On the existence, regularity and uniqueness of L^p-solutions to the steady-state 3D Boussinesq system in the whole space

Oscar Jarrín*[1]

[1]Escuela de Ciencias Físicas y Matemáticas, Universidad de Las Américas, Vía a Nayón, C.P.170124, Quito, Ecuador.

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Abstract

We consider the steady-state Boussinesq system in the whole three-dimensional space, with the action of external forces and the gravitational acceleration. First, for $3 < p \leq +\infty$ we prove the existence of weak L^p-solutions. Moreover, within the framework of a slightly modified system, we discuss the possibly non-existence of L^p-solutions for $1 \leq p \leq 3$. Then, we use the more general setting of the $L^{p,\infty}-$spaces to show that weak solutions and their derivatives are Hölder continuous functions, where the maximum gain of regularity is determined by the initial regularity of the external forces and the gravitational acceleration. As a bi-product, we get a new regularity criterion for the steady-state Navier-Stokes equations. Furthermore, in the particular homogeneous case when the external forces are equal to zero; and for a range of values of the parameter p, we show that weak solutions are not only smooth enough, but also they are identical to the trivial (zero) solution. This result is of independent interest, and it is also known as the Liouville-type problem for the steady-state Boussinesq system.

Keywords: Boussinesq system; steady-state L^p-solutions; non-existence of solutions; regularity criterion, Lorentz spaces, Liouville problem.

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1 Introduction

In this note, we consider the steady-state (time-independent) incompressible 3D Boussinesq system in the whole three-dimensional space. For $\vec{u}: \mathbb{R}^3 \to \mathbb{R}^3$ a divergence-free velocity field, $P: \mathbb{R}^3 \to \mathbb{R}$ a pressure and $\theta: \mathbb{R}^3 \to \mathbb{R}$ the temperature of the fluid we have:

$$\begin{cases} -\Delta \vec{u} + \text{div}(\vec{u} \otimes \vec{u}) + \vec{\nabla} P = \theta \vec{\mathbf{g}} + \vec{f}, \qquad \text{div}(\vec{u}) = 0, \\ -\Delta \theta + \text{div}(\theta \vec{u}) = g, \end{cases} \quad \text{in} \quad \mathbb{R}^3, \tag{1}$$

where $\vec{\mathbf{g}}: \mathbb{R}^3 \to \mathbb{R}^3$ is the gravitational acceleration and $\vec{f}: \mathbb{R}^3 \to \mathbb{R}^3$, $g: \mathbb{R}^3 \to \mathbb{R}$ denote external forces acting on each equation of this system. Moreover, with a minor loss of generality all the physical constants have been set to be equal to one.

The system (1) describes the dynamics of a viscous incompressible fluid with heat exchanges [3, 9], and moreover, it takes into account the physical phenomena carried out by a gravitational field $\vec{\mathbf{g}}$, like the Earth's gravitational field. This system arises from an approximation on a system coupling the classical Navier-Stokes equations and the equations of thermodynamics. In this approximation, the variations of the

*corresponding author: oscar.jarrin@udla.edu.ec

density due to heat transfers are neglected in the continuity equation, but are taken into account in the equation of the motion through an additional buoyancy term proportional to the temperature variations and the gravitational acceleration: $\theta\vec{\mathbf{g}}$. In our study, we shall observe that this term yields interesting qualitative properties to the system (1), in particular the possible non-existence of L^p-solutions for some range of values of the parameter p.

When we set $\theta \equiv 0$, the system (1) boils down to the classical steady-state Navier-Stokes equations:

$$-\Delta\vec{u} + \text{div}(\vec{u}\otimes\vec{u}) + \vec{\nabla}P = \vec{f}, \qquad \text{div}(\vec{u}) = 0, \qquad \text{in } \mathbb{R}^3, \tag{2}$$

which contains many challenging open problems. We refer to the books [17] and [18] for a complete theoretical study of these equations.

In the *homogeneous case* when $\vec{f} \equiv 0$ and $g \equiv 0$, the steady state Boussinesq system has been mainly studied in the setting of a *bounded and smooth domain* $\Omega \subset \mathbb{R}^3$, jointly with the Dirichlet conditions:

$$\begin{cases} -\nu\Delta\vec{u} + \text{div}(\vec{u}\otimes\vec{u}) + \vec{\nabla}P = \theta\vec{\mathbf{g}}, \qquad \text{div}(\vec{u}) = 0, \quad \text{in } \Omega, \\ -\Delta\theta + \text{div}(\theta\vec{u}) = 0, \quad \text{in } \Omega, \\ \vec{u} = \vec{u}_b, \quad \theta = \theta_b \quad \text{on } \partial\Omega. \end{cases} \tag{3}$$

Here, $\nu > 0$ is the viscosity parameter, and moreover, the given functions $\vec{u}_b : \partial\Omega \to \mathbb{R}^3$ and $\theta_b : \partial\Omega \to \mathbb{R}$ are boundary data. The qualitative properties of this system: existence, uniqueness and regularity of solutions, deeply depend on suitable hypotheses on the functions $\vec{\mathbf{g}}$, $\vec{u}_b$ and θ_b as well as on suitable hypothesis on the boundary $\partial\Omega$.

When $\partial\Omega$ is assumed to be a *connected set* of class $\mathcal{C}^2$, and additionally, when the viscosity parameter ν is large enough, in [24, 25] it is proven the existence and uniqueness of weak solutions $(\vec{u},\theta) \in L^3(\Omega)\times L^2(\Omega)$ to the system (3) with boundary data $(\vec{u}_b,\theta_b) \in L^2(\partial\Omega)\times L^2(\partial\Omega)$ and $\vec{\mathbf{g}} \in L^\infty(\Omega)$.

Thereafter, the case of more irregular boundary data was studied in [16]. For $\vec{\mathbf{g}} \in L^\infty(\Omega)$, the existence and uniqueness of weak solutions $(\vec{u},\theta) \in L^p(\Omega)\times L^q(\Omega)$ to the system (3) are proven under the smallness assumption $\|\vec{u}_b\|_{W^{-1/p,p}(\partial\Omega)} + \|\theta_b\|_{W^{-1/q,q}(\partial\Omega)} \ll 1$; and where the parameters p and q verify the following set of technical conditions: with $3 \leq p < +\infty$, $p' < q < +\infty$ (where $1/p + 1/p' = 1$) and $3p/(3+2p) \leq r$.

On the other hand, the more general case when $\partial\Omega$ is of class $\mathcal{C}^{1,1}$ but *not necessarily connected* is addressed in [1]. In this work, for a weaker assumption on gravitational acceleration: $\vec{\mathbf{g}} \in L^{3/2}(\Omega)$, it is proven the existence of weak solutions in the space $H^1(\Omega)$ just considering smallness of $\vec{u}_b$ across each connected component $\partial\Omega_i$ of the boundary $\partial\Omega$. Moreover, it is also proven a gain of regularity of weak solutions in the space $W^{1,p}(\Omega)$ with $p > 2$; and in the space $W^{2,p}(\Omega)$ with $p \geq 6/5$. For more related results, we refer to [12, 22, 23] and the references therein.

To the best of our knowledge, these qualitative properties of L^p-solutions to the system (1) (in whole space $\mathbb{R}^3$) have not been studied before. It is worth highlighting the previously cited works are not longer valid in $\mathbb{R}^3$ due to the lack of some of their key tools, for instance, the embedding properties of the $L^p(\Omega)$-spaces and the *compact* Sobolev embeddings. We thus use different approaches to study the existence, regularity and uniqueness of L^p-solutions to the system (1); and our key tools are mainly based in useful properties of the Lorentz spaces. We refer to the book [7] for a complete study of these spaces.

We recall that for a measurable function $f : \mathbb{R}^3 \to \mathbb{R}$ and for a parameter $\lambda \geq 0$ we define the distribution function

$$d_f(\lambda) = dx\left(\{x \in \mathbb{R}^3 : \ |f(x)| > \lambda\}\right),$$

where dx denotes the Lebesgue measure. Then, the re-arrangement function f^* is defined by the expression

$$f^*(t) = \inf\{\lambda \geq 0 : \ d_f(\lambda) \leq t\}.$$

By definition, for $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$ the Lorentz space $L^{p,q}(\mathbb{R}^3)$ is the space of measure functions $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\|f\|_{L^{p,q}} < +\infty$, where:

$$\|f\|_{L^{p,q}} = \begin{cases} \dfrac{q}{p}\left(\displaystyle\int_0^{+\infty} (t^{1/p} f^*(t))^q dt\right)^{1/q}, & q < +\infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t), \quad q = +\infty. \end{cases}$$

It is worth mentioning some important properties of these spaces. The quantity $\|f\|_{L^{p,q}}$ is often used as a norm, even thought it does not verify the triangle inequality. However, there exists an equivalent norm (strictly speaking) which makes these spaces into Banach spaces. On the other hand, these spaces are homogeneous of degree $-3/p$ and for $1 \leq q_1 < p < q_2 \leq +\infty$ we have the continuous embedding

$$L^{p,q_1}(\mathbb{R}^3) \subset L^p(\mathbb{R}^3) = L^{p,p}(\mathbb{R}^3) \subset L^{p,q_2}(\mathbb{R}^3).$$

Finally, for $p = +\infty$ we also have the identity $L^{\infty,\infty}(\mathbb{R}^3) = L^\infty(\mathbb{R}^3)$.

In our first main result, we find some smallness conditions on the external forces and the gravitational acceleration to construct L^p-weak solutions to the system (1). Precisely, by following some of the ideas in [2], first we solve this system in the $L^{3,\infty}-$space, where (to the best of our knowledge) we are able to apply a fixed point argument. Thereafter, we show that these $L^{3,\infty}-$solutions also belong to the space $L^p(\mathbb{R}^3)$ (with $3 < p \leq +\infty$) as long as the external forces verify additionally suitable conditions.

Theorem 1.1 *Assume that the external forces $\vec{f}$ and g satisfy*

$$(-\Delta)^{-1}\mathbb{P}(\vec{f}) \in L^{3,\infty}(\mathbb{R}^3) \quad \textit{and} \quad (-\Delta)^{-1}g \in L^{3,\infty}(\mathbb{R}^3).$$

Moreover, assume that the gravitational acceleration $\vec{\mathbf{g}}$ satisfies

$$\vec{\mathbf{g}} \in L^{3/2,1}(\mathbb{R}^3).$$

There exists a universal quantity $\varepsilon > 0$ such that if

$$\delta = \left\|(-\Delta)^{-1}\mathbb{P}(\vec{f})\right\|_{L^{3,\infty}} + \left\|(-\Delta)^{-1}g\right\|_{L^{3,\infty}} < \varepsilon \quad \textit{and} \quad \|\vec{\mathbf{g}}\|_{L^{3/2,1}} < \varepsilon,$$

then the following statements hold:

1. *The system (1) has a weak solution $(\vec{u}, \theta) \in L^{3,\infty}(\mathbb{R}^3)$ satisfying and uniquely defined by the condition $\|\vec{u}\|_{L^{3,\infty}} + \|\theta\|_{L^{3,\infty}} \leq 3\delta$.*
2. *This solution verifies $(\vec{u}, \theta) \in L^p(\mathbb{R}^3)$, with $3 < p \leq +\infty$, as long as*

$$\begin{cases} (-\Delta)^{-1}\mathbb{P}(\vec{f}) \in L^p \cap L^4(\mathbb{R}^3), \quad (-\Delta)^{-1}g \in L^p \cap L^4(\mathbb{R}^3), \quad \textit{when} \;\; 3 < p < 4, \\ (-\Delta)^{-1}\mathbb{P}(\vec{f}) \in L^p(\mathbb{R}^3), \quad (-\Delta)^{-1}g \in L^p(\mathbb{R}^3), \quad \textit{when} \;\; 4 \leq p \leq +\infty, \end{cases} \quad \textit{and} \quad \vec{\mathbf{g}} \in L^{3,1}(\mathbb{R}^3).$$

Some comments are in order. As mentioned in the first point above, the smallness assumption $\delta < \varepsilon$ yields the existence of weak $L^{3,\infty}$-solutions, where this particular space naturally appears in the involved fixed point estimates (see the Proposition 3.1 below for more details). However, the conditions $(-\Delta)^{-1}\mathbb{P}(\vec{f}) \in L^{3,\infty}(\mathbb{R}^3)$ and $(-\Delta)^{-1}g \in L^{3,\infty}(\mathbb{R}^3)$ allow us to consider very rough external forces, for instance, for suitable constants $\eta_1, \eta_2 > 0$ and for the Dirac mass δ_0, we can consider $\vec{f} = \eta_1(\delta_0, \delta_0, \delta_0)$ and $g = \eta_2\delta_0$. On the other hand, the gravitational acceleration $\vec{\mathbf{g}}$ also plays an important role in this existence result. In particular, the required condition $\vec{\mathbf{g}} \in L^{3/2,1}(\mathbb{R}^3)$ is similar to the condition $\vec{\mathbf{g}} \in L^{3/2}(\Omega)$ found in [1].

The second point above shows a persistence property of solutions in the L^p-spaces (see the Proposition 3.2 below for all the details). Under minor technical modifications, this result is also valid in the (more general) setting of the $L^{p,q}-$spaces.

As may observe, this last result is valid for the range of values $3 < p \leq +\infty$ and it is natural to look for similar persistence properties in the complementary range of values $1 \leq p \leq 3$. It is thus interesting to remark that the constraint $3 < p \leq +\infty$ is not only technical (see the Remarks 2 and 3 below), but it seems to be a phenomenological effect of the system (1). More precisely, the term $\theta\vec{\mathbf{g}}$ in the first equation of this system potentially yields the non-existence of L^p-solutions for $1 \leq p \leq 3$.

To study this phenomena, let us consider the following toy model for the system (1):

$$\begin{cases} -\Delta\vec{u} + \text{div}(\varphi\vec{u}\otimes\vec{u}) = \theta\vec{\mathbf{g}} + \vec{f}, \quad \text{div}(\vec{u} = 0), \qquad |\varphi(x)| \leq \frac{C}{1+|x|^2}, \\ -\Delta\theta + \text{div}(\theta\vec{u}) = g. \end{cases} \tag{4}$$

Here, the localizing function φ in the nonlinear term is a merely technical requirement and, in further research, we aim to get rid of this function. However, the main goal of this toy model is to highlight the effects of the term $\theta\vec{\mathbf{g}}$ in the non-existence of solutions.

Since $\varphi \in L^\infty(\mathbb{R}^3)$ the modified bilinear term $\text{div}(\varphi\vec{u}\otimes\vec{u})$ follows the same estimates of the original one in the equation (1). See the Section 3 for all the details. Consequently, the result obtained in Theorem 1.1 is valid for the system (4). In particular, one can consider external forces $\vec{f}, g \in \mathcal{S}(\mathbb{R}^3)$ and a gravitational acceleration $\vec{\mathbf{g}} \in \mathcal{S}(\mathbb{R}^3)$ verifying

$$0 \notin \text{supp}(\widehat{\vec{f}}), \quad 0 \notin \text{supp}(\widehat{g}) \quad \text{and} \quad 0 \notin \text{supp}(\widehat{\vec{\mathbf{g}}}), \tag{5}$$

hence we have

$$(-\Delta)^{-1}\vec{f} \in \mathcal{S}(\mathbb{R}^3), \quad (-\Delta)^{-1}g \in \mathcal{S}(\mathbb{R}^3) \quad \text{and} \quad (-\Delta)^{-1}\vec{\mathbf{g}} \in \mathcal{S}(\mathbb{R}^3).$$

Then, we assume the smallness condition in Theorem 1.1 and there exists a solution $(\vec{u}, \theta) \in L^{3,\infty}(\mathbb{R}^3)$ to the system (4) which verifies $(\vec{u}, \theta) \in L^p(\mathbb{R}^3)$ with $3 < p \leq +\infty$. In this context, our next result reads as follows:

Proposition 1.1 *There exist $\vec{f}, g, \vec{\mathbf{g}} \in \mathcal{S}(\mathbb{R}^3)$ well-prepared functions satisfying (5), such that the associated solution $(\vec{u}, \theta)$ to the system (4) verifies $\vec{u} \notin L^p(\mathbb{R}^3)$ for $1 \leq p \leq 3$.*

The proof is essentially based on the following estimate from below (see the Proposition 4.3 for more details)

$$\frac{1}{|x|}\left|\int_{\mathbb{R}^3}\theta(y)\vec{\mathbf{g}}(y)dy\right| \lesssim |\vec{u}(x)|, \qquad |x| \gg 1,$$

which yields $\vec{u} \notin L^p(\mathbb{R}^3)$ for the range of values $1 \leq p \leq 3$. Here, the well-prepared data $\vec{f}, g$ and $\vec{\mathbf{g}}$ ensure that $\displaystyle\int_{\mathbb{R}^3}\theta(y)\vec{\mathbf{g}}(y)dy \neq 0$. It is also interesting to compare this result to the one obtained in [2, Theorem 3.5] for the steady-state Navier-Stokes equations (2). In this work, it is proven a non-existence of L^p-solutions for $1 \leq p \leq 3/2$ and we thus observe the strongest effects of the term $\theta\vec{\mathbf{g}}$ is the Boussinesq system (1).

