From non-local to local Navier-Stokes equations

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September 26, 2023

Abstract

Inspired by some experimental (numerical) works on fractional diffusion PDEs, we develop a rigorous framework to prove that solutions to the fractional Navier-Stokes equations, which involve the fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$, converge to a solution of the classical case, with the classical Laplacian operator, when α goes to 2. Precisely, in the setting of mild solutions, we prove a uniform convergence in both the time and spatial variables and derive a convergence rate.

Keywords: Navier-Stokes equations; Fractional Laplacian operator; Mild solutions; Non-local to local convergence.

AMS Classification: 35B40, 35B30.

1 Introduction

In this note, for a velocity field $\vec{u} : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$, and for a pressure term $p : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$, we consider the three-dimensional and incompressible Navier-Stokes equations, in the whole space $\mathbb{R}^3$, and with *two different cases* in the diffusion term:

$$\partial_t \vec{u} = -(-\Delta)^{\alpha/2}\vec{u} - (\vec{u}\cdot\vec{\nabla})\vec{u} - \vec{\nabla}p, \quad \text{div}(\vec{u}) = 0, \qquad 1 < \alpha \leq 2. \tag{1}$$

When $1 < \alpha < 2$, the diffusion term is given by the fractional Laplacian operator, which is easily defined in the Fourier level by the symbol $|\xi|^\alpha$. Moreover, in the spatial variable, we have

$$(-\Delta)^{\alpha/2}\vec{u}(t,x) = C_\alpha \, \mathbf{p.v.} \int_{\mathbb{R}^3} \frac{\vec{u}(t,x) - \vec{u}(t,y)}{|x-y|^{3+\alpha}} dy,$$

where $C_\alpha > 0$ is a constant depending on α, and **p.v.** denotes the principal value. The non-local behavior of this operator allows us to call the equations (1) the non-local Navier-Stokes equations. By contrast, when $\alpha = 2$, the diffusion term is given by the classical Laplacian operator, and we shall refer to the local Navier-Stokes equations. With a minor loss of generality, we have set the viscosity constant equal to one.

Numerical solutions to the classical Navier-Stokes equations (when $\alpha = 2$) for engineering problems, turbulent fluid flows, and geophysical phenomena are not completely possible at present, see for example [4, 15]. In addition, the mathematical theory of global existence and regularity of solutions to these equations remains one of the most challenging open questions in mathematical analysis [11, 16]. In this context, the fractional Navier-Stokes equations (when $1 < \alpha < 2$) have been understood as a relevant modification

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of the classical equations for a better understanding of these mathematical and computational difficulties [8, 12, 13, 14].

In equation (1), we observe that for each value of the parameter $1 < \alpha < 2$ we obtain a corresponding fractional Navier-Stokes equation, which we shall denote by $(\vec{u}_\alpha, p_\alpha)$ the associated solution. The main objective of this note is to study the asymptotic behavior of the family of solutions $(\vec{u}_\alpha, p_\alpha)_{1<\alpha<2}$ when the parameter α tends to 2.

This question is not only interesting from the theoretical point of view, but has also been pointed out in some experimental works involving fractional Burgers equations [6] and a fractional transport-type equation [18]. More precisely, these *numerical* studies show that solutions to fractional equations behave as solutions to the classical ones (involving the Laplace operator) when α is close enough to 2. Inspired by these latter works, we aim to develop a *rigorous framework* to study the convergence

$$(\vec{u}_\alpha, p_\alpha) \to (\vec{u}_2, p_2), \quad \text{when} \quad \alpha \to 2, \tag{2}$$

where $(\vec{u}_2, p_2)$ denotes a solution to the classical Navier-Stokes equations.

It is also worth mentioning this question has been studied for some *elliptic* equations, among them the nonlinear Schrödinger equation [1] and the fractional p-Laplacian problem [5]. In these works, the authors mainly used variational methods and concentration-compactness principles to prove the convergence of *weak solutions* of the fractional problem to the classical problem. Specifically, in [1] this convergence was proven in the strong topology of the space $L^2_{loc}(\mathbb{R}^n)$ (with $n \geq 3$), whereas in [5] the authors used the (more technical) notion of Γ-convergence. For the *parabolic* setting of equation (1), we shall use a completely different approach, principally based on the explicit structure of *mild solutions*. This approach allows us to prove a *uniform convergence* in the L^∞_{tx}-space, as well as to derive a sharp convergence rate.

The main result. We shall consider the initial value problem for both the non-local (when $1 < \alpha < 2$) and local ($\alpha = 2$) Navier-Stokes equations:

$$\begin{cases} \partial_t \vec{u}_\alpha = -(-\Delta)^{\alpha/2}\vec{u}_\alpha - (\vec{u}_\alpha \cdot \vec{\nabla})\vec{u}_\alpha - \vec{\nabla} p_\alpha, \quad \text{div}(\vec{u}_\alpha) = 0, \qquad 1 < \alpha \leq 2, \\ \vec{u}_\alpha(0, \cdot) = \vec{u}_{0,\alpha}, \end{cases} \tag{3}$$

where $\vec{u}_{0,\alpha} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the (divergence-free) initial datum. Recall that *mild solutions* to equations (3) are obtained by Banach's contraction principle by solving the following integral equation (due to Duhamel's formula)

$$\vec{u}_\alpha(t, \cdot) = e^{-(-\Delta)^{\alpha/2}t}\, \vec{u}_{0,\alpha} - \int_0^t e^{-(-\Delta)^{\alpha/2}(t-\tau)}\, \mathbb{P}\left((\vec{u}_\alpha \cdot \vec{\nabla})\vec{u}_\alpha \right)(\tau, \cdot) d\tau, \quad 1 < \alpha \leq 2. \tag{4}$$

