-1}$ by $u^{\prime}$, this is the only suffix of $u$ lying in $\Gamma^{n-1}$; its associated $(n-1)$-fold extension is

$$
\mathbf{E}\left(u^{\prime}\right): 0 \rightarrow S_{n-1} \rightarrow M\left(\alpha_{n-1}\right) \rightarrow M\left(\alpha_{n-2}\right) \rightarrow \cdots \rightarrow M\left(\alpha_{1}\right) \rightarrow S_{x} \rightarrow 0 .
$$

The module $M\left(p \alpha_{n}^{-1}\right)$ is well-defined because $p$ does not end with $\alpha_{n}$. Moreover, its top contains $S_{x}$ as a direct summand because $p$ starts at $x$. Denote by $\pi_{x}$ the natural surjection $M\left(p \alpha_{n}^{-1}\right) \rightarrow S_{x}$. Considering the pullback of $\mathbf{E}\left(u^{\prime}\right)$ along $\pi_{x}$ yields the following commutative diagram whose rows are exact


Note that $S_{n-1}$ is also a direct summand of top $M\left(p \alpha_{n}^{-1}\right)$. Now, keeping the first row in the above pullback diagram and composing the rightmost non-zero morphism with the natural surjection $M\left(p \alpha_{n}^{-1}\right) \rightarrow S_{n-1}$ and the natural inclusion $S_{n-1} \hookrightarrow M\left(\alpha_{n-1}\right)$ yields the cycle $\mathbf{C}(u, p)$.

In our homological construction we started by considering $u^{\prime}=\alpha_{1} \cdots \alpha_{n-1}$, but could also start with $u^{\prime \prime}=\alpha_{2} \cdots \alpha_{n}$, the unique suffix of $u$ belonging to $\Gamma^{n-1}$, starting at the vertex 1 , the target of $\alpha_{1}$, then consider its associated extension $\mathbf{E}\left(u^{\prime \prime}\right)$ and the pushout with the map $S_{1} \rightarrow M\left(\alpha_{1}^{-1} p\right)$. This leads, again, to consider the module $M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)$. The cycle is then completed with the map $S_{y} \rightarrow M\left(\alpha_{1}^{-1} p\right)$, and we are led to

$$
M\left(\alpha_{2}\right) \rightarrow M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right) \longrightarrow M\left(\alpha_{n-1}\right) \longrightarrow \cdots \rightarrow M\left(\alpha_{2}\right)
$$

The cycle obtained in this way is "rotated", but it uses the same maps and modules.

Example. Let $A=\mathbb{k} Q / I$ where $Q$ is the quiver

and $I$ is the ideal generated by the paths of length 2 . The algebra $A$ is then a representation - finite string algebra, and one can compute

$$
\operatorname{dim} \operatorname{HH}^{i}(A)= \begin{cases}1 & \text { if } i \in\{0,2,3,4\} \\ 2 & \text { if } i=1 \\ 0 & \text { otherwise }\end{cases}
$$

The generators of the Hochschild spaces are given by (the classes of) the basic cochains $\chi_{\beta}^{\beta}$ and $\chi_{\gamma}^{\gamma}$ for $i=1 ; f^{2}=\chi_{\alpha_{2} \alpha_{3}}^{\beta}$ for $i=2, f^{3}=\chi_{\alpha_{1} \beta \alpha_{4}}^{\gamma}$ for $i=3$ and $f^{4}=\chi_{\alpha_{2} \alpha_{2} \alpha_{3} \alpha_{4}}^{\gamma}$ for $i=4$.

The Auslander-Reiten quiver of $A$ is depicted below. We indicate the modules by their composition factors and the dotted lines are the Asulander-Reiten translation.