In our second main result, we study another persistence problem associated with the system (1), which is an important question to a better mathematically comprehension of this system: the gain of regularity of weak solutions from an initial regularity in the data $\vec{f}, g$ and $\vec{\mathbf{g}}$. We shall consider here a fairly general notion of weak solutions, which is given in the following:

Definition 1.1 *Let $\vec{f}, g \in \mathcal{D}'(\mathbb{R}^3)$ and let $\vec{\mathbf{g}} \in L^2_{loc}(\mathbb{R}^3)$. A weak solution of the coupled system (1) is a triplet $(\vec{u}, P, \theta)$, where: $\vec{u} \in L^2_{loc}(\mathbb{R}^3)$, $P \in \mathcal{D}'(\mathbb{R}^3)$ and $\theta \in L^2_{loc}(\mathbb{R}^3)$, such that it verifies (1) in the distributional sense.*

It is worth observing we use minimal conditions on the functions $\vec{f}, g, \vec{\mathbf{g}}, \vec{u}, P$ and θ to ensure that all the terms in the system (1) are well defined as distributions. In addition, we let the pressure P to be a very general object since we only assume $P \in \mathcal{D}'(\mathbb{R}^3)$.

The $L^{p,\infty}$-spaces provide us a general and suitable framework to improve the regularity of weak solutions defined above. For $1 \leq p < +\infty$ and for a regularity parameter $k \in \mathbb{N}$, we introduce the Sobolev-Lorentz space

$$\mathcal{W}^{k,p}(\mathbb{R}^3) = \left\{ f \in L^{p,\infty}(\mathbb{R}^3) : \ \partial^\alpha f \in L^{p,\infty}(\mathbb{R}^3) \text{ for all multi-indice } |\alpha| \leq k \right\}.$$

Moreover, we denote by $W^{k,\infty}(\mathbb{R}^3)$ the classical Sobolev space of bounded functions with bounded weak derivatives until the order k. Finally, for $0 < s < 1$ we shall denote by $\mathscr{C}^{k,s}(\mathbb{R}^3)$ the Hölder space of $\mathcal{C}^k-$functions whose derivatives are Hölder continuous functions with parameter s. In this setting, our second result reads as follows:

Theorem 1.2 *Let $(\vec{u}, P, \theta)$ be a weak solution to the coupled system (1) given by Definition 1.1. Assume that*

$$\vec{u} \in L^{p,\infty}(\mathbb{R}^3) \quad \text{and} \quad \theta \in L^{p,\infty}(\mathbb{R}^3), \quad \text{with} \ \ 3 < p < +\infty.$$

Then, if for $k \geq 0$:

$$\vec{f},\, g \in \mathcal{W}^{k, \frac{3p}{3+p}} \cap W^{k,\infty}(\mathbb{R}^3), \tag{6}$$

and

$$\vec{\mathbf{g}} \in \mathcal{W}^{k, \frac{3p}{3+p}} \cap W^{k,\infty}(\mathbb{R}^3), \tag{7}$$

it follows that $\vec{u} \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$, $P \in \mathcal{W}^{k+1,p}(\mathbb{R}^3)$ and $\theta \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$. Moreover, we have

$$\vec{u} \in \mathscr{C}^{k+1,s}(\mathbb{R}^3), \quad P \in \mathscr{C}^{k,s}(\mathbb{R}^3) \quad \text{and} \quad \theta \in \mathscr{C}^{k+1,s}(\mathbb{R}^3), \quad \text{with} \quad s = 1 - 3/p.$$

In (6) and (7) the parameter k measures the initial regularity of the data, which yields a gain of regularity of weak solutions of the order $k+2$. This (expected) maximum gain of regularity is given by the effects of the Laplacian operator in both equations of the system (1). We refer to Remark 6 for more details.

Let us briefly explain the general strategy of the proof, which bases on two key ideas. First, by assuming $\vec{u} \in L^{p,\infty}(\mathbb{R}^3)$ and $\theta \in L^{p,\infty}(\mathbb{R}^3)$ and by using useful properties of the parabolic (time-dependent) Boussinesq system (given in the equation (59)), we prove that $\vec{u}$ and θ are bounded functions on $\mathbb{R}^3$. Thereafter, we use a bootstrap argument to show that $\vec{u} \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$ and $\theta \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$. Finally, we use the more general setting of the Morrey spaces to conclude the wished Hölder regularity.

These ideas can also be applied to the other relevant coupled system of the fluid dynamics, for instance, the *magneto-hydrodynamics equations* and some variations of the Boussinesq system as the *Bérnard system* and the *magnetic-Bérnard system*. In particular, when we set $\theta \equiv 0$ the system (1) agrees with the Navier-Stokes equations (2); and as a direct consequence of the theorem above we obtain a new regularity criterion for these equations:

Corollary 1.1 *Let $(\vec{u}, P) \in L^2_{loc}(\mathbb{R}^3) \times \mathcal{D}'(\mathbb{R}^3)$ be a weak solution to the equation (2). If $\vec{u} \in L^{p,\infty}(\mathbb{R}^3)$ with $p > 3$ and if the external force $\vec{f}$ verifies (6) for some $k \geq 0$, then we have $\vec{u} \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$ and $P \in \mathcal{W}^{k+1,p}(\mathbb{R}^3)$. In addition, we have $\vec{u} \in \mathscr{C}^{k+1,s}(\mathbb{R}^3)$ and $P \in \mathscr{C}^{k,s}(\mathbb{R}^3)$ with $s = 1 - 3/p$.*

Another interesting consequence of Theorem 1.2 arises when we consider the homogeneous system (when $\vec{f} \equiv 0$ and $g \equiv 0$):

$$\begin{cases} -\Delta \vec{u} + \text{div}(\vec{u} \otimes \vec{u}) + \vec{\nabla} P = \theta \vec{\mathbf{g}}, & \text{div}(\vec{u}) = 0, \\ -\Delta \theta + \text{div}(\theta \vec{u}) = 0, & \end{cases} \quad \text{in} \quad \mathbb{R}^3, \tag{8}$$

and then, the gain of regularity of weak solution is now determined by the gravitational acceleration $\vec{\mathbf{g}}$ as long as it verifies (7). In particular, we have $(\vec{u}, P, \theta) \in \mathcal{C}^\infty(\mathbb{R}^3)$, provided that $\vec{u}, \theta \in L^{p,\infty}(\mathbb{R}^3)$ with $3 < p < +\infty$ and $\vec{\mathbf{g}} \in \bigcap\limits_{k\geq 0} \mathcal{W}^{k,\frac{3p}{3+p}} \cap W^{k,\infty}(\mathbb{R}^3)$.

Regularity of weak solutions to the homogeneous system (8) is also one of the key assumptions when studying another relevant problem: the uniqueness of the trivial solution $(\vec{u}, P, \theta) = (0, 0, 0)$. This is problem is commonly known as the *Liouville-type problem* for the homogeneous Boussinesq system (8) and, to the best of our knowledge, it has not been studied before. Thus, in our third main result, we find a range of values of the parameter p for which $L^{p,q}(\mathbb{R}^3)$-solutions to the system (8) vanish identically.

Theorem 1.3 *Let $(\vec{u}, P, \theta)$ be a weak solution to the coupled homogeneous system (8) given by Definition 1.1. The trivial solution $(\vec{u}, P, \theta) = (0, 0, 0)$ in uniquely determined in the following cases:*

1. *When $\vec{u} \in L^{\frac{9}{2},q}(\mathbb{R}^3)$, $\theta \in L^{\frac{9}{2},q}(\mathbb{R}^3)$ with $1 \leq q < +\infty$, and $\vec{\mathbf{g}} \in \mathcal{W}^{1,9/2} \cap W^{1,\infty}(\mathbb{R}^3)$.*
2. *When $\vec{u} \in L^{p,\infty}(\mathbb{R}^3)$, $\theta \in L^{p,\infty}(\mathbb{R}^3)$ and $\vec{\mathbf{g}} \in \mathcal{W}^{1,p} \cap W^{1,\infty}(\mathbb{R}^3)$, with $3 < p < \frac{9}{2}$.*

The proof of this theorem is based on fine local estimates for each equation in the system (8), which are known as the *Caccioppoli-type estimates* (see the Proposition 6.1 below). To do this, a minimal regularity of solutions is required, more precisely, we need that $\vec{u}, \theta \in C^2_{loc}(\mathbb{R}^3)$ and $P \in \mathcal{C}^1_{loc}(\mathbb{R}^3)$. Thus, this regularity is ensured by the assumptions on $\vec{u}, \theta$ and $\vec{\mathbf{g}}$ given above; and by a direct application of Theorem 1.2.

The Liouville-type problem has been extensively studied in the homogeneous Navier-Stokes equations (2) (when $\vec{f} \equiv 0$). See, for instance, [4, 5, 6, 8, 19, 20, 21] and the references therein. In this sense, this result is a generalization of these previous works to the coupled framework of the system (8). In particular, the first point above generalizes the well-known Galdi's result [11, Theorem X.9.5]. Here, it is important to emphasize that the Liouville problem for the equation (2) in the largest space $L^{9/2,\infty}(\mathbb{R}^3)$ and in the setting of the Lebesgue or Lorentz spaces with parameter $p > 9/2$ is still an open question far from obvious. The second point above generalizes to the system (8) some previous results obtained in [13, Theorem 1] for the equation (2). Here, the constraint $3 < p < 9/2$ allows us to work in the largest space $L^{p,\infty}(\mathbb{R}^3)$.

Finally, let us mention that in future research some recent results on the Liouville problem for the equation (2), which consider more sophisticated functional spaces, could be also adapted to the system (8), but this come out the scope of this note.

Organization of the paper. In Section 2 we summarize some useful properties of the $L^{p,q}$-spaces. Section 3 and 4 are devoted to the proof of Theorem 1.1 and Proposition 1.1 respectively. Then, in Section 5 we give a proof of Theorem 1.2; and this note finishes with the proof of Theorem 1.3 given in Section 6.1.

2 Preliminaries

For the completeness of this note, we summarize here some well-known results which will be useful in the sequel. In the forthcoming estimates, $C > 0$ is a generic constant.

Lemma 2.1 (Young inequalities) *Let $1 < p, p_1, p_2 < +\infty$ and $1 \leq q, q_1, q_2 \leq +\infty$. There exists a constant $C_i > 0$, which depends on the parameters above, such that the following estimates hold:*

1. $\|f * h\|_{L^{p,q}} \leq C_1 \|f\|_{L^{p_1,q_1}} \|h\|_{L^{p_2,q_2}}$, *with* $1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2}$ *and* $C_1 = C\, p \left(\frac{p_1}{p_1 - 1}\right)\left(\frac{p_2}{p_2 - 1}\right)$.
2. $\|f * h\|_{L^{p,q}} \leq C_2 \|f\|_{L^1} \|h\|_{L^{p,q}}$, *with* $C_2 = C\, \frac{p^2}{p-1}$.
3. $\|f * h\|_{L^\infty} \leq C_3 \|f\|_{L^{p,q}} \|h\|_{L^{p',q'}}$, *with* $1 = \frac{1}{p} + \frac{1}{p'}$, $1 \leq \frac{1}{q} + \frac{1}{q'}$ *and* $C_3 = C \left(\frac{p}{p-1}\right)\left(\frac{p'}{p'-1}\right)$.

For a proof see [7, Section 1.4.3].

Lemma 2.2 (Hölder inequalities)

1. *Let $f \in L^{p_1,q_1}(\mathbb{R}^3)$ and $h \in L^{p_2,q_2}(\mathbb{R}^3)$ with $1 \leq p_1, p_2 < +\infty$, $1 \leq q_1, q_2 \leq +\infty$ verifying the relationships $1 = 1/p_1 + 1/p_2$ and $1 = 1/q_1 + 1/q_2$. Then we have $fh \in L^1(\mathbb{R}^3)$ and it holds $\|fh\|_{L^1} \leq C\|f\|_{L^{p_1,q_1}} \|h\|_{L^{p_2,q_2}}$.*

2. *Moreover, for $1 \leq p, p_1, p_2 < +\infty$, $1 \leq q, q_1, q_2 \leq +\infty$ and with the relationships $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$; we have $fh \in L^{p,q}(\mathbb{R}^3)$ and it holds $\|fh\|_{L^{p,q}} \leq C\|f\|_{L^{p_1,q_1}} \|h\|_{L^{p_2,q_2}}$.*

A proof can be found in [7, Theorems 1.2.6 and 1.4.1].

Lemma 2.3 (Interpolation inequalities)

1. *Let $f \in L^{p,\infty}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, with $1 \leq p < +\infty$. Then, for all $1 \leq \sigma < +\infty$ we have $f \in L^{p\sigma}(\mathbb{R}^3)$ and the following estimate holds: $\|f\|_{L^{p\sigma}} \leq C \|f\|^{\frac{1}{\sigma}}_{L^{p,\infty}} \|f\|^{1-\frac{1}{\sigma}}_{L^\infty}$.*

2. *Let $f \in L^{p_1,q_1}(\mathbb{R}^3) \cap L^{p_2,q_2}(\mathbb{R}^3)$ where $1 < p_1 < p < p_2 < +\infty$ and $1 \leq q, q_1, q_2 \leq +\infty$. Then we have $f \in L^{p,q}(\mathbb{R}^3)$ and it holds: $\|f\|_{L^{p,q}} \leq C\|f\|^{\sigma_1}_{L^{p_1,q_1}} \|f\|^{1-\sigma_1}_{L^{p_2,q_2}}$, where $0 < \sigma_1 < 1$ depends on p, p_1 and p_2.*

For a proof see [7, Theorem 1.1.2] and [17, Proposition 2.3].

Lemma 2.4 (Lorentz-Besov embedding) *For all $t > 0$ it holds: $t^{\frac{3}{2p}} \|e^{t\Delta} f\|_{L^\infty} \leq C \|f\|_{L^{p,\infty}}$, with $1 \leq p < +\infty$.*

This estimate is a direct consequence of the continuous embedding $L^{p,\infty}(\mathbb{R}^3) \subset \dot{B}^{-\frac{3}{p},\infty}_{\infty}(\mathbb{R}^3)$. See [18, Page 171]. We recall that the homogeneous Besov space $\dot{B}^{-\frac{3}{p},\infty}_{\infty}(\mathbb{R}^3)$ can be characterized as the space of temperate distributions $f \in \mathcal{S}'(\mathbb{R}^3)$ such that $\sup\limits_{t>0} t^{\frac{3}{2p}} \|e^{t\Delta} f\|_{L^\infty} < +\infty$.

Lemma 2.5 (Riesz transforms) *For $i = 1, 2, 3$ let $\mathcal{R}_i = \dfrac{\partial_i}{\sqrt{-\Delta}}$ be the i-th Riesz transform. Then, for $i, j = 1, 2, 3$ the operator $\mathcal{R}_i\mathcal{R}_j$ is continuous in the space $L^{p,\infty}(\mathbb{R}^3)$ (with $1 < p < +\infty$) and we have $\|\mathcal{R}_i\mathcal{R}_j(f)\|_{L^{p,\infty}} \leq C\|f\|_{L^{p,\infty}}$. In particular, the Leray's projector $\mathbb{P}$ is a bounded operator in the space $L^{p,\infty}(\mathbb{R}^3)$.*

A proof can be found in [15, Lemma 4.2]. Finally, we shall use the following result linking the Morrey spaces and the Hölder regularity of functions. For a proof see [10, Proposition 3.4]. Recall that for $1 \leq p < +\infty$ the homogeneous Morrey space $\dot{M}^{1,p}(\mathbb{R}^3)$ is defined as the space of locally finite Borel measures $d\mu$ such that

$$\sup_{x_0 \in \mathbb{R}^3,\, R>0} R^{\frac{3}{p}} \left(\frac{1}{|B(x_0, R)|} \int_{B(x_0,R)} d|\mu|(x) \right) < +\infty. \tag{9}$$

Lemma 2.6 (Hölder regularity) *Let $f \in \mathcal{S}'(\mathbb{R}^3)$ such that $\vec{\nabla} f \in \dot{M}^{1,p}(\mathbb{R}^3)$, with $p > 3$. There exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^3$ we have $|f(x) - f(y)| \leq C \|\vec{\nabla} f\|_{\dot{M}^{1,p}} |x - y|^{1-3/p}$.*

3 Proof of Theorem 1.1

First note that the system (1) can be rewritten as the following (equivalent) problem:

$$\begin{cases} -\Delta \vec{u} + \mathbb{P}(\mathrm{div}(\vec{u} \otimes \vec{u})) = \mathbb{P}(\theta \vec{\mathbf{g}}) + \mathbb{P}(\vec{f}), \qquad \mathrm{div}(\vec{u}) = 0, \\ -\Delta \theta + \mathrm{div}(\theta \vec{u}) = g, \end{cases} \tag{10}$$

hence we obtain the integral formulations:

$$\vec{u} = -\frac{1}{-\Delta} \left(\mathbb{P}(\mathrm{div}(\vec{u} \otimes \vec{u})) \right) + \frac{1}{-\Delta} \Big(\mathbb{P}\left(\theta \vec{\mathbf{g}} \right) \Big) + \frac{1}{-\Delta} \Big(\mathbb{P}(\vec{f}) \Big), \tag{11}$$

$$\theta = -\frac{1}{-\Delta} \left(\mathrm{div}(\theta \vec{u}) \right) + \frac{1}{-\Delta}(g). \tag{12}$$

In the next propositions, we shall prove each point stated in Theorem 1.1.