Here, for $1 < \alpha < 2$ we have $e^{-(-\Delta)^{\alpha/2}t} f = h_\alpha(t, \cdot) * f$, where the kernel $h_\alpha(t, x)$ is the fundamental solution to the fractional heat equation (see (15) for a definition). For $\alpha = 2$ we have $e^{\Delta t} f = h(t, \cdot) * f$, where $h(t, x)$ is the well-known heat kernel.

The operator $\mathbb{P}$ stands for Leray's projector and it is well-known that the pressure p_α can be easily deduced from the velocity $\vec{u}_\alpha = (u_{\alpha,1}, u_{\alpha,2}, u_{\alpha,3})$ since, due to the divergence-free property of this latter, we have

$$p_\alpha = \frac{1}{-\Delta} \text{div}\left((\vec{u}_\alpha \cdot \vec{\nabla})\vec{u}_\alpha \right) = \sum_{i,j=1}^{3} \mathcal{R}_i \mathcal{R}_j (u_{\alpha,i}\, u_{\alpha,j}), \tag{5}$$

where $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ denotes the Riesz transform.

In the setting of non-homogeneous Sobolev spaces $H^s(\mathbb{R}^3)$ with $s > 1/2$, the local well-posedness theory for mild solutions to the classical Navier-Stokes equations is a well-known issue because of [2]. In our next proposition, we revisit this result for the generalized case of equation (3). Moreover, we shall consider the space $H^s(\mathbb{R}^3)$, with $s > 3/2$, where the (technical) constraint $s > 3/2$ is required to prove our *key tool* (given in Lemma 2.1 below) in the study of convergence (2).

We emphasize that the proof of the proposition below is classical, but we aim to determine how the existence time of the mild solution $\vec{u}_\alpha$, denoted by T_α, explicitly depends on the parameter α.

Proposition 1.1 *Let $1 < \alpha \leq 2$ fixed. Let $s > 3/2$ and let $\vec{u}_{0,\alpha} \in H^s(\mathbb{R}^3)$ be a divergence-free initial datum. There exists a time*

$$0 < T_\alpha = \frac{1}{2}\left(\frac{1-\frac{1}{\alpha}}{4C\|\vec{u}_{0,\alpha}\|_{H^s}}\right)^{\frac{\alpha}{\alpha-1}}, \tag{6}$$

where $C > 0$ is a generic constant, and there exists a unique mild solution $\vec{u}_\alpha$ to equation (3) such that $\vec{u}_\alpha \in \mathcal{C}\big([0,T_\alpha], H^s(\mathbb{R}^3)\big)$ and $p_\alpha \in \mathcal{C}\big([0,T_\alpha], H^s(\mathbb{R}^3)\big)$.

Remark 1 *Note that $0 < T_\alpha$ as long as $1 < \alpha$.*

Once we have stated this proposition, we rigorously studied the convergence presented in (2). For the fractional case (when $1 < \alpha < 2$), we consider the family of initial data $(\vec{u}_{0,\alpha})_{1<\alpha<2} \subset H^s(\mathbb{R}^3)$ and denote by $(\vec{u}_\alpha, p_\alpha)_{1<\alpha<2} \subset \mathcal{C}([0,T_\alpha], H^s(\mathbb{R}^3))$ the corresponding family of solutions given by Proposition 1.1. Similarly, for the classical case (when $\alpha = 2$), we consider the initial datum $\vec{u}_{0,2} \in H^s(\mathbb{R}^3)$ and $(\vec{u}_2, p_2) \in \mathcal{C}([0,T_2], H^s(\mathbb{R}^3))$ its associated solution.

We shall assume the following strong convergence of the initial data:

$$\vec{u}_{0,\alpha} \to \vec{u}_{0,2}, \quad \alpha \to 2, \quad \text{in} \quad H^s(\mathbb{R}^3). \tag{7}$$

On the one hand, this convergence will allow us to find a quantity $0 < \varepsilon <\ll 1$ and a time T_0, only depending on ε, such that

$$T_0 \leq T_\alpha, \quad \text{for all} \quad 1+\varepsilon < \alpha \leq 2. \tag{8}$$

See Appendix B for further details. Consequently, for $1+\varepsilon < \alpha \leq 2$ each solution $(\vec{u}_\alpha, p_\alpha)$ is at least defined on the time interval $[0, T_0]$, and this fact will be used when studying (2).