Further, the admissible cycles corresponding to these generators are given below. Note that the cycle corresponding to $f^{2}=\chi_{\alpha_{2} \alpha_{3}}^{\beta}$ is reduced to a single module. We indicate, in each case, the
admissible cycle associated with each cochain, as well as a corresponding cycle starting and ending at a simple module.

$$
\begin{align*}
& \mathbf{C}\left(\beta, \alpha_{2} \alpha_{3}\right): \\
& \text { Cycle at } S_{3}: \quad S_{3} \longrightarrow{ }_{4}^{32} \longrightarrow S_{3} \\
& \mathbf{C}\left(a_{1} \beta \alpha_{4}, \gamma\right): \quad{ }_{4}^{2} \longrightarrow{ }_{2}^{1}{ }_{2}^{4} \longrightarrow{ }_{4}^{2}  \tag{2.3.3}\\
& \text { Cycle at } S_{4}: \quad S_{4} \longrightarrow{ }_{4}^{2} \longrightarrow{ }_{2}^{1} 4 \\
& \mathbf{C}\left(\alpha_{1} \alpha_{2} \alpha_{3} \alpha_{4}, \gamma\right): \quad{ }_{4}^{3} \longrightarrow{ }_{3}^{2} \longrightarrow{ }_{2}^{1} 4{ }_{2}^{4} \longrightarrow{ }_{4}^{3} \\
& \text { Cycle at } S_{4}: \quad S_{4} \longrightarrow{ }_{4}^{3} \longrightarrow{ }_{3}^{2} \longrightarrow{ }_{2}^{1} 44 \text {. }
\end{align*}
$$

Clearly this construction does not apply to non-reduced basic cochains. When $\chi_{u}^{p}$ is not reduced one may try to associate a cycle to it by performing the above construction to $\chi_{v}^{q}$ where $(v, q)$ is obtained from $(u, p)$ by removing from $u$ and $p$ their largest common suffix and prefix. However, this is not always possible because the resulting reduced basic cochain $\chi_{v}^{q}$ may have degree smaller than 2.

Example. Let $Q$ be the quiver

bound by $I=\left\langle a_{1} a_{2}, a_{1}^{\prime} a_{2}\right\rangle$. The cochain $\chi_{a_{1} a_{2}}^{a_{1} b}$ is not reduced. Removing from $a_{1} b$ and $a_{1} a_{2}$ their largest common prefix and suffix yields the reduced cochain $\chi_{a_{2}}^{b}$. The latter has degree 1. Hence, the previous construction of admissible cycles does not apply to it.

We now turn our attention to the compositions of $\Gamma$-paths, and how these are reflected by the cycle assignment, and to the compositions of cochains that give rise to the Gerstenhaber bracket.
2.4. Compositions of cochains. Let $\chi_{u}^{p}$ and $\chi_{v}^{q}$ be two basic cochains with $u=\alpha_{1} \cdots \alpha_{n} \in \Gamma^{n}$ and $v=\beta_{1} \cdots \beta_{m} \in \Gamma^{m}\left(\alpha_{i}, \beta_{j} \in Q_{1}\right)$. Saying that the cochain $\chi_{u}^{p} \circ_{s} \chi_{v}^{q}$ is non-zero amounts to say that there is a path $\gamma_{1} \cdots \gamma_{n+m-1} \in \Gamma^{n+m-1}$ such that

$$
\chi_{u}^{p} \circ_{s} \chi_{v}^{q}\left(\gamma_{1} \cdots \gamma_{n+m-1}\right) \neq 0
$$

Since the only non-zero value that $\chi_{u}^{p}$ takes on $\Gamma$-paths is $p$, this means that

$$
\chi_{u}^{p}\left(\gamma_{1} \cdots \gamma_{s} \chi_{v}^{q}\left(\gamma_{s} \cdots \gamma_{s+m-1}\right) \gamma_{s+m} \cdots \gamma_{n+m-1}\right)=p
$$

and this forces $\gamma_{1} \cdots \gamma_{s} q \gamma_{s+m} \cdots \gamma_{n+m-1}=u$ so $(u, v)$ is a $\Gamma$-bypass, and further

$$
\chi_{u}^{p} \circ_{s} \chi_{v}^{q}=\chi_{u \circ_{s} v}^{p}
$$

We thus have a situation like the one illustrated in (2.4.1).