Proposition 3.1 *Assume that* $\frac{1}{-\Delta}\Big(\mathbb{P}(\vec{f})\Big) \in L^{3,\infty}(\mathbb{R}^3)$, $\frac{1}{-\Delta}(g) \in L^{3,\infty}(\mathbb{R}^3)$ *and* $\vec{\mathbf{g}} \in L^{3/2,1}(\mathbb{R}^3)$. *There exists a universal constant* $\varepsilon_1 > 0$ *(defined in (21)) such that if*

$$\delta = \left\| \frac{1}{-\Delta}\Big(\mathbb{P}(\vec{f})\Big) \right\|_{L^{3,\infty}} + \left\| \frac{1}{-\Delta}(g) \right\|_{L^{3,\infty}} < \varepsilon_1 \quad \textit{and} \quad \|\vec{\mathbf{g}}\|_{L^{3/2,1}} < \varepsilon_1, \tag{13}$$

then the system (11)-(12) has a solution $(\vec{u}, \theta) \in L^{3,\infty}(\mathbb{R}^3)$ *satisfying and uniquely defined by* $\|\vec{u}\|_{L^{3,\infty}} + \|\theta\|_{L^{3,\infty}} \leq 3\delta$.

Proof. To prove this proposition, we shall use the next version of the Picard's fixed point theorem. The proof is rather standard but, for the reader's convenience we give a sketch.

Lemma 3.1 (Picard's fixed point) *Let* $(E, \|\cdot\|_E)$ *be a Banach space and let* $u_0 \in E$ *be an initial datum such that* $\|u_0\|_E \leq \delta$. *Assume that* $B : E \times E \to E$ *is a bilinear application and that* $L : E \to E$ *is a linear application. Assume moreover the following controls:*

$$\|B(u, v)\|_E \leq C_B \|u\|_E \|u\|_E, \qquad \|L(u)\|_E \leq C_L \|u\|_E, \tag{14}$$

for all $u, v \in E$, *where the continuity constants of these applications verify:*

$$0 < 3C_L < 1, \quad 0 < 9\delta C_B < 1 \quad \textit{and} \quad C_L + 6\delta C_B < 1. \tag{15}$$

Then the equation $u = u_0 + B(u, u) + L(u)$ *admits a unique solution* $u \in E$ *such that* $\|u\|_E \leq 3\delta$.

Proof. From the initial datum u_0 we consider the iterative equation: $u_{n+1} = u_0 + B(u_n, u_n) + L(u_n)$ for $n \in \mathbb{N}$. By the boundness of the operators $B(\cdot, \cdot)$ and $L(\cdot)$ given in (14), and moreover, by the first and the second condition in (15), we can prove that $\|u_{n+1}\|_E \leq 3\delta$ for all $n \in \mathbb{N}$. On the other hand, always by (14) we have the estimate $\|u_{n+1} - u_n\|_E \leq (C_L + 6\delta C_B)^n \|u_1 - u_0\|_E$. Finally, since $0 < C_L + 6\delta C_B < 1$ the rest of the proof follows from well-known arguments. ■

Withing the framework of this lemma, we set the Banach space

$$E = \big\{ (\vec{u}, \theta) \in L^{3,\infty}(\mathbb{R}^3) : \ \mathrm{div}(\vec{u}) = 0 \big\},$$

with its natural norm $\|(\vec{u}, \theta)\|_{L^{3,\infty}} = \|\vec{u}\|_{L^{3,\infty}} + \|\theta\|_{L^{3,\infty}}$. Moreover, from the equations (11) and (12) we set

$$u = \begin{pmatrix} \vec{u} \\ \theta \end{pmatrix}, \quad u_0 = \begin{pmatrix} \frac{1}{-\Delta}\Big(\mathbb{P}(\vec{f})\Big) \\ \frac{1}{-\Delta}(g) \end{pmatrix}, \tag{16}$$

and

$$B(u,u)=\begin{pmatrix} -\frac{1}{-\Delta}\left(\mathbb{P}(\text{div}(\vec{u}\otimes\vec{u}))\right) \\ -\frac{1}{-\Delta}\left(\text{div}(\theta\vec{u})\right)\end{pmatrix},\quad L(u)=\begin{pmatrix} \frac{1}{-\Delta}\Big(\mathbb{P}\left(\theta\vec{\mathbf{g}}\right)\Big) \\ 0\end{pmatrix}. \tag{17}$$

We shall construct a solution to the fixed point problem:

$$u=u_0+B(u,u)+L(u). \tag{18}$$

By the hypothesis of Proposition 3.1 we directly have $u_0\in E$, so we must verify the controls given in (14) with constants $C_B, C_L>0$ satisfying (15); and where the quantity $\delta>0$ is defined in (13).

Let us start by studying each component of the bilinear term $B(u,u)$. We remark that in the Fourier level the operator $\dfrac{\text{div}}{-\Delta}\Big(\mathbb{P}\left(\cdot\right)\Big)$ has a symbol $\widehat{m}_1(\xi)=\big(\widehat{m}_{1,i,j,k}(\xi)\big)_{1\leq i,j,k\leq 3}$, where each $\widehat{m}_{1,i,j,k}$ is a $\mathcal{C}^\infty-$ function outside the origin, and moreover, it is a homogeneous function of degree -1. Consequently, in the spatial variable, the action of this operator can be seen as a product of convolution with the tensor $m_1=\big(m_{1,i,j,k}\big)_{1\leq i,j,k\leq 3}$, where $m_{1,i,j,k}$ is a $\mathcal{C}^\infty-$ function outside the origin and homogeneous of degree -2. In particular we have $m_1\in L^{\frac{3}{2},\infty}(\mathbb{R}^3)$.

Then, by the first point of Lemma 2.1 (with $p=3$, $p_1=p_2=3/2$ and $q=q_1=q_2=+\infty$) and by the second point of Lemma 2.2 (with $p=3/2$, $p_1=p_2=1/3$ and $q=q_1=q_2=+\infty$) we write

$$\begin{aligned}\left\|\frac{1}{-\Delta}\left(\mathbb{P}(\text{div}(\vec{u}\otimes\vec{u}))\right)\right\|_{L^{3,\infty}} &= \|m_1*(\vec{u}\otimes\vec{u})\|_{L^{3,\infty}}\leq\, C_1\|m_1\|_{L^{3/2,\infty}}\|\vec{u}\otimes\vec{u}\|_{L^{3/2,\infty}}\\ &\leq C_1\|m_1\|_{L^{3/2,\infty}}\,\|\vec{u}\|^2_{L^{3,\infty}},\end{aligned}$$

where we set the constant $C_{B,1}=C_1\|m_1\|_{L^{3/2,\infty}}$.

The other term $-\dfrac{1}{-\Delta}\left(\text{div}(\theta\vec{u})\right)$ follows the same estimates: we denote by $m_2\in L^{3/2,\infty}(\mathbb{R}^3)$ the convolution tensor of the operator $\dfrac{1}{-\Delta}\left(\text{div}(\cdot)\right)$ and we write

$$\left\|-\frac{1}{-\Delta}\left(\text{div}(\theta\vec{u})\right)\right\|_{L^{3,\infty}}\leq C_1\|m_2\|_{L^{3/2,\infty}}\,\|\theta\|_{L^{3,\infty}}\,\|\vec{u}\|_{L^{3,\infty}},\quad C_{B,2}=C_1\|m_2\|_{L^{3/2,\infty}}.$$

With these estimates at hand, we set the constant $C_B=\max(C_{B,1},C_{B,2})$ and we get

$$\|B(u,u)\|_{L^{3,\infty}}\leq C_B\,\|u\|^2_{L^{3,\infty}}. \tag{19}$$

We study now the linear term $L(u)$. In this case we remark that the operator $\dfrac{1}{-\Delta}\left(\mathbb{P}(\cdot)\right)$ has a symbol $\widehat{m}_3(\xi)$ which is a tensor containing C^∞- functions outside the origin and homogeneous of degree -2. Then, in the spatial variable we have $\dfrac{1}{-\Delta}\left(\mathbb{P}(\theta\vec{\mathbf{g}})\right)=m_3*\theta\vec{\mathbf{g}}$, where the tensor m_3 is smooth outside the origin and homogeneous of degree -1. Consequently, it verifies $m_3\in L^{3,\infty}(\mathbb{R}^3)$.

By the second point of Lemma 2.1 and by the first point of Lemma 2.2 (with $p_1=3/2$, $p_2=3$, $q_1=1$, $q_2=+\infty$) we write

$$\begin{aligned}\left\|\frac{1}{-\Delta}\left(\mathbb{P}(\theta\vec{\mathbf{g}})\right)\right\|_{L^{3,\infty}} &= \|m_3*\theta\vec{\mathbf{g}}\|_{L^{3,\infty}}\leq C_2\|m_3\|_{L^{3,\infty}}\|\theta\vec{\mathbf{g}}\|_{L^1}\\ &\leq C_2\|m_3\|_{L^{3,\infty}}\|\vec{\mathbf{g}}\|_{L^{3/2,1}}\,\|\theta\|_{L^{3,\infty}}.\end{aligned}$$

We set the constant $C_L=C_2\|m_3\|_{L^{3,\infty}}\|\vec{\mathbf{g}}\|_{L^{3/2,1}}$ and we have the estimate

$$\|L(u)\|_{L^{3,\infty}}\leq C_L\|u\|_{L^{3,\infty}}. \tag{20}$$

Once we have the constants C_B and C_L given above, we must verify the constraints given in (15). To verify the first constraint we set $\|\vec{\mathbf{g}}\|_{L^{3/2,1}} < \dfrac{1}{3C_2\|m_3\|_{L^{3,\infty}}}$ to get $C_L < \frac{1}{3}$. With this inequality the second and the third constraint in (15) are satisfied as long as $\delta < \dfrac{1}{9C_B}$. We thus set

$$\varepsilon_1 < \min\left(\frac{1}{3C_2\|m_3\|_{L^{3,\infty}}}, \frac{1}{9C_B}\right), \tag{21}$$

and Proposition 3.1 follows from the assumption (13) and Lemma 3.1. ■

With this first result, now we are able to prove that the solution $(\vec{u},\theta) \in L^{3,\infty}(\mathbb{R}^3)$ also belongs to the space $L^p(\mathbb{R}^3)$ with $3 < p < +\infty$, provided that the external forces verify suitable supplementary hypotheses. In the case of the Lorentz space $L^{p,q}(\mathbb{R}^3)$ (with $1 \leq q \leq +\infty$) the proof follows the same lines, so it is enough to focus on the Lebesgue spaces.

Proposition 3.2 *Let $(\vec{u},\theta) \in L^{3,\infty}(\mathbb{R}^3)$ be the solution to the equations (11)-(12) constructed in Proposition 3.1. Let $\delta > 0$ be the quantity given in (13) and $\varepsilon_1 > 0$ defined in (21).*

Assume in addition that

$$\begin{cases} \frac{1}{-\Delta}\left(\mathbb{P}(\vec{f})\right) \in L^p \cap L^4(\mathbb{R}^3), \quad \frac{1}{-\Delta}(g) \in L^p \cap L^4(\mathbb{R}^3), \quad 3 < p < 4, \\ \frac{1}{-\Delta}\left(\mathbb{P}(\vec{f})\right) \in L^p(\mathbb{R}^3), \quad \frac{1}{-\Delta}(g) \in L^p(\mathbb{R}^3), \quad 4 \leq p \leq +\infty, \end{cases} \quad \text{and} \quad \vec{\mathbf{g}} \in L^{3,1}(\mathbb{R}^3). \tag{22}$$

There exists a universal quantity $0 < \varepsilon < \varepsilon_1$ (defined in (34)), such that if $\delta < \varepsilon$ then we have $(\vec{u},\theta) \in L^p(\mathbb{R}^3)$ with $3 < p \leq +\infty$.

Proof. Recall that the solution $(\vec{u},\theta) \in L^{3,\infty}(\mathbb{R}^3)$ to the equations (11)-(12) is obtained as the limit (in the strong topology of the space $L^{3,\infty}(\mathbb{R}^3)$) of the sequence $(u_n)_{n\in\mathbb{N}}$, which defined by the iterative expression:

$$u_{n+1} = u_0 + B(u_n,u_n) + L(u_n), \qquad \text{for} \;\; n \geq 0, \tag{23}$$

and where u_0, $B(u_n,u_n)$ and $L(u_n)$ are given in the expressions (16) and (17) respectively. We shall use this sequence to prove that $(\vec{u},\theta) \in L^p(\mathbb{R}^3)$. We thus write

$$\|u_{n+1}\|_{L^p} \leq \|u_0\|_{L^p} + \|B(u_n u_n)\|_{L^p} + \|L(u_n)\|_{L^p}, \tag{24}$$

where, by the assumption (22) we have $\|u_0\|_{L^p} < +\infty$. On the other hand, in order to estimate the bilinear term we recall that its first component writes down as $m_1 * (\vec{u} \otimes \vec{u})$ with $m_1 \in L^{3/2,\infty}(\mathbb{R}^3)$. Then, by the first point of Lemma 2.1 (with $p = q$, $p_1 = 3/2$, $p_2 = 3p/(3+p)$, $q_1 = +\infty$ and $q_2 = p_2$) we get

$$\|m_1 * (\vec{u} \otimes \vec{u})\|_{L^p} \leq C_1(p)\, \|m_1\|_{L^{3/2,\infty}} \|\vec{u} \otimes \vec{u}\|_{L^{\frac{3p}{3+p}}}, \quad \text{with} \;\; 0 < C_1(p) = C\frac{3p^2}{2p-3}. \tag{25}$$

Remark 1 *Note that the constant $C_1(p)$ given above blows-up at $p = \frac{3}{2}$ and $p = +\infty$. This fact yields the first constraint $\frac{3}{2} < p < +\infty$, which is the same in [2].*

Then, from the estimate (25) and by second point of Lemma 2.2 (remark that $(3+p)/3p = 1/3 + 1/p$) we write

$$\|m_1 * (\vec{u} \otimes \vec{u})\|_{L^p} \leq C_1(p)\|m_1\|_{L^{3/2,\infty}}\, \|\vec{u}\|_{L^p}\, \|\vec{u}\|_{L^{3,\infty}}.$$

Moreover, recall that the sequence $(u_n)_{n\in\mathbb{N}}$ defined in (23) verifies $\|u_n\|_{L^{3,\infty}} \leq 3\delta$ and we thus obtain

$$\|m_1 * (\vec{u} \otimes \vec{u})\|_{L^p} \leq 3\delta C_1(p)\, \|m_1\|_{L^{3/2,\infty}}\, \|\vec{u}\|_{L^p}.$$

The second component of the bilinear form $B(u_n, u_n)$ essentially follows the same arguments and we have

$$\left\| \frac{1}{-\Delta} \text{div}(\theta \vec{\mathbf{g}}) \right\|_{L^p} = \|m_2 * (\theta \vec{u})\|_{L^p} \leq 3\delta C_1(p) \, \|m_2\|_{L^{3/2,\infty}} \, \|\theta\|_{L^p}. \quad (26)$$

With these estimates at our disposal, for the sake of simplicity we introduce the constant $C_m = \max(\|m_1\|_{L^{3/2,\infty}}, \|m_2\|_{L^{3/2,\infty}})$ and we obtain the estimate

$$\|B(u_n, u_n)\|_{L^p} \leq 3\delta C_m C_1(p) \|u_n\|_{L^p}. \quad (27)$$

We study now the linear term $L(u_n)$. Let $1 < p_1 < 3/2$ defined by the relationship $1/p_1 = 2/3 + 1/p$.