On the other hand, since $s > 3/2$, the space $H^s(\mathbb{R}^3)$ is continuously embedded in the space $L^\infty(\mathbb{R}^3)$ and the convergence (7) also holds in $L^\infty(\mathbb{R}^3)$. Thus, for the family of velocities $\vec{u}_\alpha$ we shall prove the *uniform convergence*:

$$\vec{u}_\alpha \to \vec{u}_2, \quad \alpha \to 2, \quad \text{in} \quad L^\infty([0,T_0] \times \mathbb{R}^3). \tag{9}$$

Recall that the pressures p_α are defined through Riesz transforms and the velocities $\vec{u}_\alpha$ in the expression (5). Nevertheless, since Riesz transforms are not bound in the $L^\infty-$space, we need to consider the larger space $BMO(\mathbb{R}^3)$, see [7, Chapter 3] for a definition and some properties of this space. In this setting, convergence (9) yields:

$$p_\alpha \to p_2, \quad \alpha \to 2, \quad \text{in} \quad L^\infty([0,T_0], BMO(\mathbb{R}^3)). \tag{10}$$

Furthermore, we are interested in studying a *convergence rate* of (9) and (10). For this, we introduce a parameter $\gamma > 0$ and we shall assume the estimate (11) below, which is a given convergence rate of the initial data in the space $L^\infty(\mathbb{R}^3)$. We aim to determine when the family of solutions follows this prescribed convergence rate. In this context, our main result is as follows:

Theorem 1.1 *Let $(\vec{u}_{0,\alpha})_{1+\varepsilon<\alpha\leq 2}$ be an initial data family, where $\vec{u}_{0,\alpha} \in H^s(\mathbb{R}^3)$ with $s > 3/2$. Let $(\vec{u}_\alpha, p_\alpha)_{1+\varepsilon<\alpha\leq 2} \subset \mathcal{C}\big([0,T_0], H^s(\mathbb{R}^3)\big)$ be the corresponding family of solutions to equation (3), given by Proposition 1.1.*

We assume the convergence given in (7). Moreover, for a parameter $\gamma > 0$, assume the estimate

$$\|\vec{u}_{0,\alpha} - \vec{u}_{0,2}\|_{L^\infty} \leq \mathbf{c}(2-\alpha)^\gamma, \tag{11}$$

where $\mathbf{c} > 0$ *is a generic constant. There exists a constant* $\mathbf{C} > 0$*, depending on* $\vec{u}_{0,2}, \varepsilon$ *and* $\mathbf{c}$*, such that the following estimate holds:*

$$\begin{aligned}&\sup_{0\leq t\leq T_0}\big(\|\vec{u}_\alpha(t,\cdot)-\vec{u}_2(t,\cdot)\|_{L^\infty}+\|p_\alpha(t,\cdot)-p_2(t,\cdot)\|_{BMO}\big)\\&\quad\leq \mathbf{C}(1+T_0+T_0^2)\max\big((2-\alpha)^\gamma,2-\alpha\big),\end{aligned}\qquad 1+\varepsilon<\alpha<2. \tag{12}$$

Some remarks and comments have been provided in order here.

(A) The uniform convergence (in both the temporal and the spatial variables) obtained in (12) is stronger than the ones obtained in the aforementioned works [1, 5]. Moreover, in contrast to these works, we also derive a convergence rate, given in (12), which is essentially determined by a *competition* between the quantities $(2-\alpha)^\gamma$ and $(2-\alpha)$.

To make a deeper discussion of this fact, let us briefly explain the general idea of the proof. As pointed out, we shall consider mild solutions to equation (3), which are given by the expression (4). Inequality (12) is then obtained by the following estimates

$$\|h_\alpha(t,\cdot)*\vec{u}_{0,\alpha}-h(t,\cdot)*\vec{u}_{0,2}\|_{L^\infty}\lesssim \max\Big((2-\alpha)^\gamma,2-\alpha\Big),$$

and

$$\left\|\int_0^t h_\alpha(t-s,\cdot)*\mathbb{P}\big((\vec{u}_\alpha\cdot\vec{\nabla})\vec{u}_\alpha\big)(\tau,\cdot)d\tau-h(t-s,\cdot)*\mathbb{P}\big((\vec{u}_2\cdot\vec{\nabla})\vec{u}_2\big)(\tau,\cdot)d\tau\right\|_{L^\infty}\lesssim\max\Big((2-\alpha)^\gamma,2-\alpha\Big),$$

for the linear and nonlinear terms, respectively. For simplicity, we will only explain the estimates for the linear term in more detail. The estimates for the nonlinear term are much more delicate, but they follow similar ideas. Thus, we split the linear term as

$$\|h_\alpha(t,\cdot)*\vec{u}_{0,\alpha}-h(t,\cdot)*\vec{u}_{0,2}\|_{L^\infty}\leq\|(h_\alpha(t,\cdot)-h(t,\cdot))*\vec{u}_{0,\alpha}\|_{L^\infty}+\|h(t,\cdot)*(\vec{u}_{0,\alpha}-\vec{u}_{0,2})\|_{L^\infty},$$

where we have

$$\|(h_\alpha(t,\cdot)-h(t,\cdot))*\vec{u}_{0,\alpha}\|_{L^\infty}\lesssim(2-\alpha),\quad\text{and}\quad\|h(t,\cdot)*(\vec{u}_{0,\alpha}-\vec{u}_{0,2})\|_{L^\infty}\lesssim(2-\alpha)^\gamma.$$

Here, the quantity $(2-\alpha)^\gamma$ is the convergence rate assumed for the initial data, whereas the quantity $(2-\alpha)$ is the convergence rate of the kernels $h_\alpha(t,x)\to h(t,x)$ (when $\alpha\to 2$), which is rigorously proven in Lemma 2.1.