It follows from the linearity in the definition of the bracket at the cochain level that:
Proposition. Let $\chi_{u}^{p}$ and $\chi_{v}^{q}$ be basic cochains of degrees $n$ and $m$, respectively. Let $s$ be an integer. Then

$$
\chi_{u}^{p} \circ_{s} \chi_{v}^{q}= \begin{cases}\chi_{u \circ_{s} v}^{p} & \text { if }(u, v) \text { is an }(s, s+1) \Gamma \text {-bypass } \\ 0 & \text { otherwise } .\end{cases}
$$

2.5. Compositions of cycles. We now turn our attention to cycles in mod $-A$. We will define operations between admissible cycles called compositions (at some specific place) and we will see that for reduced basic cochains $\chi_{u}^{p}$ and $\chi_{v}^{q}$ of degree at least two and such that $(u, v)$ is an $(s, s+1)$ $\Gamma$-bypass, the composition of the admissible cycles $\mathbf{C}(u, p)$ and $\mathbf{C}(v, q)$ is the cycle $\mathbf{C}\left(u \circ_{s} v, p\right)$.

Consider two admissible cycles

$$
\begin{array}{ll}
\mathbf{C}: & M\left(\alpha_{n-1}\right) \rightarrow \cdots \rightarrow M\left(\alpha_{2}\right) \rightarrow M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right) \rightarrow M\left(\alpha_{n-1}\right) \\
\mathbf{C}^{\prime}: & M\left(\beta_{m-1}\right) \rightarrow \cdots \rightarrow M\left(\beta_{2}\right) \rightarrow M\left(\beta_{1}^{-1} q \beta_{m}^{-1}\right) \rightarrow M\left(\beta_{m-1}\right)
\end{array}
$$

and assume there is an integer $s \in\{1, \ldots, n\}$ satisfying :
(a) $\beta_{1} \cdots \beta_{m}$ is parallel to $\alpha_{s}$
(b) $q=\alpha_{s}$.

Recall from subsection 2.3 that the admissible cycles $\mathbf{C}$ and $\mathbf{C}^{\prime}$ together with $s$ determine $\beta_{1} \cdots \beta_{m}$, $\alpha_{s}$ and $q$, hence (a) and (b) are indeed assumptions on the admissible cycles $\mathbf{C}$ and $\mathbf{C}^{\prime}$. We hence have the diagram (2.4.1)

The definition of the $s$-th composition of $\mathbf{C}$ with $\mathbf{C}^{\prime}$, denoted by $\mathbf{C}{ }_{s} \mathbf{C}^{\prime}$, is given below according to whether $s=1,1<s<n$, or $s=n$. Recall from subsection 2.3 that the cycles $\mathbf{C}$ and $\mathbf{C}^{\prime}$ determine the sequences of modules $M\left(\alpha_{1}\right), \ldots, M\left(\alpha_{n}\right), M(p)$ and $M\left(\beta_{1}\right), \ldots, M\left(\beta_{m}\right), M(q)$. In particular, $\mathbf{C} \circ_{s} \mathbf{C}^{\prime}$ depends on $\mathbf{C}$ and $\mathbf{C}^{\prime}$ only.

If $s=1$ then $\mathbf{C} \circ_{s} \mathbf{C}^{\prime}$ is defined as the following admissible cycle where all the arrows are the natural morphisms


This admissible cycle involves the module $M\left(\beta_{1}^{-1} p \alpha_{n}^{-1}\right)$. As an object of $\bmod -A$, this module is obtained from $\mathbf{C}$ and $\mathbf{C}^{\prime}$ in two steps as follows

1. $M\left(p \alpha_{n}^{-1}\right)$ is the cokernel of the natural morphism $\operatorname{rad}\left(M\left(\alpha_{1}\right)\right) \rightarrow M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right)$,
2. $M\left(\beta_{1}^{-1} p \alpha_{n}^{-1}\right)$ is the middle term of a short exact sequence of the shape

$$
0 \rightarrow \operatorname{rad}\left(M\left(\beta_{1}\right)\right) \rightarrow M\left(\beta_{1}^{-1} p \alpha_{n}^{-1}\right) \rightarrow M\left(p \alpha_{n}^{-1}\right) \rightarrow 0
$$