Remark 2 *Observe that the relationship $1/p_1 = 2/3 + 1/p$ yields the new constraint $3 < p < +\infty$.*

By the first point of Lemma 2.1 (since $1/p_1 = 2/3 + 1/p$ we have $1 + 1/p = 1/3 + 1/p_1$, hence we set $p_2 = 3$, $q_2 = +\infty$ and $q = p$, $q_1 = p_1$) and by using again the second point of Lemma 2.2 (recall that $1/p_1 = 2/3 + 1/p$) we write

$$\begin{aligned} \left\| \frac{1}{-\Delta} \left(\mathbb{P}(\theta \vec{\mathbf{g}}) \right) \right\|_{L^p} &= \|m_3 * (\theta \vec{u})\|_{L^p} \leq C'_1(p) \|m_3\|_{L^{3,\infty}} \, \|\theta \vec{\mathbf{g}}\|_{L^{p_1}} \leq C_1^{'}(p) \|m_3\|_{L^{3,\infty}} \, \|\vec{\mathbf{g}}\|_{L^{3/2}} \, \|\theta\|_{L^p} \\ &\leq C_1^{'}(p) \|m_3\|_{L^{3,\infty}} \, \|\vec{\mathbf{g}}\|_{L^{3/2,1}} \, \|\theta\|_{L^p}. \end{aligned}$$

Remark 3 *We have $C_1^{'}(p) = C \frac{3p^2}{p-3}$, which is well-defined as long as $3 < p < +\infty$.*

For the sake of simplicity, we introduce the constant $C_m^{'} = \|m_3\|_{L^{3,\infty}}$ and we thus get

$$\|L(u_n)\|_{L^p} \leq \left(C_m^{'} C_1^{'}(p) \|\vec{\mathbf{g}}\|_{L^{3/2,1}} \right) \, \|u_n\|_{L^p}. \quad (28)$$

Once we have the estimates (27) and (28), we get back to the inequality (24) to write

$$\|u_{n+1}\|_{L^p} \leq \|u_0\|_{L^p} + \left(3\delta C_m C_1(p) + C_m^{'} C_1^{'}(p) \, \|\vec{\mathbf{g}}\|_{L^{3/2,1}} \right) \|u_n\|_{L^p}, \quad \text{for} \;\; n \geq 0 \;\; \text{and} \;\; 3 < p < +\infty. \quad (29)$$

At this point, we need to find additional constraints on the quantities δ and $\|\vec{\mathbf{g}}\|_{L^{3/2,1}}$ to obtain

$$\left(3\delta C_m C_1(p) + C_m^{'} C_1^{'}(p) \, \|\vec{\mathbf{g}}\|_{L^{3/2,1}} \right) < \frac{1}{2}. \quad (30)$$

With this inequality we will able to prove that the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in the space $L^p(\mathbb{R}^3)$ (see the estimate (33) below) and consequently its limit verifies $u \in L^p(\mathbb{R}^3)$. However, there is a problem to overcome: the quantities $C_1(p)$ and $C_1^{'}(p)$ (defined in the expression (25) and the Remark 3 respectively) are not bounded in the whole interval $3 < p < +\infty$. So, we shall use the following strategy: first we will prove the wished inequality (30) in the interval $4 \leq p \leq 7$ to obtain $u \in L^p(\mathbb{R}^3)$ for these values of p. Thereafter, we shall use this information to get that $u \in L^p(\mathbb{R}^3)$ in the intervals $3 < p < 4$, $7 < p < +\infty$ and the value $p = +\infty$.

- **The case** $4 \leq p \leq 7$. We get back to the expression of the quantity $C_1(p)$ given in (25) and we define $0 < M = \max\limits_{4 \leq p \in 7} C_1(p) < +\infty$. Then, we set the new constraint on the parameter δ (which already verifies (13)) as follows:

$$3\delta C_m M < \frac{1}{4}. \quad (31)$$

Similarly, we get back to the expression of the quantity $C_1^{'}(p)$ given in Remark 3 and we define $0 < M' = \max_{4\leq p\leq 7} C_1^{'}(p) < +\infty$. We thus set the new constraint on the quantity $\|\vec{\mathbf{g}}\|_{L^{3/2,1}}$ (which also verifies (13)) as the next one:

$$C_m^{'} M' \|\vec{\mathbf{g}}\|_{L^{3/2,1}} < \frac{1}{4}. \tag{32}$$

By these new constraints, for all $4 \leq p \leq 7$ we have the inequality (30); and we get back to the inequality (29) to write

$$\|u_{n+1}\|_{L^p} \leq \|u_0\|_{L^p} + \frac{1}{2}\|u_n\|_{L^p}, \quad \text{for all} \quad n \geq 0.$$

Hence, we iterate to obtain the uniform control

$$\|u_{n+1}\|_{L^p} \leq \left(\sum_{k=0}^{+\infty} \frac{1}{2^k}\right) \|u_0\|_{L^p}, \quad \text{for all} \quad n \geq 0, \tag{33}$$

and then we have $u = (\vec{u}, \theta) \in L^p(\mathbb{R}^3)$ with $4 \leq p \leq 7$.

Moreover, we emphasize that this fact holds true as long as the quantities δ and $\|\vec{\mathbf{g}}\|_{L^{3/2,1}}$ verify the set of constraints (13), (31) and (32), which can be jointly written as

$$\delta, \|\vec{\mathbf{g}}\|_{L^{3/2,1}} < \min\left(\frac{1}{12 C_m M}, \frac{1}{4 C_m^{'} M'}, \varepsilon_1\right) = \varepsilon. \tag{34}$$

- **The case** $3 < p < 4$. By the assumption (22) we have $u_0 \in L^4(\mathbb{R}^3)$ and by the uniform control (33) we have $u \in L^4(\mathbb{R}^3)$. Moreover, recall that by Proposition (3.1) we also have $u \in L^{3,\infty}(\mathbb{R}^3)$ and by the second point of Lemma 2.3 we have $u = (\vec{u}, \theta) \in L^p(\mathbb{R}^3)$ with $3 < p < 4$.
- **The case** $7 < p < +\infty$. First recall that by assumption (22) we have $u_0 \in L^p(\mathbb{R}^3)$ and since we also $u_0 \in L^{3,\infty}(\mathbb{R}^3)$ by the standard interpolation inequalities we obtain $u_0 \in L^4 \cap L^7(\mathbb{R}^3)$. Thus, always by the uniform control (33) we obtain $u \in L^4 \cap L^7(\mathbb{R}^3)$ and then by the second point of Lemma 2.3 we have $u \in L^{6,2}(\mathbb{R}^3)$.

With this information we can prove that $B(u,u) \in L^\infty(\mathbb{R}^3)$. Indeed, this bilinear form essentially writes down as $m * (u \otimes u)$ where the tensor $m \in L^{3/2,\infty}(\mathbb{R}^3)$ denotes each tensor m_i (with $i = 1, 2$) in each component of $B(u,u)$ (see the expression (17)). Then, since $u \in L^{6,2}(\mathbb{R}^3)$ by the second point of Lemma 2.2 we obtain $u \otimes u \in L^{3,1}(\mathbb{R}^3)$; and since $m \in L^{3/2,\infty}(\mathbb{R}^3)$ by the third point of Lemma 2.1 we have

$$B(u,u) \in L^\infty(\mathbb{R}^3). \tag{35}$$

As we also have $B(u,u) \in L^{3,\infty}(\mathbb{R}^3)$ (see the estimate (19)) and by using again the second point of Lemma 2.3 we obtain $B(u,u) \in L^p(\mathbb{R}^3)$ with $p \in (7, +\infty)$.

Now, we shall verify that $L(u) \in L^\infty(\mathbb{R}^3)$. Indeed, by the third point of Lemma 2.1 we write

$$\left\|\frac{1}{-\Delta}\left(\mathbb{P}(\theta\vec{\mathbf{g}})\right)\right\|_{L^\infty} = \|m_3 * (\theta\vec{\mathbf{g}})\|_{L^\infty} \leq C_3 \|m_3\|_{L^{3,\infty}} \|\theta\vec{\mathbf{g}}\|_{L^{3/2,1}}.$$

But, since $\theta \in L^{3,\infty}(\mathbb{R}^3)$ and $\vec{\mathbf{g}} \in L^{3,1}(\mathbb{R}^3)$ (see the assumption (22)) by the second point of Lemma 2.2 we get $\|\theta\vec{\mathbf{g}}\|_{L^{3/2,1}} \leq \|\theta\|_{L^{3,\infty}} \|\vec{\mathbf{g}}\|_{L^{3,1}}$. We thus obtain

$$L(u) \in L^\infty(\mathbb{R}^3). \tag{36}$$

As before, since we also have $L(u) \in L^{3,\infty}(\mathbb{R}^3)$ (see the estimate (20)) we use using the second point of Lemma 2.3 to conclude that $L(u) \in L^p(\mathbb{R}^3)$ with $p \in (7, +\infty)$. Consequently, we have $u = (\vec{u}, \theta) \in L^p(\mathbb{R}^3)$ with $7 < p < +\infty$.

- **The case** $p = +\infty$. We just remark that from the information given in (35) and (36) we have $u = (\vec{u}, \theta) \in L^\infty(\mathbb{R}^3)$ as long as $u_0 \in L^\infty(\mathbb{R}^3)$, which holds true under the assumption $\frac{1}{-\Delta}\Big(\mathbb{P}(\vec{f})\Big) \in L^\infty(\mathbb{R}^3)$ and $\frac{1}{-\Delta}(g) \in L^\infty(\mathbb{R}^3)$. Proposition 3.2 is proven. ■

Once we have Propositions 3.1 and 3.2, Theorem 1.1 is now proven. ■

4 Proof of Proposition 1.1

The proof is divided into three main steps, which we shall study separately in the next (technical) propositions. In addition, we recall that in all this section the external forces $\vec{f}, g \in \mathcal{S}(\mathbb{R}^3)$ and the gravitational acceleration $\vec{\mathbf{g}} \in \mathcal{S}(\mathbb{R}^3)$ verify (5) and we have $(-\Delta)^{-1}\vec{f} \in \mathcal{S}(\mathbb{R}^3), (-\Delta)^{-1}g \in \mathcal{S}(\mathbb{R}^3)$ and $(-\Delta)^{-1}\vec{\mathbf{g}} \in \mathcal{S}(\mathbb{R}^3)$.

Step 1. Pointwise decaying properties. We start by proving the following pointwise decaying properties of solutions to the system (4).

Proposition 4.1 *Let $\vec{f}, g, \vec{\mathbf{g}} \in \mathcal{S}(\mathbb{R}^3)$ verifying (5). Moreover, let $\varepsilon > 0$ be the quantity defined in (34).*

There exists a universal quantity $0 < \varepsilon_2 < \varepsilon$, such that if

$$\delta_2 = \|(-\Delta)^{-1}\vec{f}\|_{L^{3,\infty}} + \|\,|x|(-\Delta)^{-1}\vec{f}\|_{L^\infty} + \|(-\Delta)^{-1}g\|_{L^{3,\infty}} + \|\,|x|(-\Delta)^{-1}g\|_{L^\infty} < \varepsilon_2, \tag{37}$$

and

$$\|\vec{\mathbf{g}}\|_{L^{3/2,1}} + \|(-\Delta)^{-1}|\vec{\mathbf{g}}|\|_{L^\infty} < \varepsilon_2, \tag{38}$$

then the solution $(\vec{u}, \theta) \in L^{3,\infty}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ to the system (4), given by Theorem 1.1, verifies the following pointwise estimates:

$$|\vec{u}(x)| \leq \frac{C_1}{1+|x|}, \quad |\theta(x)| \leq \frac{C_2}{1+|x|}, \qquad x \in \mathbb{R}^3, \tag{39}$$

with constants $C_1, C_2 > 0$ depending on $\vec{f}, g, \vec{\mathbf{g}}, \vec{u}$ and θ, but independent on x.

Remark 4 *It is worth emphasizing this result also holds true for the (original) system (1), since here we only need the information $\varphi \in L^\infty(\mathbb{R}^3)$; and we thus can set $\varphi \equiv 1$.*

Proof. First, remark that the solution $(\vec{u}, \theta) \in L^{3,\infty}(\mathbb{R}^3)$ is constructed as in the proof of Proposition 3.1: we assume the stronger inequality (37) (remark that $\delta < \delta_2$, where δ is defined in (13)) and we solve the fixed point equation

$$u = u_0 + B_\varphi(u, u) + L(u) \tag{40}$$

in the Banach space $L^{3,\infty}(\mathbb{R}^3)$. Here, the terms u, u_0 are given in (16) and the term $L(u)$ is given in (17) (without the Leray's projector $\mathbb{P}$). Moreover we have

$$B_\varphi(u, u) = \begin{pmatrix} -\frac{1}{-\Delta}\left(\mathbb{P}(\text{div}(\varphi\vec{u} \otimes \vec{u}))\right) \\ -\frac{1}{-\Delta}\left(\text{div}(\theta\vec{u})\right) \end{pmatrix}. \tag{41}$$

This solution is uniquely defined by the condition $\|u\|_{L^{3,\infty}} \leq 3\delta_2$.

On the other hand, we introduce the weighted L^∞- space:

$$L^\infty_1(\mathbb{R}^3) = \big\{ f : \mathbb{R}^3 \to \mathbb{R}^3 \;/\; f \text{ is measurable and } \; |x|f \in L^\infty(\mathbb{R}^3) \big\}, \tag{42}$$

which verifies $L^\infty_1(\mathbb{R}^3) \subset L^{3,\infty}(\mathbb{R}^3)$. Thus, we shall solve the problem (40) in the smaller Banach space $L^\infty_1(\mathbb{R}^3)$ with the norm

$$\|f\|_1 = \|\,|x|f\|_{L^\infty} + \|f\|_{L^{3,\infty}}.$$

Since $\varphi \in L^\infty(\mathbb{R}^3)$ the quantity $\|B_\varphi(u,u)\|_{L^{3,\infty}}$ is similarly estimated as in (19) and we have

$$\|B_\varphi(u,u)\|_{L^{3,\infty}} \leq \|\varphi\|_{L^\infty} C_B \|u\|^2_{L^{3,\infty}}. \tag{43}$$

Moreover, the quantity $\|L(u)\|_{L^{3,\infty}}$ was already estimated in (20). So, it remains to estimate the quantities $\|\,|x|B_\varphi(u,u)\|_{L^\infty}$ and $\|\,|x|L(u)\|_{L^\infty}$. The first quantity follows the same estimates in [2, Lemma 3.6] and we can write

$$\|\,|x|B_\varphi(u,u)\|_{L^\infty} \leq C\|\varphi\|_{L^\infty}\|\,|x|u\|^2_{L^\infty}. \tag{44}$$

To study the second quantity, for $x \neq 0$ fixed we write

$$\begin{aligned}\left|\frac{1}{-\Delta}(\theta\vec{\mathbf{g}})(x)\right| &\leq \int_{\mathbb{R}^3}\frac{1}{|x-y|}|\theta(y)|\,|\vec{\mathbf{g}}(y)|dy \leq \left(\int_{|y|\leq|x|/2}+\int_{|y|>|x|/2}\right)\frac{1}{|x-y|}|\theta(y)|\,|\vec{\mathbf{g}}(y)|dy\\ &= I_1+I_2.\end{aligned}$$

To estimate the term I_1, we remark that since $|y|\leq|x|/2$ we have $|x-y|\geq|x|/2$ and by the first point of Lemma 2.2 we can write

$$I_1 \leq \frac{C}{|x|}\int_{|y|\leq|x|/2}|\theta(y)|\,|\vec{\mathbf{g}}(y)|dy \leq \|\theta\vec{\mathbf{g}}\|_{L^1} \leq \frac{C}{|x|}\|\theta\|_{L^{3,\infty}}\|\vec{\mathbf{g}}\|_{L^{3/2,1}}.$$

To estimate the term I_2, since $|y|>|x|/2$ we can use expression $\|\theta\|_{L^\infty_1}$ to get

$$\begin{aligned}I_2 &\leq C\|\theta\|_{L^\infty_1}\int_{|y|>|x|/2}\frac{1}{|x-y|\,|y|}|\vec{\mathbf{g}}(y)|dy \leq \frac{C}{|x|}\|\theta\|_{L^\infty_1}\int_{|y|>|x|/2}\frac{1}{|x-y|}|\vec{\mathbf{g}}(y)|dy\\ &\leq \frac{C}{|x|}\|\theta\|_{L^\infty_1}\int_{\mathbb{R}^3}\frac{1}{|x-y|}|\vec{\mathbf{g}}(y)|dy \leq \frac{C}{|x|}\|\theta\|_{L^\infty_1}(-\Delta)^{-1}|\vec{\mathbf{g}}(x)| \leq \frac{C}{|x|}\|\theta\|_{L^\infty_1}\|(-\Delta)^{-1}|\vec{\mathbf{g}}|\|_{L^\infty}.\end{aligned}$$

From these estimate we obtain

$$\|\,|x|L(u)\|_{L^\infty} \leq C\left(\|\vec{\mathbf{g}}\|_{L^{3/2,1}}+\|(-\Delta)^{-1}|\vec{\mathbf{g}}|\|_{L^\infty}\right)\|\theta\|_1. \tag{45}$$

Once we have the estimates (43), (43) and (45), and moreover, by the estimate (20) we are able to write

$$\|B_\varphi(u,u)\|_1 \leq \max(C_B,C)\|\varphi\|_{L^\infty}\|u\|_1^2, \quad \|L(u)\| \leq \max(C,C_L)(\|\vec{\mathbf{g}}\|_{L^{3/2,1}}+\|(-\Delta)^{-1}|\vec{\mathbf{g}}|\|_{L^\infty})\|u\|_1.$$

We set a quantity $0<\varepsilon_2<\varepsilon$ small enough such that the constraints given in (15) hold. We also assume (37) and (38). Consequently, by Lemma 3.1 there exists a solution $u=(\vec{u},\theta)\in L^\infty_1$ to the equation (40) which is uniquely determined by the condition $\|u\|_1\leq 3\delta_2$. But, since $\|u\|_{L^{3,\infty}}\leq\|u\|_1$ this solution is the same to the one given by Theorem 1.1.