Since we have $1<\alpha<2$, and therefore $0<2-\alpha<1$, estimate (12) yields the following conclusions by considering two cases of the parameter γ:

- When $0<\gamma\leq 1$, we have $\max\Big((2-\alpha)^\gamma,2-\alpha\Big)=(2-\alpha)^\gamma$, and consequently, the velocities $\vec{u}_\alpha(t,x)$ converge to the velocity $\vec{u}_2(t,x)$ with the same convergence rate as that of the initial data.
- When $1<\gamma$, we have $\max\Big((2-\alpha)^\gamma,2-\alpha\Big)=2-\alpha$. In this case, it is interesting to observe that the convergence rate of the solutions does not follow that of the initial data. More precisely, the velocities $\vec{u}_\alpha(t,x)$ converge to the velocity $\vec{u}_2(t,x)$ at a rate of order $2-\alpha$, which is *slower* than the convergence rate of the initial data $(2-\alpha)^\gamma$.

In summary, the increasing of the parameter γ makes the assumption (11) strong but not the result given in (12). This is an interesting *phenomenological effect*, which is given by the convergence rate of the kernels $h_\alpha(t,\cdot)\to h(t,\cdot)$.

(B) In the particular case of the same initial data for whole the family of equations (3): $\vec{u}_{0,\alpha} = \vec{u}_{0,2}$ for all $1+\varepsilon < \alpha < 2$, the estimate (12) becomes

$$\sup_{0\leq t\leq T_0} \big(\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty} + \|p_\alpha(t,\cdot) - p_2(t,\cdot)\|_{BMO} \big) \lesssim 2-\alpha,$$

where the convergence rate is purely given by the convergence of the kernels $h_\alpha(t,x) \to h(t,x)$ (see Lemma 2.1).

(C) In the case of small initial data: $\sup\limits_{1+\varepsilon<\alpha\leq 2} \|\vec{u}_{0,\alpha}\|_{H^s} \ll 1$, it is well known that mild solutions to the equation (3) are global in time, see [11, Theorem 7.3]. In this setting, our main estimate (12) writes down as

$$\begin{aligned} &\sup_{0\leq t\leq T} \big(\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty} + \|p_\alpha(t,\cdot) - p_2(t,\cdot)\|_{BMO} \big) \\ &\quad \leq \mathbf{C}(1+T+T^2) \max\big((2-\alpha)^\gamma, 2-\alpha\big), \end{aligned} \qquad 1+\varepsilon<\alpha<2, \quad \text{for all } 0<T<+\infty. \quad (13)$$

See Remark 2 below for all details.

(D) The convergence result given in Theorem 1.1 also allows us to study the convergence from non-local to local Navier-Stokes equation in the space $L^p((0,T_0), L^q(\mathbb{R}^3))$.

Corollary 1.1 *With the same hypothesis of Theorem 1.1, for $1 \leq p \leq +\infty$ and $2 < q < +\infty$ the estimate holds:*

$$\begin{aligned} &\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^p_t L^q_x} + \|p_\alpha(t,\cdot) - p_2(t,\cdot)\|_{L^p_t L^q_x} \\ &\leq \mathbf{C}_{p,q}(1+T_0+T_0^2) \max\Big((2-\alpha)^{\gamma(1-1/q)}, (2-\alpha)^{1-1/q}\Big), \end{aligned} \qquad 1+\varepsilon<\alpha<2. \qquad (14)$$

In the setting of the $L^p_t L^q_x$-spaces, the convergence rate is only driven by the parameters γ and q, this latter describes the decaying properties of solutions in the spatial variable.

(E) Finally, let us mention that Theorem 1.1 could be adapted to the two-dimensional case, where the regularity constraint $s > 3/2$ is relaxed to $s > 1$. In addition, Theorem 1.1 can be generalized to some relevant coupled systems in fluid dynamics, for instance, the magneto-hydrodynamic equations:

$$\begin{cases} \partial_t \vec{u} = -(-\Delta)^{\alpha/2}\vec{u} - (\vec{u}\cdot\vec{\nabla})\vec{u} + (\vec{b}\cdot\vec{\nabla})\vec{b} - \vec{\nabla}p, \quad \text{div}(\vec{u}) = 0, \\ \partial_t \vec{b} = -(-\Delta)^{\alpha/2}\vec{b} - (\vec{u}\cdot\vec{\nabla})\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{u}, \quad \text{div}(\vec{b}) = 0, \end{cases}$$

where $\vec{b} : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ is a magnetic field.

To close this section, we make some final comments: as mentioned, the strategy developed to prove Theorem 1.1 is strongly based on mild solutions of the equations (3). In future research, we aim to develop a different approach to study the convergence (2) for another relevant type of solution, for instance, Leray's solutions. Moreover, by following some of the ideas in [1, 5], we think it would be interesting to study this convergence for the elliptic case of *stationary* (time-independent) solutions.

Organization of the note. Section 2 is essentially devoted to the proof of the key Lemma 2.1. In Section 3, for the sake of completeness, we provide a brief proof of Proposition 1.1. Finally, in Section 4 we prove our main results: Theorem 1.1 and its Corollary 1.1.