If $1<s<n$ then $\mathbf{C} \circ_{s} \mathbf{C}^{\prime}$ is defined as the following admissible cycle where all the arrows are the natural morphisms


If $s=n$ then $\mathbf{C} \circ_{s} \mathbf{C}^{\prime}$ is the following admissible cycle where all the arrows are the natural morphisms


This admissible cycle involves the module $M\left(\alpha_{1}^{-1} p \beta_{n}^{-1}\right)$. As an object of $\bmod -A$, this module is obtained from $\mathbf{C}$ and $\mathbf{C}^{\prime}$ in two steps as follows

1. $M\left(\alpha_{1}^{-1} p\right)$ is the kernel of the natural surjection $M\left(\alpha_{1}^{-1} p \alpha_{n}^{-1}\right) \rightarrow M\left(\alpha_{n}\right) / \operatorname{soc}\left(M\left(\alpha_{n}\right)\right)$,
2. $M\left(\alpha_{1}^{-1} p \beta_{n}^{-1}\right)$ is the middle term of a short exact sequence of the shape

$$
0 \rightarrow M\left(\alpha_{1}^{-1} p\right) \rightarrow M\left(\alpha_{1}^{-1} p \beta_{n}^{-1}\right) \rightarrow M\left(\beta_{n}^{-1}\right) / \operatorname{soc}\left(M\left(\beta_{n}^{-1}\right)\right) \rightarrow 0
$$

We are now able to state our main result.

Theorem. Let $\chi_{u}^{p}$ and $\chi_{v}^{q}$ be two reduced basic cochains of degrees $n \geqslant 2$ and $m \geqslant 2$ respectively. Let $s$ be an integer such that $1 \leqslant s \leqslant n$. Assume that $(u, v)$ is an $(s, s+1) \Gamma$-bypass. Then, the $s$-th composition of $\mathbf{C}(u, p)$ with $\mathbf{C}(v, q)$ is defined and $\mathbf{C}(u, p) \circ_{s} \mathbf{C}(v, q)=\mathbf{C}\left(u \circ_{s} v, p\right)$.

Proof. This follows readily from the definition of the compositions $u \circ_{s} v$ and $\mathbf{C}(u, p) \circ_{s} \mathbf{C}(v, p)$. Note that, by assumption, $q$ is the $s$-th arrow of the path $u$.

Example. Consider the first example of section 2.3. The generators of the Hochchild comology groups were already described. Further, a direct computation shows that $f^{3} \circ_{2} f^{2}=f^{4}$, and that $f^{3} \circ f^{2}=-f^{3} \circ_{2} f^{2}$, so that $\left[\mathrm{HH}^{3}(A), \mathrm{HH}^{2}(A)\right]=\mathrm{HH}^{4}(A)$.

From the module categoy point of view, one sees that there are two cycles at the simple module $S_{4}$. One of them corresponds to $\mathbf{C}\left(\alpha_{1} \beta \alpha_{4}, \gamma\right)$, the other to $\mathbf{C}\left(\alpha_{1} \alpha_{2} \alpha_{3} \alpha_{4}, \gamma\right)$. The second is obtained by composing the first one with the cycle at $S_{3}$. See the diagrams (2.3.3).
Remark. While our original interest was to understand the Gerstenhaber bracket in Hochschild cohomoloy, it must be noted that a non-zero element in some $\operatorname{HH}^{j}(A)$ for $j \geqslant 2$ is not necessary to obtain oriented cycles in mod $-A$. Indeed, our construction requires only a reduced basic cochain $\chi_{u}^{p}$, it may not be a cocycle, or it could be a coboundary. Note that in this situation, the subcategory formed by the arrows of $u$ and those of $p$ form a clockwork cycle, in the sense of [5], where the representation-finite case was studied. Therein it is shown that a clockwork cycle gives rise to a cycle in the Auslander - Reiten quiver of $A$, so cycles do appear in a broader setting than that of Hochschild cohomology.

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