Finally, recall that this solution also verifies $u\in L^\infty(\mathbb{R}^3)$, hence we obtain the wished estimates (39). Proposition 4.1 is proven. ■

Step 2. Asymptotic profile. Once we have the estimates (39), in the next proposition we develop an asymptotic profile for the velocity $\vec{u}$.

Proposition 4.2 *Within the framework of Proposition 4.1, the velocity $\vec{u}$ has the asymptotic behavior*

$$\vec{u}(x) = \frac{1}{|x|}\left(\int_{\mathbb{R}^3}\theta(y)\vec{\mathbf{g}}(y)dy\right)+o\left(\frac{1}{|x|}\right), \quad |x|\to+\infty. \tag{46}$$

Remark 5 *To the best of our knowledge, the identity (46) strongly depends on the well-prepared function φ. In particular, to justify our estimates, here we need the pointwise inequality $|\varphi(x)|\lesssim\frac{1}{1+|x|^2}$.*

Proof. Recall that $\vec{u}$ verifies the fixed point equation

$$\vec{u}(x)=\frac{1}{-\Delta}(\theta\vec{\mathbf{g}})(x)-\frac{1}{-\Delta}(\text{div}(\varphi\vec{u}\otimes\vec{u}))(x)+\frac{1}{-\Delta}(\vec{f})(x). \tag{47}$$

Our starting point is to prove that the first term on the right-hand side splits as follows:

$$\frac{1}{-\Delta}(\theta\vec{\mathbf{g}})(x)=\frac{1}{|x|}\left(\int_{\mathbb{R}^3}\theta(y)\vec{\mathbf{g}}(y)dy\right)+o\left(\frac{1}{|x|}\right),\quad |x|\to+\infty. \tag{48}$$

Indeed, for $|x|>0$ fixed large enough we write

$$\begin{aligned}\frac{1}{-\Delta}(\theta\vec{\mathbf{g}})(x)&=\int_{\mathbb{R}^3}\frac{1}{|x-y|}\theta(y)\vec{\mathbf{g}}(y)dy\\ &=\frac{1}{|x|}\left(\int_{\mathbb{R}^3}\theta(y)\vec{\mathbf{g}}(y)dy\right)+\int_{\mathbb{R}^3}\left(\frac{1}{|x-y|}-\frac{1}{|x|}\right)\theta(y)\vec{\mathbf{g}}(y)dy.\end{aligned}$$

The, for $i=1,2,3$ we shall prove that $\displaystyle\int_{\mathbb{R}^3}\left(\frac{1}{|x-y|}-\frac{1}{|x|}\right)\theta(y)\mathbf{g}_i(y)dy=o\left(\frac{1}{|x|}\right)$. We thus write

$$\int_{\mathbb{R}^3}\left(\frac{1}{|x-y|}-\frac{1}{|x|}\right)\theta(y)\mathbf{g}_i(y)dy=\left(\int_{|y|\leq|x|/2}+\int_{|y|>|x|/2}\right)\left(\frac{1}{|x-y|}-\frac{1}{|x|}\right)\theta(y)\mathbf{g}_i(y)dy=I_1+I_2.$$

To estimate the term I_1, recall that the function $\frac{1}{|x|}$ is smooth outside the origin and by the mean value theorem, for $w=\alpha(x-y)+(1-\alpha)x$ with $0<\alpha<1$, we write $\left|\frac{1}{|x-y|}-\frac{1}{|x|}\right|\leq\frac{1}{|w|^2}|y|$. Moreover, remark that $w=x-\alpha y$ and as we have $|y|\leq|x|/2$ then we get $|w|\geq|x|-\alpha|y|\geq|x|-|y|\geq|x|/2$. We thus obtain $\left|\frac{1}{|x-y|}-\frac{1}{|x|}\right|\leq\frac{C}{|x|^2}|y|$. With this inequality at hand, and moreover, by recalling that $\mathbf{g}_i\in\mathcal{S}(\mathbb{R}^3)$ and $\theta\in L^\infty(\mathbb{R}^3)$, the term I_1 verifies

$$I_1\leq\frac{C}{|x|^2}\|\theta\|_{L^\infty}\int_{|y|\leq|x|/2}|y||\mathbf{g}_i(y)|dy\leq\frac{C}{|x|^2}\|\theta\|_{L^\infty}\|\,|\cdot|\mathbf{g}_i\|_{L^1}. \tag{49}$$

To estimate the term I_2 we write

$$I_2\leq\int_{|y|>|x|/2}\frac{1}{|x-y|}|\theta(y)||\mathbf{g}_i(y)|dy+\frac{1}{|x|}\int_{|y|>|x|/2}|\theta(y)||\mathbf{g}_i(y)|dy=I_{2,1}+I_{2,2}.$$

Then, in order to study the term $I_{2,1}$, we recall that θ verifies the pointwise estimate (39), and moreover, since $\mathbf{g}_i\in\mathcal{S}(\mathbb{R}^3)$ we have $|\mathbf{g}_i(y)|\leq\frac{C}{|y|^\beta}$, with $2<\beta<3$. We thus get

$$I_{2,1}\leq C\int_{|y|>|x|/2}\frac{dy}{|x-y||y||y|^\beta}\leq\frac{C}{|x|}\int_{\mathbb{R}^3}\frac{dy}{|x-y||y|^\beta}\leq\frac{C}{|x|^{\beta-1}}. \tag{50}$$

Thereafter, in order to estimate the term $I_{2,2}$, we use again the previous arguments to write

$$I_{2,2}\leq\frac{C}{|x|^2}\int_{|y|>|x|/2}|\mathbf{g}_i(y)|dy\leq\frac{C}{|x|^2}\|\mathbf{g}_i\|_{L^1}. \tag{51}$$

With the estimates (49), (50) and (51) at hand, we finally obtain the wished identity (48).

Now, we need to verify that

$$\frac{1}{-\Delta}(\text{div}(\varphi\vec{u}\otimes\vec{u}))(x)=o\left(\frac{1}{|x|}\right),\qquad |x|\to+\infty. \tag{52}$$

Since $|\varphi(x)| \leq \frac{C}{1+|x|^2}$ one can follow the same estimates in the proof of [2, Theorem 3.1, equation (2.3)] to obtain the asymptotic profile

$$\frac{1}{-\Delta}(\text{div}(\varphi\vec{u}\otimes\vec{u}))(x) = m_2(x)\left(\int_{\mathbb{R}^3}\varphi(y)\vec{u}\otimes\vec{u}(y)dy\right) + O\left(\frac{\log(|x|)}{|x|^3}\right), \quad |x| \to +\infty,$$

where m_2 denotes the tensor of the operator $\frac{\text{div}(\cdot)}{-\Delta}$; and it verifies the estimate $|m_2(x)| \leq \frac{C}{|x|^2}$ outside the origin. Hence we have the identity (52).

Finally, recall that the condition (5) yields $\frac{1}{-\Delta}(\vec{f}) \in \mathcal{S}(\mathbb{R}^3)$. With this information and by the identities (48) and (52) we obtain the wished asymptotic profile (46). Proposition 4.2 is proven. ■

Step 3. Estimate from below. In out last step, the asymptotic profile (46) will allow us to obtain the next estimate from below on the velocity $\vec{u}$, which yields $\vec{u} \notin L^p(\mathbb{R}^3)$ for $1 \leq p \leq 3$.

Proposition 4.3 *Within the framework of Proposition 4.1, assume that*

$$\vec{f} = \kappa\vec{\mathfrak{f}}, \quad g = \kappa\mathfrak{g} \quad \textit{and} \quad \vec{\mathbf{g}} = \kappa\vec{\mathfrak{g}}, \tag{53}$$

with $\vec{\mathfrak{f}}, \mathfrak{g}, \vec{\mathfrak{g}} \in \mathcal{S}(\mathbb{R}^3)$ verifying (5); and with $\kappa > 0$ a suitable quantity defined in the expression (58) below. Moreover, for the external force $\mathfrak{g}$ and gravitational acceleration $\vec{\mathfrak{g}} = (\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3)$, assume that

$$\int_{\mathbb{R}^3}\frac{1}{-\Delta}(\mathfrak{g})(x)\mathfrak{g}_i(x)dx \neq 0, \qquad i = 1,2,3. \tag{54}$$

Then there exists two quantities $M_1 = M_1(\theta, \vec{\mathbf{g}}) > 0$ and $M_2 = M(\theta, \vec{\mathbf{g}}) > 0$, which depend on θ and g, such that the velocity $\vec{u}$ verifies the estimate from below:

$$\frac{M_1}{2|x|} \leq |\vec{u}(x)|, \qquad M_2 < |x|. \tag{55}$$

Then, $\vec{u} \notin L^p(\mathbb{R}^3)$ for $1 \leq p \leq 3$.

Proof. From the asymptotic profile (46), for $|x| > 0$ large enough we can write

$$|\vec{u}(x)| = \left|\frac{1}{|x|}\left(\int_{\mathbb{R}^3}\theta(y)\vec{\mathbf{g}}(y)dy\right) + o\left(\frac{1}{|x|}\right)\right| \geq \frac{1}{|x|}\left|\int_{\mathbb{R}^3}\theta(y)\vec{\mathbf{g}}(y)dy\right| - \left|o\left(\frac{1}{|x|}\right)\right|. \tag{56}$$

We thus set the quantity $M_1 = \left|\int_{\mathbb{R}^3}\theta(y)\vec{\mathbf{g}}(y)dy\right|$ and, in order to verify that $M_1 > 0$, we have the following technical lemma:

Lemma 4.1 *The assumptions (53) and (54) imply that $M_1 > 0$.*

Proof. Recall that the temperature θ verifies the fixed point equation

$$\theta = \frac{1}{-\Delta}(g) - \frac{1}{-\Delta}(\text{div}(\theta\vec{u})),$$

where we denote $\frac{1}{-\Delta}(g) = \theta_0$ and $\frac{1}{-\Delta}(\text{div}(\theta\vec{u})) = B_2(\theta, \vec{u})$. Moreover, since $g = \kappa\mathfrak{g}$ we shall denote $\theta_0 = \kappa\tilde{\theta}_0 = \kappa\frac{1}{-\Delta}(\mathfrak{g})$. Then, by the identities (53), for $i = 1,2,3$ we write

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\theta\mathbf{g}_i dy\right| &= \left|\int_{\mathbb{R}^3}(\theta\mathbf{g}_i - \theta_0\mathbf{g}_i + \theta_0\mathbf{g}_i)dy\right| \geq \left|\int_{\mathbb{R}^3}\theta_0\mathbf{g}_i dy\right| - \left|\int_{\mathbb{R}^3}(\theta - \theta_0)\mathbf{g}_i dy\right| \\ &\geq \kappa^2\left|\int_{\mathbb{R}^3}\tilde{\theta}_0\mathfrak{g}_i dy\right| - \kappa\left|\int_{\mathbb{R}^3}(\theta - \theta_0)\mathfrak{g}_i dy\right|.\end{aligned} \tag{57}$$

We must estimate the last term on the right-hand side. To do this, we recall that $\theta - \theta_0 = B_2(\theta, \vec{u})$; and by the Hölder inequalities we write

$$\left| \int_{\mathbb{R}^3} (\theta - \theta_0)\mathfrak{g}_i dy \right| \leq \|\theta - \theta_0\|_{L^4} \|\vec{\mathfrak{g}}\|_{L^{4/3}} \leq \|B_2(\theta, \vec{u})\|_{L^4} \|\vec{\mathfrak{g}}\|_{L^{4/3}}.$$

Here, we still need to estimate the term $\|B_2(\theta, \vec{u})\|_{L^4}$. By the estimate (26) (with $\kappa\delta_2$ instead of δ; and where δ_2 is given in (37) with the functions $\vec{\mathfrak{f}}$ and $\mathfrak{g}$) we have

$$\|B_2(\theta, \vec{u})\|_{L^4} \leq 3\kappa\delta_2 C_1(4) \|m_2\|_{L^{3/2,\infty}} \|\theta\|_{L^4} = C\kappa \|\theta\|_{L^4}.$$

Finally, we must estimate the term $\|\theta\|_{L^4}$. We use the uniform control (33), where we recall that u_0 is defined in (16); and for $\mathfrak{u}_0 = \left((-\Delta)^{-1}\vec{\mathfrak{f}}, (-\Delta)^{-1}\mathfrak{g} \right)$ we have $u_0 = \kappa \mathfrak{u}_0$. Moreover, for the sake of simplicity we shall denote $C' = \left(\sum_{k=0}^{+\infty} \frac{1}{2^k} \right)$. We thus have $\|\theta\|_{L^4} \leq C' \|u_0\|_{L^4} = C'\kappa \|\mathfrak{u}_0\|_{L^4}$. From these estimates we finally obtain

$$\left| \int_{\mathbb{R}^3} (\theta - \theta_0)\mathfrak{g}_i dy \right| \leq CC'\kappa^2 \|\mathfrak{u}_0\|_{L^4} \|\vec{\mathfrak{g}}\|_{L^{4/3}}.$$

We thus get back to the estimate (57) to write

$$\left| \int_{\mathbb{R}^3} \theta \mathbf{g}_i dy \right| \geq \kappa^2 \left| \int_{\mathbb{R}^3} \tilde{\theta}_0 \mathfrak{g}_i dy \right| - \kappa^3 CC' \|\mathfrak{u}_0\|_{L^4} \|\vec{\mathfrak{g}}\|_{L^{4/3}},$$

hence we have

$$\kappa^2 \left| \int_{\mathbb{R}^3} \tilde{\theta}_0 \mathfrak{g}_i dy \right| - \kappa^3 CC' \|\mathfrak{u}_0\|_{L^4} \|\vec{\mathfrak{g}}\|_{L^{4/3}} > 0,$$

as long as

$$\kappa < \frac{\left| \int_{\mathbb{R}^3} \tilde{\theta}_0 \mathfrak{g}_i dy \right|}{CC' \|\mathfrak{u}_0\|_{L^4} \|\vec{\mathfrak{g}}\|_{L^{4/3}}} \quad \text{and} \quad 0 < \left| \int_{\mathbb{R}^3} \tilde{\theta}_0 \mathfrak{g}_i dy \right|, \tag{58}$$

where this last inequality is ensured by the assumption (54). ■

Once we have $M_1 > 0$, there exists quantity $M_2 > 0$ large enough such that for $|x| > M_2$ we have

$$\left| o\left(\frac{1}{|x|} \right) \right| \leq \frac{M_1}{2|x|}.$$

Then, we get back to the inequality (56) to obtain the wished estimate from below (55). Proposition 4.3 is proven. ■

As noticed, Proposition 1.1 is now proven by Propositions 4.1, 4.2 and 4.3. ■

5 Proof of Theorem 1.2

For the sake of clearness, we shall divide the proof in three main steps.

Step 1. The parabolic system. Our starting point is the study the time-dependent Boussinesq system. For the velocity $\vec{v} : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ and the temperature $\vartheta : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$, and moreover, for the time-independent functions $\vec{f}, g, \vec{\mathbf{g}}$, we consider:

$$\begin{cases} \partial_t \vec{v} - \Delta \vec{v} + \text{div}(\vec{v} \otimes \vec{v}) + \vec{\nabla} P = \vartheta \vec{\mathbf{g}} + \vec{f}, \qquad \text{div}(\vec{v}) = 0, \\ \partial_t \vartheta - \Delta \vartheta + \text{div}(\vartheta \vec{v}) = g, \\ \vec{v}(0, \cdot) = \vec{v}_0, \quad \vartheta(0, \cdot) = \vartheta_0, \end{cases} \tag{59}$$

where $\vec{v}_0 : \mathbb{R}^3 \to \mathbb{R}^3$, $\vartheta_0 : \mathbb{R}^3 \to \mathbb{R}$ are the initial data. For a time $0 < T < +\infty$, we denote $\mathcal{C}_*([0, T], L^{p,\infty}(\mathbb{R}^3))$ the functional space of bounded and weak$-*$ continuous functions from $[0, T]$ with values in the Lorentz space $L^{p,\infty}(\mathbb{R}^3)$. We prove now the following:

Proposition 5.1 *Consider the initial value problem (59) where $\vec{f}, g$ verify (6) and $\vec{\mathbf{g}}$ verifies (7). Let $p>3$ and let $\vec{v}_0 \in L^{p,\infty}(\mathbb{R}^3)$, $\vartheta_0 \in L^{p,\infty}(\mathbb{R}^3)$ be the initial data. There exists a time $T_0>0$, depending on $\vec{v}_0$, ϑ_0, $\vec{f}, g$ and $\vec{\mathbf{g}}$; and there exists a unique solution $(\vec{v}, \vartheta) \in \mathcal{C}_*([0,T_0], L^{p,\infty}(\mathbb{R}^3))$ to the equation (59). Moreover this solution verifies:*

$$\sup_{0<t\leq T_0} t^{\frac{3}{2p}} \|\vec{v}(t,\cdot)\|_{L^\infty} + \sup_{0<t\leq T_0} t^{\frac{3}{2p}} \|\vartheta(t,\cdot)\|_{L^\infty} < +\infty. \tag{60}$$

Proof. Mild solutions of the system (59) write down as the integral formulation:

$$\vec{v}(t,\cdot) = e^{t\Delta}\vec{v}_0 + \int_0^t e^{(t-s)\Delta}\,\mathbb{P}(\vec{f})ds + \underbrace{\int_0^t e^{(t-s)\Delta}\mathbb{P}(div(\vec{v}\otimes\vec{v}))(s,\cdot)ds}_{\mathcal{B}_1(\vec{v},\vec{v})} + \underbrace{\int_0^t e^{(t-s)\Delta}\mathbb{P}(\theta\vec{\mathbf{g}}(s,\cdot))ds}_{\mathcal{L}(\theta)}, \tag{61}$$

and

$$\vartheta(t,\cdot) = e^{t\Delta}\vartheta_0 + \int_0^t e^{(t-s)\Delta}\,g ds + \underbrace{\int_0^t e^{(t-s)\Delta}\text{div}(\vartheta\vec{v})(s,\cdot)ds}_{\mathcal{B}_2(\vartheta,\vec{v})}. \tag{62}$$

By the Picard's fixed point argument, we will solve both problems (61) and (62) in the Banach space $\mathcal{X}$, where

$$\mathcal{X} = \left\{ f\in\mathcal{C}_*([0,T], L^{p,\infty}(\mathbb{R}^3)) : \sup_{0<t\leq T} t^{\frac{3}{2p}}\|f(t,\cdot)\|_{L^\infty} < +\infty \right\},$$

with the norm

$$\|f\|_{\mathcal{X}} = \sup_{0\leq t\leq T}\|f(t,\cdot)\|_{L^{p,\infty}} + \sup_{0<t\leq T} t^{\frac{3}{2p}}\|f(t,\cdot)\|_{L^\infty}.$$

Let us mention that for $f_1, f_2 \in \mathcal{X}$, for simplicity we shall write $\|(f_1,f_2)\|_{\mathcal{X}} = \|f_1\|_{\mathcal{X}} + \|f_2\|_{\mathcal{X}}$.