2 Preliminaries

Let us consider the following linear and homogeneous fractional heat equation:

$$\partial_t h_\alpha + (-\Delta)^{\alpha/2} h_\alpha = 0, \quad 1 < \alpha < 2, \quad t > 0.$$

Recall that the fundamental solution of this equation, denoted by $h_\alpha(t,x)$, can be easily computed via the Fourier transform by $\widehat{h_\alpha}(t,\xi) = e^{-t\,|\xi|^\alpha}$. Moreover, in the spatial variable, the fundamental solution h_α is given by

$$h_\alpha(t,x) = \frac{1}{t^{\frac{1}{\alpha}}} H_\alpha\left(\frac{x}{t^{\frac{1}{\alpha}}}\right), \tag{15}$$

where the function H_α is the inverse Fourier transform of $e^{-|\xi|^\alpha}$. It is well-known that for $1 < \alpha < 2$, the functions H_α are smooth and positive. See [9, Chapter 13] for further details.

In the following lemma, we study the strong convergence of the kernel $h_\alpha(t,x)$ to the heat kernel $h(t,x)$, when $\alpha \to 2$. This result will be our key tool in the sequel.

Lemma 2.1 (Non-local to local heat equation) *Let $s > 3/2$. There exists a constant $C = C_s > 0$ such that, for all $1 < \alpha < 2$ and for all time $0 < T < +\infty$ the following estimate holds:*

$$\sup_{0\leq t\leq T} \|h_\alpha(t,\cdot) - h(t,\cdot)\|_{H^{-s}} \leq C\,T(2-\alpha).$$

Proof. First, we verify that the quantity $\|h_\alpha(t,\cdot) - h(t,\cdot)\|^2_{H^{-s}}$ is continuous with respect to the variable t. For $0 \leq t_0, t \leq T$ we have

$$\|h_\alpha(t,\cdot) - h(t,\cdot)\|^2_{H^{-s}} - \|h_\alpha(t_0,\cdot) - h(t_0,\cdot)\|^2_{H^{-s}} = \int_{\mathbb{R}^3} \left(\left| e^{-|\xi|^\alpha t} - e^{-|\xi|^2 t} \right|^2 - \left| e^{-|\xi|^\alpha t_0} - e^{-|\xi|^2 t_0} \right|^2 \right) \frac{d\xi}{(1+|\xi|^2)^s}.$$

As $s > 3/2$, we have $\displaystyle\int_{\mathbb{R}^3} \frac{d\xi}{(1+|\xi|^2)^s} < +\infty$, and we can apply the dominated convergence theorem to obtain

$$\lim_{t\to t_0} \left(\|h_\alpha(t,\cdot) - h(t,\cdot)\|^2_{H^{-s}} - \|h_\alpha(t_0,\cdot) - h(t_0,\cdot)\|^2_{H^{-s}} \right) = 0.$$

Once we have proven this continuity property, there exists a time $0 < t_1 \leq T$ such that

$$\sup_{0\leq t\leq T} \|h_\alpha(t,\cdot) - h(t,\cdot)\|_{H^{-s}} = \|h_\alpha(t_1,\cdot) - h(t_1,\cdot)\|_{H^{-s}}.$$

Now, we prove the estimate $\|h_\alpha(t_1,\cdot) - h(t_1,\cdot)\|_{H^{-s}} \leq C\,T(2-\alpha)$. Thus, we write

$$\|h_\alpha(t_1,\cdot) - h(t_1,\cdot)\|^2_{H^{-s}} = \int_{\mathbb{R}^3} |e^{-|\xi|^\alpha t_1} - e^{-|\xi|^2 t_1}|^2 \frac{d\xi}{(1+|\xi|^2)^s}. \tag{16}$$

For $\xi \in \mathbb{R}^n \setminus \{0\}$ fixed, and for $1 < \alpha < 2 + \delta$ (with $\delta > 0$) we define the function

$$f_\xi(\alpha) = e^{-t_1|\xi|^\alpha},$$

and by computing its derivative with respect to the variable α we get

$$f'_\xi(\alpha) = -t_1\, e^{-t_1|\xi|^\alpha}\, |\xi|^\alpha \ln(|\xi|).$$

Then, by the mean value theorem (in the variable α) we can write

$$|f_\xi(\alpha) - f_\xi(2)| \leq \|f'_\xi\|_{L^\infty([1,2+\delta])}\, |2-\alpha|.$$

Moreover, we can prove the uniform estimate with respect to the variable ξ:

$$\Big\| \|f'_\xi\|_{L^\infty([1,2+\delta])} \Big\|_{L^\infty(\mathbb{R}^3)} \leq C\,T. \tag{17}$$

The proof of this estimate is not difficult and is given in detail in Appendix A. We thus have

$$|f_\xi(\alpha) - f_\xi(2)| \leq C\,T|2-\alpha|.$$

Finally, we get back to the identity (16) and write

$$\begin{aligned}\|h_\alpha(t_1,\cdot) - h(t_1,\cdot)\|^2_{H^{-s}} &= \int_{\mathbb{R}^n} |f_\xi(\alpha) - f_\xi(2)|^2 \frac{d\xi}{(1+|\xi|^2)^s} \\ &\leq C\,T^2\,|2-\alpha|^2 \int_{\mathbb{R}^3} \frac{d\xi}{(1+|\xi|^2)^s} \leq C_s\,T^2\,(2-\alpha)^2.\end{aligned}$$

Lemma 2.1 is proven. ■

3 Proof of Proposition 1.1

The proof is rather standard, so we will only detail the main estimates. For a time $0 < T < +\infty$, we consider the Banach space $\mathcal{C}\big([0,T], H^s(\mathbb{R}^3)\big)$, endowed with its natural norm $\|\cdot\|_{L^\infty_t H^s_x}$. On the right-hand side of equation (4), the linear term is easy to estimate and we have $\Big\| e^{-(-\Delta)^{\alpha/2}t}\,\vec{u}_{0,\alpha} \Big\|_{L^\infty_t H^s_x} \leq \|\vec{u}_{0,\alpha}\|_{H^s}$.

Thereafter, for $0 < t < T$ fixed, the bilinear term is estimated as follows

$$\left\| \int_0^t e^{-(-\Delta)^{\alpha/2}(t-\tau)}\,\mathbb{P}\left((\vec{u}_\alpha \cdot \vec{\nabla})\vec{u}_\alpha \right)(\tau,\cdot) d\tau \right\|_{H^s} \leq C \int_0^t \|\vec{\nabla} h_\alpha(t-\tau,\cdot)\|_{L^1}\,\|\vec{u}_\alpha \otimes \vec{u}_\alpha(\tau,\cdot)\|_{H^s} d\tau.$$

From [17, Lemma 2.2], we have $\|\vec{\nabla} h_\alpha(t-\tau,\cdot)\|_{L^1} \leq C(t-\tau)^{-\frac{1}{\alpha}}$. On the other hand, since $s > 3/2$, by the product laws in Sobolev spaces we write $\|\vec{u}_\alpha \otimes \vec{u}_\alpha(\tau,\cdot)\|_{H^s} \leq C\|\vec{u}_\alpha(\tau,\cdot)\|^2_{H^s}$. We thus obtain

$$C\int_0^t \|\vec{\nabla} h_\alpha(t-\tau,\cdot)\|_{L^1}\,\|\vec{u}_\alpha \otimes \vec{u}_\alpha(\tau,\cdot)\|_{H^s} d\tau \leq C\left(\int_0^t (t-\tau)^{-\frac{1}{\alpha}} \right) \|\vec{u}_\alpha\|^2_{L^\infty_t H^s_x} \leq C\,\frac{T^{1-\frac{1}{\alpha}}}{1-\frac{1}{\alpha}}\,\|\vec{u}_\alpha\|^2_{L^\infty_t H^s_x}.$$

The existence and uniqueness of a mild solution $\vec{u}_\alpha$ follows from Picard's iterative schema, as long as $4C\|\vec{u}_{0,\alpha}\|_{H^s}\dfrac{T^{1-\frac{1}{\alpha}}}{1-\frac{1}{\alpha}} < 1$, which yields the definition of the time T_α as in (6). Proposition 1.1 is proven. ■

4 From non-local to local

In the following, $C > 0$ denotes a generic constant that may change in each line, but it does not depend on the parameter α.

4.1 Proof of Theorem 1.1

For a time $0 < T \leq T_0$ fixed, we write

$$\begin{aligned}&\sup_{0\leq t\leq T} \|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty} \\ \leq& \sup_{0\leq t\leq T} \Big\| e^{-(-\Delta)^{\alpha/2}t}\vec{u}_{0,\alpha} - e^{\Delta t}\vec{u}_{0,2} \Big\|_{L^\infty} \\ &+ \sup_{0\leq t\leq T} \left\| \int_0^t e^{-(-\Delta)^{\alpha/2}(t-\tau)}\mathbb{P}\left((\vec{u}_\alpha\cdot\vec{\nabla})\vec{u}_\alpha \right)(\tau,\cdot)d\tau - \int_0^t e^{\Delta(t-\tau)}\mathbb{P}\left((\vec{u}_2\cdot\vec{\nabla})\vec{u}_2 \right)(\tau,\cdot)d\tau \right\|_{L^\infty} \\ =& I_\alpha + J_\alpha.\end{aligned} \tag{18}$$

We begin by estimating each term on the right. For the term I_α, we get

$$\begin{aligned} I_\alpha &\leq \sup_{0\leq t\leq T} \left\| \left(e^{-(-\Delta)^{\alpha/2} t} - e^{\Delta t} \right) \vec{u}_{0,\alpha} \right\|_{L^\infty} + \sup_{0\leq t\leq T} \left\| e^{\Delta t} \left(\vec{u}_{0,\alpha} - \vec{u}_{0,2} \right) \right\|_{L^\infty} \\ &= \sup_{0\leq t\leq T} \left\| \Big(h_\alpha(t,\cdot) - h(t,\cdot) \Big) * \vec{u}_{0,\alpha} \right\|_{L^\infty} + \sup_{0\leq t\leq T} \left\| h(t,\cdot) * (\vec{u}_{0,\alpha} - \vec{u}_{0,2}) \right\|_{L^\infty} \\ &= I_{\alpha,1} + I_{\alpha,2}. \end{aligned} \tag{19}$$

Afterwards, to estimate the term $I_{\alpha,1}$, one can apply the Bessel potential operators $(1-\Delta)^{-s/2}$ and $(1-\Delta)^{s/2}$ to deduce

$$I_{\alpha,1} = \sup_{0\leq t\leq T} \left\| (1-\Delta)^{-s/2} \Big(h_\alpha(t,\cdot) - h(t,\cdot) \Big) * (1-\Delta)^{s/2} \vec{u}_{0,\alpha} \right\|_{L^\infty}.$$