We start by studying the terms involving the data in the equations (61) and (62).

Lemma 5.1 *The next estimate holds:*

$$\left\|\left(e^{t\Delta}\vec{v}_0, e^{t\Delta}\vartheta_0\right)\right\|_{\mathcal{X}} + \left\|\int_0^t e^{(t-s)\Delta}\left(\vec{f}, g\right) ds\right\|_{\mathcal{X}} \leq C\|(\vec{v}_0,\vartheta_0)\|_{L^{p,\infty}} + CT\left\|\left(\vec{f},g\right)\right\|_{L^{p,\infty}}. \tag{63}$$

Proof. First, for the initial data $(\vec{v}_0,\varphi)\in L^{p,\infty}(\mathbb{R}^3)$, by the second point of Lemma 2.1 we have the estimate $\|(e^{t\Delta}\vec{v}_0, e^{t\Delta}\vartheta_0)\|_{L^{p,\infty}} \leq C_2\|(\vec{v}_0,\vartheta_0)\|_{L^{p,\infty}}$. We thus get $(e^{t\Delta}\vec{v}_0, e^{t\Delta}\vartheta_0)\in\mathcal{C}_*([0,T], L^{p,\infty}(\mathbb{R}^3))$. On the other hand, by Lemma 2.4 we write $\sup_{0<t\leq T} t^{\frac{3}{2p}}\left\|e^{t\Delta}\vec{v}_0\right\|_{L^\infty} \leq C\|\vec{v}_0\|_{L^{p,\infty}}$ and $\sup_{0<t\leq T} t^{\frac{3}{2p}}\left\|e^{t\Delta}\vartheta_0\right\|_{L^\infty} \leq C\|\vartheta_0\|_{L^{p,\infty}}$. We thus have $(e^{t\Delta}\vec{v}_0, e^{t\Delta}\vartheta_0)\in\mathcal{X}$ and the following estimate holds:

$$\left\|\left(e^{t\Delta}\vec{v}_0, e^{t\Delta}\vartheta_0\right)\right\|_{\mathcal{X}} \leq C\|(\vec{v}_0,\vartheta_0)\|_{L^{p,\infty}}. \tag{64}$$

Thereafter, recall that the external forces $\vec{f}, g$ are time-independent functions. Since we assume (6) by the second point of Lemme 2.1 we can write:

$$\left\|\int_0^t e^{(t-s)\Delta}\left(\mathbb{P}(\vec{f}),g\right)ds\right\|_{L^{p,\infty}} \leq \int_0^t \left\|e^{(t-s)\Delta}\left(\mathbb{P}(\vec{f}),g\right)\right\|_{L^{p,\infty}} ds \leq C_2\left\|\left(\mathbb{P}(\vec{f}),g\right)\right\|_{L^{p,\infty}}\left(\int_0^t ds\right),$$

and we get

$$\sup_{0\leq t\leq T}\left\|\int_0^t e^{(t-s)\Delta}\left(\mathbb{P}(\vec{f}),g\right)ds\right\|_{L^{p,\infty}} \leq CT\left\|\left(\mathbb{P}(\vec{f}),g\right)\right\|_{L^{p,\infty}}.$$

On the other hand, to estimate the expression $\displaystyle\sup_{0<t\leq T} t^{\frac{3}{2p}}\left\| \int_0^t e^{(t-s)\Delta}\mathbb{P}(\vec{f})ds\right\|_{L^\infty}$ we remark that by Lemma 2.4 we have

$$\left\| e^{(t-s)\Delta}\mathbb{P}(\vec{f})\right\|_{L^\infty} \leq C(t-s)^{-\frac{3}{2p}}\left\|\mathbb{P}(\vec{f})\right\|_{L^{p,\infty}},$$

and then we can write

$$\begin{aligned} &t^{\frac{3}{2p}}\left\| \int_0^t e^{(t-s)\Delta}\mathbb{P}(\vec{f})ds\right\|_{L^\infty} \leq t^{\frac{3}{2p}}\int_0^t \left\| e^{(t-s)\Delta}\mathbb{P}(\vec{f})\right\|_{L^\infty} ds \\ &\leq C\, t^{\frac{3}{2p}}\int_0^t (t-s)^{-\frac{3}{2p}}\left\|\mathbb{P}(\vec{f})\right\|_{L^{p,\infty}} ds \leq C t^{\frac{3}{2p}}\left\|\mathbb{P}(\vec{f})\right\|_{L^{p,\infty}}\left(\int_0^t (t-s)^{-\frac{3}{2p}}\, ds\right) \leq C\, t\left\|\mathbb{P}(\vec{f})\right\|_{L^{p,\infty}}. \end{aligned}$$

We thus obtain

$$\sup_{0<t\leq T} t^{\frac{3}{2p}}\left\| \int_0^t e^{(t-s)\Delta}\mathbb{P}(\vec{f})ds\right\|_{L^\infty} \leq CT\left\|\mathbb{P}(\vec{f})\right\|_{L^{p,\infty}}.$$

The other term involving the external force g follows similar estimates and we have

$$\sup_{0<t\leq T} t^{\frac{3}{2p}}\left\| \int_0^t e^{(t-s)\Delta}g ds\right\|_{L^\infty} \leq CT\left\|g\right\|_{L^{p,\infty}}.$$

By the estimates above we get

$$\left\| \int_0^t e^{(t-s)\Delta}\left(\mathbb{P}(\vec{f}), g\right) ds\right\|_{\mathcal{X}} \leq CT\left\|\left(\mathbb{P}(\vec{f}), g\right)\right\|_{L^{p,\infty}}. \tag{65}$$

Thus, the wished estimate (63) follows from the estimates (64) and (65). ■

We study now the bilinear terms in (61) and (62).

Lemma 5.2 *The following estimates hold:*

$$\|\mathcal{B}_1(\vec{v},\vec{v})\|_{\mathcal{X}} \leq C\, T^{\frac{1}{2}-\frac{3}{2p}}\|\vec{v}\|^2_{\mathcal{X}}, \quad \|\mathcal{B}_2(\vartheta,\vec{v})\|_{\mathcal{X}} \leq C\, T^{\frac{1}{2}-\frac{3}{2p}}\|\vartheta\|_{\mathcal{X}}\|\vec{v}\|_{\mathcal{X}}, \tag{66}$$

where $\frac{1}{2}-\frac{3}{2p}>0$ *as long as* $p>3$.

Proof. First, the term $\mathcal{B}_1(\vec{u},\vec{u})$ in (61) is estimated as follows:

$$\sup_{0\leq t\leq T}\|\mathcal{B}_1(\vec{v},\vec{v})\|_{L^{p,\infty}} \leq C\, T^{\frac{1}{2}-\frac{3}{2p}}\,\|\vec{v}\|^2_{E_T}. \tag{67}$$

Indeed, by the second point of Lemma 2.1, by the well-known estimate on the heat kernel: $\|\vec{\nabla} h_{(t-s)}(\cdot)\|_{L^1} \leq \frac{c}{(t-s)^{1/2}}$, and moreover, by Lemma 2.4 we have

$$\begin{aligned} \sup_{0\leq t\leq T}\|\mathcal{B}_1(\vec{v},\vec{v})\|_{L^{p,\infty}} &\leq C\sup_{0\leq t\leq T}\int_0^t \left\| e^{(t-s)\Delta}\mathbb{P}\text{div}(\vec{v}\otimes\vec{v})(s,\cdot)\right\|_{L^{p,\infty}} ds \\ &\leq C\sup_{0\leq t\leq T}\int_0^t \frac{1}{(t-s)^{1/2}}\|\vec{v}(s,\cdot)\otimes\vec{v}(s,\cdot)\|_{L^{p,\infty}} ds \\ &\leq C\sup_{0\leq t\leq T}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}\, s^{\frac{3}{2p}}}(s^{\frac{3}{2p}}\|\vec{v}(s,\cdot)\|_{L^\infty})\|\vec{v}(s,\cdot)\|_{L^{p,\infty}} ds \\ &\leq C\, T^{\frac{1}{2}-\frac{3}{2p}}\,\|\vec{v}\|^2_{\mathcal{X}}. \end{aligned}$$

Thereafter, we have also the estimate

$$\sup_{0<t\leq T} t^{\frac{3}{2p}}\|\mathcal{B}_1(\vec{v},\vec{v})\|_{L^\infty} \leq C\, T^{\frac{1}{2}-\frac{3}{2p}}\,\|\vec{v}\|^2_{\mathcal{X}}. \tag{68}$$

Indeed, we write:

$$\sup_{0<t\leq T} t^{\frac{3}{2p}} \|\mathcal{B}_1(\vec{v},\vec{v})\|_{L^\infty} \leq \sup_{0<t<T} t^{\frac{3}{2p}} \int_0^t \left\| e^{(t-s)\Delta}\mathbb{P}\text{div}\,(\vec{v}\otimes\vec{v})\,(s,\cdot)\right\|_{L^\infty} ds = (a).$$

Here, we recall that the operator $e^{(t-s)\Delta}\mathbb{P}(div(\cdot))$ writes down as a tensor of convolution operators (in the spatial variable) whose kernels $K_{i,j,k}$ verify $|K_{i,j,k}(t-s,x)| \leq \dfrac{c}{((t-s)^{1/2}+|x|)^4}$. See [17, Proposition 11.1]. Then, we have $\|K_{i,j}(t-s,\cdot)\|_{L^1} \leq \dfrac{c}{(t-s)^{1/2}}$; and we can write:

$$\begin{aligned}
(a) &\leq C \sup_{0<t\leq T} t^{\frac{3}{2p}} \int_0^t \frac{1}{(t-s)^{1/2}} \left(\|\vec{v}(s,\cdot)\otimes\vec{v}(s,\cdot)\|_{L^\infty}\right) ds\\
&\leq C \sup_{0<t\leq T} t^{\frac{3}{2p}} \int_0^t \frac{ds}{(t-s)^{1/2}s^{\frac{3}{p}}} \left(s^{\frac{3}{2p}}\|\vec{v}(s,\cdot)\|_{L^\infty}\right)^2 ds\\
&\leq C \left(\sup_{0<t\leq T} t^{\frac{3}{2p}} \int_0^t \frac{ds}{(t-s)^{1/2}s^{\frac{3}{p}}}\right) \|\vec{v}\|^2_{\mathcal{X}}.\\
&\leq C\,T^{\frac{1}{2}-\frac{3}{2p}}\,\|\vec{v}\|^2_{\mathcal{X}}.
\end{aligned}$$

By the estimates (67) and (68) we obtain

$$\|\mathcal{B}_1(\vec{v},\vec{v})\|_{\mathcal{X}} \leq c\,T^{\frac{1}{2}-\frac{3}{2p}}\,\|\vec{v}\|^2_{\mathcal{X}}. \tag{69}$$

We estimate now the term $\mathcal{B}_2(\vartheta,\vec{v})$. By following the same arguments in the prove of the estimate (67) (where we use the information $s^{\frac{3}{2p}}\|\vec{v}(s,\cdot)\|_{L^\infty} < +\infty$ and $\|\vartheta(s,\cdot)\|_{L^{p,\infty}} < +\infty$) we have

$$\sup_{0\leq t\leq T} \|\mathcal{B}_2(\vartheta,\vec{v})\|_{L^{p,\infty}} \leq C\,T^{\frac{1}{2}-\frac{3}{2p}}\,\|\vartheta\|_{\mathcal{X}}\|\vec{v}\|_{\mathcal{X}}. \tag{70}$$

Moreover, by following similar ideas in the proof of the estimate (68) we can prove:

$$\sup_{0<t\leq T} t^{\frac{3}{2p}}\|\mathcal{B}_2(\vartheta,\vec{v})\|_{L^\infty} \leq C\,T^{\frac{1}{2}-\frac{3}{2p}}\|\vartheta\|_{\mathcal{X}}\|\vec{v}\|_{\mathcal{X}}. \tag{71}$$

Indeed, we just write

$$\sup_{0<t\leq T} t^{\frac{3}{2p}}\|\mathcal{B}_2(\vartheta,\vec{v})\|_{L^\infty} \leq C\left(\sup_{0<t\leq T} t^{\frac{3}{2p}} \int_0^t \frac{ds}{(t-s)^{1/2}s^{\frac{3}{2p}+\frac{3}{2p}}}\right)\|\vartheta\|_{\mathcal{X}}\|\vec{v}\|_{\mathcal{X}} \leq C\,T^{\frac{1}{2}-\frac{3}{2p}}\,\|\vartheta\|_{\mathcal{X}}\|\vec{v}\|_{\mathcal{X}}.$$

By the estimates (70) and (71) we get

$$\|\mathcal{B}_2(\vartheta,\vec{v})\|_{\mathcal{X}} \leq C\,T^{\frac{1}{2}-\frac{3}{2p}}\,\|\vartheta\|_{\mathcal{X}}\|\vec{v}\|_{\mathcal{X}}. \tag{72}$$

To finish the proof, we gather the estimates (69) and (72) to obtain the wished estimate (66). ■

Finally, we must study the linear term $\mathcal{L}(\vartheta)$ in the equation (61). Recall that by the assumption (7) we have $\vec{\mathbf{g}} \in L^\infty(\mathbb{R}^3)$.

Lemma 5.3 *The following estimate holds:*

$$\|\mathcal{L}(\vartheta)\|_{\mathcal{X}} \leq C\|\vec{\mathbf{g}}\|_{L^\infty}T\,\|\vartheta\|_{\mathcal{X}}. \tag{73}$$

Proof. We start with the following estimate, where by the second point of Lemma 2.1 and the Hölder inequalities we write

$$\sup_{0\leq t\leq T}\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}(\vartheta\vec{\mathbf{g}})(s,\cdot)ds\right\|_{L^{p,\infty}} \leq \sup_{0\leq t\leq T}\int_0^t \left\|e^{(t-s)\Delta}\mathbb{P}(\vartheta\vec{\mathbf{g}})(s,\cdot)\right\|_{L^{p,\infty}} ds$$
$$\leq C \sup_{0\leq t\leq T}\int_0^t \|\vartheta\vec{\mathbf{g}}(s,\cdot)\|_{L^{p,\infty}} ds \leq C\|\vec{\mathbf{g}}\|_{L^\infty} T\|\vartheta\|_{\mathcal{X}}.$$

Moreover, by the third point of Lemma 2.1 and by using again the Hölder inequalities, the next estimate holds

$$\sup_{0<t\leq T} t^{\frac{3}{2p}}\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}(\theta\vec{\mathbf{g}})(s,\cdot)ds\right\|_{L^\infty} \leq C \sup_{0<t\leq T} t^{\frac{3}{2p}}\int_0^t (t-s)^{-\frac{3}{2p}}\|\mathbb{P}(\vartheta\vec{\mathbf{g}})(s,\cdot)\|_{L^{p,\infty}} ds$$
$$\leq C\left(\sup_{0<t\leq T} t^{\frac{3}{2p}}\int_0^t (t-s)^{-\frac{3}{2p}} ds\right)\|\vartheta\vec{\mathbf{g}}\|_{\mathcal{X}} \leq C\|\vec{\mathbf{g}}\|_{L^\infty} T\|\vartheta\|_{\mathcal{X}}.$$

We thus obtain the wished estimate (73). ∎

With the Lemmas 5.1, 5.2 and 5.3 at our disposal, we set a time $T_0>0$ small enough and Proposition 5.1 follows from standard arguments. ∎

Step 2. Global boundness of $\vec{u}$ and θ. Now, we get back to steady-state system (1). With the help of Proposition 5.1, we are able to prove the following:

Proposition 5.2 *Let $(\vec{u},P,\theta)$ be a weak solution of the system (1) given in Definition 1.1. If $\vec{u}\in L^{p,\infty}(\mathbb{R}^3)$ and $\theta\in L^{p,\infty}(\mathbb{R}^3)$ with $p>3$, then we have $\vec{u}\in L^\infty(\mathbb{R}^3)$ and $\theta\in L^\infty(\mathbb{R}^3)$.*

Proof. In the initial value problem (59), we set the initial data $(\vec{v}_0,\vartheta_0)=(\vec{u},\theta)$. Then, by Proposition 5.1 there exists a time $0<T_0$ and there exists a unique arising solution $(\vec{v},\vartheta)\in\mathcal{C}_*([0,T_0],L^{p,\infty}(\mathbb{R}^3))$ to the equation (59).

On the other hand, we have the following key remark. Since $\vec{u}$ and θ are time-independent functions we have $\partial_t\vec{u}=0$ and $\partial_t\theta=0$. Thus, the couple $(\vec{u},\theta)$ is also a solution of the initial value problem (59) with the same initial data $(\vec{u},\theta)$, and moreover we have $(\vec{u},\theta)\in\mathcal{C}_*([0,T_0],L^{p,\infty}(\mathbb{R}^3))$.

Consequently, in the space $\mathcal{C}_*([0,T_0],L^{p,\infty}(\mathbb{R}^3))$ we have two solutions of (59) with the same initial data: on the one hand, the solution $(\vec{v},\vartheta)$ given by Proposition 5.1 and, on the other hand, the (steady-sate) solution $(\vec{u},\theta)$. By uniqueness we have the identity $(\vec{v},\vartheta)=(\vec{u},\theta)$ and by (60) we can write

$$\sup_{0<t\leq T_0} t^{\frac{3}{2p}}\|\vec{u}\|_{L^\infty}+\sup_{0<t\leq T_0} t^{\frac{3}{2p}}\|\theta\|_{L^\infty}<+\infty.$$

But, as the solution $(\vec{u},\theta)$ does not depend on the time variable we have $\vec{u}\in L^\infty(\mathbb{R}^3)$ and $\theta\in L^\infty(\mathbb{R}^3)$. Proposition 5.2 is now proven. ∎

Step 3. Estimates on high order derivatives in Lorentz spaces. The global boundness of $\vec{u}$ and θ yields the following:

Proposition 5.3 *For $k\geq 0$ assume that $\vec{f},g$ verify (6) and $\vec{\mathbf{g}}$ verifies (7). Moreover, assume that $\vec{u}\in L^{p,\infty}(\mathbb{R}^3)$ and $\theta\in L^{p,\infty}(\mathbb{R}^3)$ with $p>3$. Then we have $\vec{u}\in\mathcal{W}^{k+2,p}(\mathbb{R}^3)$, $\theta\in\mathcal{W}^{k+2,p}(\mathbb{R}^3)$ and $P\in\mathcal{W}^{k+1,p}(\mathbb{R}^3)$.*

Proof. We will use the integral formulations (11) and (12) to prove that for all multi-indice $|\alpha|\leq k+2$ and for all $1\leq\sigma<+\infty$ we have $\partial^\alpha\vec{u}\in L^{p\sigma,\infty}(\mathbb{R}^3)$ and $\partial^\alpha\theta\in L^{p\sigma,\infty}(\mathbb{R}^3)$. We shall prove this fact by an iteration process respect to the order of the multi-indices α, which we will denote as $|\alpha|$. For the sake of clearness, in the following couple of technical lemmas we separately prove each step in the iterative argument.

Lemma 5.4 (The case initial case) *Recall that by Proposition 5.2 we have* $\vec{u}, \theta \in L^\infty(\mathbb{R}^3)$. *Then, for* $|\alpha| \leq 2$ *and for all* $1 \leq \sigma < +\infty$ *we have* $\partial^\alpha \vec{u} \in L^{p\sigma,\infty}(\mathbb{R}^3)$ *and* $\partial^\alpha \theta \in L^{q\sigma,\infty}(\mathbb{R}^3)$.

Proof. Due to the coupled structure of the equations (11)-(12), we must study first the equation (12) and then we study the equation (11).

- Let $|\alpha| = 1$. By the equation (12) we write

$$\partial^\alpha \theta = -\frac{\partial^\alpha}{-\Delta}(\text{div}(\theta\vec{u})) + \frac{\partial^\alpha}{-\Delta}(g). \tag{74}$$

Then, remark that since $\theta \in L^{p,\infty}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ by the first point of Lemma 2.3 we have $\theta \in L^{p\sigma,\infty}(\mathbb{R}^3)$ for all $1 \leq \sigma < \infty$. Moreover, since $\vec{u} \in L^\infty(\mathbb{R}^3)$ we have $\theta\vec{u} \in L^{p\sigma,\infty}(\mathbb{R}^3)$. With this information, for the first term on the right-hand side, remark that since $|\alpha| = 1$ the operator $-\frac{\partial^\alpha}{-\Delta}(\text{div}(\cdot))$ writes down as a combination of the Riesz Transforms $\mathcal{R}_i\mathcal{R}_j$ (with $i, j = 1, 2, 3$). Thus, as we have $\theta\vec{u} \in L^{p\sigma,\infty}(\mathbb{R}^3)$, by Lemma 2.5 we obtain $-\frac{\partial^\alpha}{-\Delta}(\text{div}(\theta\vec{u})) \in L^{p\sigma,\infty}(\mathbb{R}^3)$.

On the other hand, for the second term on the right-hand side, recall that the operator $\frac{\partial^\alpha}{-\Delta}(\cdot)$ writes down as a product of convolution with the tensor $m_2 \in L^{3/2,\infty}(\mathbb{R}^3)$. Moreover, recall that by the assumption (6) we have $g \in L^{\frac{3p}{3+p},\infty}(\mathbb{R}^3)$. We thus apply the first point of Lemma 2.1 (with $1 + 1/p = 2/3 + (3+p)/3p$) to obtain $\frac{\partial^\alpha}{-\Delta}(g) \in L^{p,\infty}(\mathbb{R}^3)$. Thereafter, also by (6) and by the second point of Lemma 2.3 we have $g \in L^{3,1}(\mathbb{R}^3)$ (remark that $\frac{3p}{3+p} < 3$) and by the third point of Lemma 2.1 we get $\frac{\partial^\alpha}{-\Delta}(g) \in L^\infty(\mathbb{R}^3)$. Finally, by the first point of Lemma 2.3 we have $\frac{\partial^\alpha}{-\Delta}(g) \in L^{p\sigma,\infty}(\mathbb{R}^3)$. Consequently, we get $\partial^\alpha\theta \in L^{p\sigma,\infty}(\mathbb{R}^3)$ for all $1 \leq \sigma < +\infty$.

With this information, we can study now the equation (11). As before, we write

$$\partial^\alpha \vec{u} = -\frac{\partial^\alpha}{-\Delta}\left(\mathbb{P}(\text{div}(\vec{u} \otimes \vec{u}))\right) + \frac{\partial^\alpha}{-\Delta}\Big(\mathbb{P}\left(\theta\vec{\mathbf{g}}\right)\Big) + \frac{\partial^\alpha}{-\Delta}\Big(\mathbb{P}(\vec{f})\Big), \tag{75}$$

and we shall prove that each term on right-hand side belongs to the space $L^{p\sigma,\infty}(\mathbb{R}^3)$ for all $1 \leq \sigma < +\infty$. Indeed, for the first term we recall that since $\vec{u} \in L^{p,\infty}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ by Lemma 2.3 we have $\vec{u} \in L^{p\sigma,\infty}(\mathbb{R}^3)$; and consequently $\vec{u} \otimes \vec{u} \in L^{p\sigma,\infty}(\mathbb{R}^3)$. Then, we follow the same ideas above to conclude that $-\frac{\partial^\alpha}{-\Delta}\left(\mathbb{P}(\text{div}(\vec{u} \otimes \vec{u}))\right) \in L^{p\sigma,\infty}(\mathbb{R}^3)$.

For the second term, recall that the operator $\frac{\partial^\alpha}{-\Delta}\mathbb{P}(\cdot)$ writes down as a product of convolution with the tensor $m_1 \in L^{3/2,\infty}(\mathbb{R}^3)$. Then, by the first point of Lemma 2.1 we write $\left\| \frac{\partial^\alpha}{-\Delta}\mathbb{P}(\theta\vec{\mathbf{g}}) \right\|_{L^{p,\infty}} \leq C_1 \|m_1\|_{L^{3/2,\infty}} \|\theta\vec{\mathbf{g}}\|_{L^{\frac{3p}{3+p},\infty}}$. Then, by the assumption (7) and the second point of Lemma 2.2 we get $\|m_1\|_{L^{3/2,\infty}} \|\theta\vec{\mathbf{g}}\|_{L^{\frac{3p}{3+p},\infty}} \leq C\|\theta\|_{L^{p,\infty}} \|\vec{\mathbf{g}}\|_{L^{3,\infty}}$. We thus get $\frac{\partial^\alpha}{-\Delta}\mathbb{P}(\theta\vec{\mathbf{g}}) \in L^{p,\infty}(\mathbb{R}^3)$. On the other hand, by the third point of Lemma 2.1 we can write $\|\frac{\partial^\alpha}{-\Delta}\mathbb{P}(\theta\vec{\mathbf{g}})\|_{L^\infty} \leq C_3\|m_1\|_{L^{3/2,\infty}} \|\theta\vec{\mathbf{g}}\|_{L^{3,1}} \leq \|\theta\|_{L^\infty} \|\vec{\mathbf{g}}\|_{L^{3,1}}$. We thus get $\frac{\partial^\alpha}{-\Delta}\mathbb{P}(\theta\vec{\mathbf{g}}) \in L^\infty(\mathbb{R}^3)$; and consequently we have $\frac{\partial^\alpha}{-\Delta}\mathbb{P}(\theta\vec{\mathbf{g}}) \in L^{p\sigma,\infty}(\mathbb{R}^3)$.

The first term is similarly estimated as the term $\frac{\partial^\alpha}{-\Delta}(g)$ and we have $\frac{\partial^\alpha}{-\Delta}\Big(\mathbb{P}(\vec{f})\Big) \in L^{p\sigma,\infty}(\mathbb{R}^3)$. We thus obtain $\partial^\alpha\vec{u} \in L^{p\sigma,\infty}(\mathbb{R}^3)$ for all $1 \leq \sigma < +\infty$.

- Let $|\alpha|=2$. We get back to the expression $\partial^\alpha\theta$ defined in (74). For the first term on right-hand side we write
$$-\frac{\partial^\alpha}{-\Delta}(\text{div}(\theta\vec{u}))=-\frac{\partial^{\alpha_1}}{-\Delta}(\text{div}(\partial^{\alpha_2}(\theta\vec{u}))),\quad \text{where}\;\; \alpha=\alpha_1+\alpha_2,\;\; |\alpha_1|=|\alpha_2|=1.$$
Here, we shall verify that $\partial^{\alpha_2}(\theta\vec{u})\in L^{p\sigma,\infty}(\mathbb{R}^3)$. Indeed, for $i=1,2,3$ we write $\partial^{\alpha_2}(\theta u_i)=(\partial^{\alpha_2}\theta)u_i+\theta(\partial^{\alpha_2}u_i)$ and since we get $\partial^{\alpha_2}\theta, \partial^{\alpha_2}\vec{u}\in L^{p\sigma,\in}$, $\vec{u},\theta\in L^\infty(\mathbb{R}^3)$ we directly obtain $\partial^{\alpha_2}(\theta\vec{u})\in L^{p\sigma,\infty}(\mathbb{R}^3)$. Thereafter, recall that the operator $-\frac{\partial^{\alpha_1}}{-\Delta}(\text{div}(\cdot))$ writes down as a combination of the Riesz transforms $\mathcal{R}_i\mathcal{R}_j$ and by applying the Lemma 2.5 we have $-\frac{\partial^\alpha}{-\Delta}(\text{div}(\theta\vec{u}))\in L^{p\sigma,\infty}(\mathbb{R}^3)$. The second term on the right-hand side of the expression (74) also belongs to the space $L^{p\sigma,\infty}(\mathbb{R}^3)$ due to the hypothesis (6).

  We study now the expression $\partial^\alpha\vec{u}$ defined in (75). The first term on the right-hand side follows the same arguments above and we have $-\frac{\partial^\alpha}{-\Delta}\left(\mathbb{P}(\text{div}(\vec{u}\otimes\vec{u}))\right)\in L^{p\sigma,\infty}(\mathbb{R}^3)$. To study the second term on the right-hand side, we also remark that in this case the operator $\frac{\partial^\alpha}{-\Delta}(\mathbb{P}(\cdot))$ writes down as a linear combination of the Riesz transforms, and moreover, since $\vec{\mathbf{g}}\in L^\infty(\mathbb{R}^3)$ we have $\frac{\partial^\alpha}{-\Delta}(\mathbb{P}(\theta\vec{\mathbf{g}}))\in L^{p\sigma,\infty}(\mathbb{R}^3)$. Finally, always by the hypothesis (6) we have $\frac{\partial^\alpha}{-\Delta}\Big(\mathbb{P}(\vec{f})\Big)\in L^{p\sigma,\infty}(\mathbb{R}^3)$. ■

Lemma 5.5 (The iterative process) *For all $1\le m\le k$ (with $k\ge 2$) and for all multi-indice $|\alpha|\le k$, assume that $\partial^\alpha\vec{u}\in L^{p\sigma,\infty}(\mathbb{R}^3)$ and $\partial^\alpha\theta\in L^{p\sigma,\infty}(\mathbb{R}^3)$, for all $1\le\sigma<+\infty$. Then it holds true for all multi-indice $|\alpha|\le k+2$.*

Proof. We shall similar ideas in the proof of the previous lemma.

- Let $|\alpha|=k+1$. We start by studying the expression $\partial^\alpha\theta$ defined in (74). For the first term on the right-hand side we write
$$-\frac{\partial^\alpha}{-\Delta}(\text{div}(\theta\vec{u}))=-\frac{\partial^{\alpha_1}}{-\Delta}(\text{div}(\partial^{\alpha_2}(\theta\vec{u}))),\quad \alpha=\alpha_1+\alpha_2,\;\; |\alpha_1|=1,\;\; |\alpha_2|=k.$$
In this last expression, we must verify that $\partial^{\alpha_2}(\theta\vec{u})\in L^{p\sigma,\infty}(\mathbb{R}^3)$. Indeed, for $i=1,2,3$ by the Leibniz rule we write $\partial^{\alpha_2}(\theta u_i)=\sum\limits_{|\beta|\le k}c_{\alpha_2,\beta}\partial^\beta\theta\,\partial^{\alpha_2-\beta}u_i$, where the constant $c_{\alpha_2,\beta}>0$ depends on the multi-indices α_2 and β. Recall that by hypothesis we have $\partial^\beta\theta$, $\partial^{\alpha_2-\beta}u_i\in L^{p\sigma,\infty}(\mathbb{R}^3)$ and by the Hölder inequalities we get $\partial^\beta\theta\,\partial^{\alpha_2-\beta}u_i\in L^{p\sigma,\infty}(\mathbb{R}^3)$. We thus have $\partial^{\alpha_2}(\theta\vec{u})\in L^{p\sigma,\infty}(\mathbb{R}^3)$ and consequently $-\frac{\partial^\alpha}{-\Delta}(\text{div}(\theta\vec{u}))\in L^{p\sigma,\infty}(\mathbb{R}^3)$. For the second term on the right-hand side, we write
$$\frac{\partial^\alpha}{-\Delta}(g)=\frac{\partial^{\alpha_1}}{-\Delta}(\partial^{\alpha_2}g),\quad \text{where}\;\; \alpha=\alpha_1+\alpha_2,\;\; |\alpha_1|=2,\;\; |\alpha_2|=k-1,$$
and by the assumption (6) we get $\frac{\partial^\alpha}{-\Delta}(g)\in L^{p\sigma,\infty}(\mathbb{R}^3)$.