Thus, thanks to Young inequalities (with $1 + 1/\infty = 1/2 + 1/2$), we can write

$$\begin{aligned} I_{\alpha,1} &\leq C \sup_{0\leq t\leq T} \left(\left\| (1-\Delta)^{-s/2} \Big(h_\alpha(t,\cdot) - h(t,\cdot) \Big) \right\|_{L^2} \left\| (1-\Delta)^{s/2} \vec{u}_{0,\alpha} \right\|_{L^2} \right) \\ &\leq C \left(\sup_{0\leq t\leq T} \| h_\alpha(t,\cdot) - h(t,\cdot) \|_{H^{-s}} \right) \left(\sup_{1+\varepsilon<\alpha<2} \| \vec{u}_{0,\alpha} \|_{H^s} \right), \end{aligned} \tag{20}$$

where each of the terms above must be estimated separately. Note that, for the first term on the right-hand side, it is natural to apply Lemma 2.1, whereas the second term on the right-hand side can be controlled by the fact that the family $(u_{0,\alpha})_{1+\varepsilon<\alpha<2}$ is bounded in $H^s(\mathbb{R}^3)$.

Therefore, the term $I_{\alpha,1}$ given in (19) can be estimated as follows

$$I_{\alpha,1} \leq C\, T\, (2-\alpha). \tag{21}$$

It is now time to study the term $I_{\alpha,2}$ in (19). By Young inequalities (with $1 + 1/\infty = 1 + 1/\infty$), the well-known properties of the heat kernel, and the assumption given in (11), we have

$$I_{\alpha,2} \leq \mathbf{c}(2-\alpha)^\gamma. \tag{22}$$

Consequently, we set the constant $\mathbf{C}_1 = \max(C, \mathbf{c})$, and by using equations (21) and (22), we can derive the following estimate

$$I_\alpha \leq \mathbf{C}_1\, (1+T)\, \max\Big((2-\alpha)^\gamma, 2-\alpha \Big). \tag{23}$$

Similarly, the term J_α in (18) can also be studied separately.

$$\begin{aligned} J_\alpha \leq& \sup_{0\leq t\leq T} \left\| \int_0^t h_\alpha(t-\tau,\cdot) * \mathbb{P}\left((\vec{u}_\alpha \cdot \vec{\nabla}) \vec{u}_\alpha \right)(\tau,\cdot) d\tau - \int_0^t h(t-\tau,\cdot) * \mathbb{P}\left((\vec{u}_\alpha \cdot \vec{\nabla}) \vec{u}_\alpha \right)(\tau,\cdot) d\tau \right\|_{L^\infty} \\ &+ \sup_{0\leq t\leq T} \left\| \int_0^t h(t-\tau,\cdot) * \mathbb{P}\left((\vec{u}_\alpha \cdot \vec{\nabla}) \vec{u}_\alpha \right)(\tau,\cdot) d\tau - \int_0^t h(t-\tau,\cdot) * \mathbb{P}\left((\vec{u}_2 \cdot \vec{\nabla}) \vec{u}_2 \right)(\tau,\cdot) d\tau \right\|_{L^\infty} \\ \leq& \sup_{0\leq t\leq T} \left\| \int_0^t \Big(h_\alpha(t-\tau,\cdot) - h(t-\tau,\cdot) \Big) * \mathbb{P}\left((\vec{u}_\alpha \cdot \vec{\nabla}) \vec{u}_\alpha \right)(\tau,\cdot) d\tau \right\|_{L^\infty} \\ &+ \sup_{0\leq t\leq T} \left\| \int_0^t h(t-\tau,\cdot) * \mathbb{P}\left((\vec{u}_\alpha \cdot \vec{\nabla}) \vec{u}_\alpha - (\vec{u}_2 \cdot \vec{\nabla}) \vec{u}_2 \right)(\tau,\cdot) d\tau \right\|_{L^\infty} = J_{\alpha,1} + J_{\alpha,2}. \end{aligned} \tag{24}$$

For the term $J_{\alpha,1}$, we can take advantage of the Leray's projector $\mathbb{P}$ properties, and once again we apply the operators $(1-\Delta)^{-s/2}$ and $(1-\Delta)^{s/2}$ along with the Young inequalities (with $1 + 1/\infty = 1/2 + 1/2$) to get the following estimates

$$
\begin{aligned}
J_{\alpha,1} &\leq \sup_{0\leq t\leq T} \left(\int_0^t \left\| \Big(h_\alpha(t-\tau,\cdot) - h(t-\tau,\cdot) \Big) * \mathbb{P}\left(\operatorname{div}(\vec{u}_\alpha \otimes \vec{u}_\alpha)\right)(\tau,\cdot) \right\|_{L^\infty} d\tau \right) \\
&\leq \sup_{0\leq t\leq T} \left(\int_0^t \left\| \mathbb{P}\Big(\vec{\nabla} h_\alpha(t-\tau,\cdot) - \vec{\nabla} h(t-\tau,\cdot) \Big) \right\|_{H^{-s}} \|(\vec{u}_\alpha \otimes \vec{u}_\alpha)(\tau,\cdot)\|_{H^s} \, d\tau \right) \\
&\leq \sup_{0\leq t\leq T} \left(\int_0^t \left\| \vec{\nabla} h_\alpha(t-\tau,\cdot) - \vec{\nabla} h(t-\tau,\cdot) \right\|_{H^{-s}} \|(\vec{u}_\alpha \otimes \vec{u}_\alpha)(\tau,\cdot)\|_{H^s} \, d\tau \right) \\
&\leq T \left(\sup_{0\leq t\leq T} \left\| \vec{\nabla} h_\alpha(t,\cdot) - \vec{\nabla} h(t,\cdot) \right\|_{H^{-s}} \right) \left(\sup_{0\leq t\leq T} \|(\vec{u}_\alpha \otimes \vec{u}_\alpha)(t,\cdot)\|_{H^s} \right).
\end{aligned}
\tag{25}
$$