  Once we have $\partial^\alpha\theta\in L^{p\sigma,\infty}(\mathbb{R}^3)$ for all $|\alpha|\le k+1$, by the expression (75) and by following the same arguments above we obtain $\partial^\alpha\vec{u}\in L^{p\sigma,\infty}(\mathbb{R}^3)$ for all $|\alpha|\le k+1$. We just mention that to treat the term $\frac{\partial^\alpha}{-\Delta}\Big(\mathbb{P}\left(\theta\vec{\mathbf{g}}\right)\Big)$ we split
$$\frac{\partial^\alpha}{-\Delta}\Big(\mathbb{P}\left(\theta\vec{\mathbf{g}}\right)\Big)=\frac{\partial^\alpha_1}{-\Delta}\Big(\mathbb{P}\left(\partial^{\alpha_2}(\theta\vec{\mathbf{g}})\right)\Big),\;\; |\alpha_1|=2,\;\; |\alpha_2|=k-1,$$
where we use again the Leibniz rule to study the term $\partial^{\alpha_2}(\theta\vec{\mathbf{g}})$.

- Let $|\alpha| = k+2$. With the information $\partial^\alpha \theta \in L^{p\sigma,\infty}(\mathbb{R}^3)$ and $\partial^\alpha \vec{u} \in L^{p\sigma,\infty}(\mathbb{R}^3)$ (for all $|\alpha| \leq k+1$) at our disposal, we just repeat again the arguments above to get $\partial^\alpha \theta \in L^{p\sigma,\infty}(\mathbb{R}^3)$ and $\partial^\alpha \vec{u} \in L^{p\sigma,\infty}(\mathbb{R}^3)$ for $|\alpha| = k+2$. ■

Remark 6 *By the hypothesis (6) on the external forces $\vec{f}$ and g, and moreover, by the terms $\frac{\partial^\alpha}{-\Delta}\mathbb{P}(\vec{f})$ and $\frac{\partial^\alpha}{-\Delta}(g)$ in the expressions (75) and (74) respectively, we get that $k+2$ is the maximum gain of regularity expected for $\partial^\alpha \vec{u}$ and $\partial^\alpha \theta$.*

It remains to study the pressure term P in the first equation of the coupled system (1). To do this, we apply the divergence operator to this equations to obtain that P is related to the velocity $\vec{u}$, the temperature θ and the force $\vec{f}$ through the expression

$$P = \frac{1}{-\Delta}\text{div}(\text{div}(\vec{u} \otimes \vec{u})) - \frac{1}{-\Delta}(\text{div}(\theta \vec{\mathbf{g}})) - \frac{1}{-\Delta}(\text{div}(\vec{f})). \tag{76}$$

By following the same arguments in the proof the previous lemmas, we can prove the next one:

Lemma 5.6 (The pressure term) *For all multi-indice $|\alpha| \leq k+1$ we have $\partial^\alpha P \in L^{p\sigma,\infty}(\mathbb{R}^3)$.*

Proof. Let $|\alpha| \leq k+1$. By the expression (76) we write

$$\partial^\alpha P = \frac{\partial^\alpha}{-\Delta}\text{div}(\text{div}(\vec{u} \otimes \vec{u})) - \frac{\partial^\alpha}{-\Delta}(\text{div}(\theta \vec{\mathbf{g}})) - \frac{\partial^\alpha}{-\Delta}(\text{div}(\vec{f})), \tag{77}$$

where we must study each term on the right-hand side. For the first term, we recall that $\partial^\alpha \vec{u} \in L^{p\sigma,\infty}(\mathbb{R}^3)$ for all $1 \leq \sigma < +\infty$. Then, we can use the Hölder inequalities as well as the Leibniz rule to obtain $\frac{\partial^\alpha}{-\Delta}\text{div}(\text{div}(\vec{u} \otimes \vec{u})) = \frac{1}{-\Delta}\text{div}(\text{div}(\partial^\alpha(\vec{u} \otimes \vec{u}))) \in L^{p\sigma,\infty}(\mathbb{R}^3)$. The second term is similarly treated by using now the information $\partial^\alpha \theta \in L^{p\sigma,\infty}(\mathbb{R}^3)$ and the assumption (7). Finally, always by the hypothesis (6) the third term verifies $\frac{\partial^\alpha}{-\Delta}(\text{div}(\text{div}(\mathbb{F}))) \in L^{p\sigma,\infty}(\mathbb{R}^3)$. ■

Now, we are able to finish the proof of Theorem 1.1. For this we recall the Lorentz space $L^{p,\infty}(\mathbb{R}^3)$ embeds in the Morrey space $\dot{M}^{1,p}(\mathbb{R}^3)$ defined by the expression (9). Consequently, for all multi-indice $|\alpha| \leq k+1$, by Lemma 2.6 the functions $\partial^\alpha \vec{u}$ and $\partial^\alpha \theta$ are Hölder continuous with parameter $s = 1 - 3/p$. Moreover, for $|\alpha| \leq k$ this fact also holds true for the function $\partial^\alpha P$. Theorem 1.2 is now proven. ■

6 Proof of Theorem 1.3

This result is based on the following Caccioppoli-type estimate for the temperature θ. For $R > 0$ we shall denote $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ and $C(R/2, R) = \{x \in \mathbb{R}^3 : R/2 < |x| < R\}$.

Proposition 6.1 *Let $(\vec{u}, P, \theta) \in \mathcal{C}^2_{loc}(\mathbb{R}^3) \times \mathcal{C}^1_{loc}(\mathbb{R}^3) \times \mathcal{C}^2_{loc}(\mathbb{R}^3)$ be a smooth solution of the coupled system (8). Let $p > 3$. There exists a constant $C > 0$ such that for all $R \geq 1$ the following estimate hold:*

$$\int_{B_{R/2}} |\vec{\nabla}\theta|^2 dx \leq C \left(\int_{C(R/2,R)} |\theta|^p dx \right)^{2/p} \left[R^{1-\frac{6}{p}} + R^{2-\frac{9}{p}} \left(\int_{C(R/2,R)} |\vec{u}|^p dx \right)^{1/p} \right]. \tag{78}$$

Proof. To prove the estimate (78), we introduce the following cut-off function: let $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^3)$ be a positive and radial function verifying $\varphi(x) = 1$ when $|x| < 1/2$ and $\varphi(x) = 0$ when $|x| \geq 1$. Then, for $R \geq 1$ we define $\varphi_R(x) = \varphi(x/R)$. Consequently, the cut-off function φ_R satisfies the next useful properties: $\varphi_R(x) = 1$ when $|x| < R/2$, $\varphi_2(x) = 0$ when $|x| \geq R$, and moreover, for all multi-indice α we have $\text{supp}(\partial^\alpha \varphi_R) \subset C(R/2, R)$.

Now, we multiply each term in the second equation in (8) by $\varphi_R\theta$; and we integrate on the ball B_R (remark that $\text{supp}(\varphi_R\theta)\subset B_R$) to get

$$\int_{B_R}(-\Delta\theta)\varphi_R\theta dx+\int_{B_R}\text{div}(\theta\vec{u})\varphi_R\theta dx=0. \tag{79}$$

Here it is worth highlighting each term above is well-defined since we have $\vec{u},\theta\in\mathcal{C}^2_{loc}(\mathbb{R}^3)$. To study the first term we integrate by parts to write

$$\begin{aligned}&\int_{B_R}(-\Delta\theta)\varphi_R\theta dx=-\sum_{i=1}^{3}\int_{B_R}(\partial_j^2\theta)(\varphi_R\theta)dx=\sum_{i,j=1}^{3}\int_{B_R}\partial_i\theta\partial_i(\varphi_R\theta)dx\\&=\sum_{i=1}^{3}\int_{B_R}(\partial_i\theta)(\partial_i\varphi_R)\theta dx+\sum_{i=1}^{3}\int_{B_R}(\partial_i\theta)\varphi_R(\partial_i\theta)dx=\sum_{i=1}^{3}\int_{B_R}(\partial_i\varphi_R)(\partial_i\theta)\theta dx+\sum_{i=1}^{3}\int_{B_R}\varphi_R(\partial_i\theta)^2dx\\&=\sum_{i=1}^{3}\int_{B_R}(\partial_i\varphi_R)\partial_i\left(\frac{\theta^2}{2}\right)dx+\int_{B_R}\varphi_R|\vec{\nabla}\theta|^2dx=-\int_{B_R}\Delta\varphi_R\left(\frac{\theta^2}{2}\right)dx+\int_{B_R}\varphi_R|\vec{\nabla}\theta|^2dx.\end{aligned}\tag{80}$$

Thereafter, to study the second term, we remark first that since $\text{div}(\vec{u})=0$ we have $\text{div}(\theta\vec{u})=\vec{u}\cdot\vec{\nabla}\theta$. Then, we use again the integration by parts to get

$$\int_{B_R}\vec{u}\cdot\vec{\nabla}\theta\cdot(\varphi_R\theta)dx=\sum_{i=1}^{3}\int_{B_R}u_i(\partial_i\theta)(\varphi_R\theta)dx=\sum_{i=1}^{3}\int_{B_R}\varphi_Ru_i(\partial_i\theta)\theta dx=\sum_{i=1}^{3}\int_{B_R}\varphi_Ru_i\,\partial_i\left(\frac{\theta^2}{2}\right)dx,$$

but, always by the fact that $div(\vec{u})=0$, we can write

$$\sum_{i=1}^{3}\int_{B_R}\varphi_Ru_i\,\partial_i\left(\frac{\theta^2}{2}\right)dx=\sum_{i=1}^{3}\int_{B_R}\varphi_R\partial_i\left(u_i\frac{\theta^2}{2}\right)dx=-\int_{B_R}\vec{\nabla}\varphi_R\cdot\left(\frac{\theta^2}{2}\vec{u}\right)dx. \tag{81}$$

With the identities (80) and (81) at our disposal, we get back to the identity (79) and we obtain

$$\int_{B_R}\varphi_R|\vec{\nabla}\theta|^2dx=\int_{B_R}\Delta\varphi_R\left(\frac{\theta^2}{2}\right)dx+\int_{B_R}\vec{\nabla}\varphi_R\cdot\left(\frac{\theta^2}{2}\vec{u}\right)dx.$$

Moreover, since $\varphi_R(x)=1$ in the ball $B_{R/2}$, $\text{supp}(\vec{\nabla}\varphi_R)\subset C(R/2,R)$ and $\text{supp}(\Delta\varphi_R)\subset C(R/2,R)$; we have

$$\int_{B_{R/2}}|\vec{\nabla}\theta|^2dx\leq\int_{C(R/2,R)}\Delta\varphi_R\left(\frac{\theta^2}{2}\right)dx+\int_{C(R/2,R)}\vec{\nabla}\varphi_R\cdot\left(\frac{\theta^2}{2}\vec{u}\right)dx. \tag{82}$$

In this estimate, we shall study each term on the right-hand side. For the first term, we use the Hölder inequalities with $\frac{1}{q}+\frac{2}{p}=1$ (hence we have $\frac{3}{q}-2=1-\frac{6}{p}$) and the fact that $\varphi_R(x)=\varphi(x/R)$ to write

$$\begin{aligned}\int_{C(R/2,R)}\Delta\varphi_R\left(\frac{\theta^2}{2}\right)dx\leq&C\left(\int_{C(R/2,R)}|\Delta\varphi_R|^qdx\right)^{1/q}\left(\int_{C(R/2,R)}|\theta|^pdx\right)^{2/p}\\\leq&C\,R^{\frac{3}{q}-2}\|\Delta\varphi\|_{L^q(C(1/2,1))}\left(\int_{C(R/2,R)}|\theta|^pdx\right)^{2/p}\\\leq&C\,R^{1-\frac{6}{p}}\left(\int_{C(R/2,R)}|\theta|^pdx\right)^{2/p}.\end{aligned}\tag{83}$$

For the second term, we use again the Hölder inequalities with $\frac{1}{r}+\frac{3}{p}=1$ (hence we have $\frac{3}{r}-1=2-\frac{9}{p}$) and with $\frac{3}{p}=\frac{2}{p}+\frac{1}{p}$ to write

$$\begin{aligned}\int_{C(R/2,R)} \vec{\nabla}\varphi_R\cdot\left(\frac{\theta^2}{2}\vec{u}\right)dx \leq & C\left(\int_{C(R/2,R)}|\vec{\nabla}\varphi_R|^r dx\right)^{1/r}\left(\int_{C(R/2,R)}|\theta^2\,\vec{u}|^{p/3}dx\right)^{3/p}\\ \leq & C\,R^{\frac{3}{r}-1}\|\vec{\nabla}\varphi\|_{L^r(C(1/2,1))}\left(\int_{C(R/2,R)}|\theta|^p dx\right)^{2/p}\left(\int_{C(R/2,R)}|\vec{u}|^p dx\right)^{1/p} \\ \leq & C\,R^{2-\frac{9}{p}}\left(\int_{C(R/2,R)}|\theta|^p dx\right)^{2/p}\left(\int_{C(R/2,R)}|\vec{u}|^p dx\right)^{1/p}.\end{aligned} \tag{84}$$

Thus, the wished estimate (78) follows from the estimates (82), (83) and (84). Proposition 6.1 is proven. ■

With the Proposition 6.1 at our disposal, we are able to proof the Theorem 1.3. The main idea is to use this Caccioppoli-type estimate, together with the information below on $\vec{u}$ and θ, to prove first that $\theta=0$. This identity then yields $\vec{u}=0$ and $P=0$.

6.1 The case $\vec{u}\in L^{\frac{9}{2},q}(\mathbb{R}^3)$, $\theta\in L^{\frac{9}{2},q}(\mathbb{R}^3)$ with $1\leq q<+\infty$ and $\vec{\mathbf{g}}\in\mathcal{W}^{1,9/2}\cap W^{1,\infty}(\mathbb{R}^3)$.

With this information on $\vec{u}$, θ and $\vec{\mathbf{g}}$, by Theorem 1.2 we have $(\vec{u},P,\theta)\in\mathcal{C}^2(\mathbb{R}^3)\times\mathcal{C}^1(\mathbb{R}^3)\times\mathcal{C}^2(\mathbb{R}^3)$. Then, we set $p=\frac{9}{2}$ and by Proposition 6.1 for all $R\geq 1$ we have the estimate

$$\int_{B_{R/2}}|\vec{\nabla}\theta|^2 dx\leq c\left(\int_{C(R/2,R)}|\theta|^{\frac{9}{2}}dx\right)^{\frac{4}{9}}\left[R^{-\frac{1}{3}}+\left(\int_{C(R/2,R)}|\vec{u}|^{\frac{9}{2}}dx\right)^{\frac{2}{9}}\right]. \tag{85}$$

At this point, we recall the following estimate (see [7, Proposition 1.110] for a proof):

$$\int_{B_R}|f|^{\frac{9}{2}}dx\leq c\,R^{3(1-9/2q)}\|f\|^{\frac{9}{2}}_{L^{9/2,\infty}}\leq c\,R^{3(1-9/2q)}\|f\|^{\frac{9}{2}}_{L^{9/2,q}}. \tag{86}$$

We thus set $f=\mathbb{1}_{C(R/2,R)}\theta$ and $f=\mathbb{1}_{C(R/2,R)}\vec{u}$; and we get back to the previous estimate to write

$$\int_{B_{R/2}}|\vec{\nabla}\theta|^2 dx\leq c\|\mathbb{1}_{C(R/2,R)}\theta\|^2_{L^{9/2,q}}\left[1+\|\vec{u}\|_{L^{9/2,q}}\right]. \tag{87}$$

Thereafter, since $\theta\in L^{9/2,q}(\mathbb{R}^3)$ with $1\leq q<+\infty$ we can apply the dominated convergence theorem in the Lorentz spaces (see [7, Theorem 1.2.8]) to obtain that $\displaystyle\lim_{R\to+\infty}\|\mathbb{1}_{C(R/2,R)}\theta\|^2_{L^{9/2,q}}=0$. Consequently, in the last estimate we take the limit when $R\to+\infty$ to conclude that $\|\vec{\nabla}\theta\|_{L^2}=0$. From this identity and the fact that $\theta\in L^{9/2,q}(\mathbb{R}^3)$ we finally get the identity $\theta=0$.

Once we have $\theta=0$, the coupled system (59) reduces to the stationary Navier-Stokes equation

$$-\Delta\vec{u}+\text{div}(\vec{u}\otimes\vec{u})+\vec{\nabla}P=0,\quad \text{div}(\vec{u})=0. \tag{88}$$

Since $\vec{u}\in L^{9/2,q}(\mathbb{R}^3)$ (with $9/2<q<+\infty$) is a smooth solution by [13, Theorem 1] we obtain the identities $\vec{u}=0$ and $P=0$.

6.2 The case $\vec{u}\in L^{p,\infty}(\mathbb{R}^3)$, $\theta\in L^{p,\infty}(\mathbb{R}^3)$ and $\vec{\mathbf{g}}\in\mathcal{W}^{1,p}\cap W^{1,\infty}(\mathbb{R}^3)$ with $3<p<\frac{9}{2}$.

With this information on $\vec{u}$ and θ; and by Proposition 5.2 we have $\vec{u},\theta\in L^\infty(\mathbb{R}^3)$. Then, by Lemma 2.3 and by the well-known embedding of the Lebesgue spaces into the Lorentz spaces we obtain $\vec{u},\theta\in L^{9/2,q}(\mathbb{R}^3)$ with $9/2<q<+\infty$. Thereafter, by the point 6.1 above we con directly conclude the identities $(\vec{u},P,\theta)=(0,0,0)$.

Theorem 1.3 is proven. ■

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