To control the first term on the right-hand side, we can adapt Lemma 2.1 to the function $f_\xi(\alpha) = i\xi_j e^{-t|\xi|\alpha}$, with $j = 1,2,3$, this manner, we obtain

$$
\sup_{0\leq t\leq T} \left\| \vec{\nabla} h_\alpha(t,\cdot) - \vec{\nabla} h(t,\cdot) \right\|_{H^{-s}} \leq CT|2-\alpha|.
$$

For the remaining term on the right-hand side, we prove that there exists a constant $\mathbf{C}_2 = \mathbf{C}_2(\vec{u}_{2,0},\varepsilon)$ that is sufficiently large and depends only on $\vec{u}_{0,2}$ and ε, such that the following uniform estimate holds:

$$
\sup_{1+\varepsilon<\alpha<2} \sup_{0\leq t\leq T} \|(\vec{u}_\alpha \otimes \vec{u}_\alpha)(t,\cdot)\|_{H^s} \leq \mathbf{C}_2.
$$

Indeed, recall that the solution $\vec{u}_\alpha \in \mathcal{C}\big([0,T], H^s(\mathbb{R}^3)\big)$ obtained in Proposition 1.1 by the Picard's iterative argument verifies

$$
\sup_{0\leq t\leq T} \|\vec{u}_\alpha(t,\cdot)\|_{H^s} \leq \sup_{0\leq t\leq T_\alpha} \|\vec{u}_\alpha(t,\cdot)\|_{H^s} \leq C \|\vec{u}_{0,\alpha}\|_{H^s}, \quad \text{where } T \leq T_0 \leq T_\alpha.
$$

Moreover, based on the assumption (7), we have $\sup\limits_{1+\varepsilon<\alpha<2} \|\vec{u}_{0,\alpha}\|_{H^s} \leq \mathbf{C}_2$. Then we obtain

$$
\sup_{1+\varepsilon<\alpha<2} \sup_{0\leq t\leq T} \|\vec{u}_\alpha(t,\cdot)\|_{H^s} \leq \mathbf{C}_2. \tag{26}
$$

Thus, the wished estimate follows from the fact that $s > 3/2$, and by the product laws in Sobolev spaces we can write

$$
\sup_{0\leq t\leq T} \|(\vec{u}_\alpha \otimes \vec{u}_\alpha)(t,\cdot)\|_{H^s} \leq C \sup_{0\leq t\leq T} \|\vec{u}_\alpha(t,\cdot)\|^2_{H^s} \leq C \left(\sup_{0\leq t\leq T} \|\vec{u}_\alpha(t,\cdot)\|_{H^s} \right)^2 \leq \mathbf{C}_2.
$$

Returning to estimate (25), the above inequality allows us to write

$$
J_{\alpha,1} \leq \mathbf{C}_2\, T^2\, |2-\alpha| \leq \mathbf{C}_2\, T^2 \max\Big((2-\alpha)^\gamma, 2-\alpha \Big). \tag{27}
$$

Subsequently, we study the term $J_{\alpha,2}$ given in (24). For this propose we combine Leray's projector $\mathbb{P}$ properties and Young inequalities (with $1 + 1/\infty = 1 + 1/\infty$) as follows

$$
\begin{aligned}
J_{\alpha,2} &\leq \sup_{0\leq t\leq T} \int_0^t \left\| h(t-\tau,\cdot) * \mathbb{P}\Big(\operatorname{div}(\vec{u}_\alpha \otimes \vec{u}_\alpha) - \operatorname{div}(\vec{u}_2 \otimes \vec{u}_2) \Big)(\tau,\cdot) \right\|_{L^\infty} d\tau \\
&\leq C \sup_{0\leq t\leq T} \int_0^t \|\nabla h(t-\tau,\cdot)\|_{L^1} \|\mathbb{P}\big((\vec{u}_\alpha \otimes \vec{u}_\alpha) - (\vec{u}_2 \otimes \vec{u}_2) \big)(\tau,\cdot)\|_{L^\infty} d\tau.
\end{aligned}
\tag{28}
$$

Owing to the well-known properties of the heat kernel $h(t,\cdot)$ we have $\|\nabla h(t-\tau,\cdot)\|_{L^1} \leq C(t-\tau)^{-1/2}$. Meanwhile, to estimate the term $\|\mathbb{P}\big((\vec{u}_\alpha \otimes \vec{u}_\alpha) - (\vec{u}_2 \otimes \vec{u}_2)\big)(\tau,\cdot)\|_{L^\infty}$, we use Leray's projector $\mathbb{P}$ properties, the uniform estimate inequality (26) and the fact that $s > 3/2$. Thus,

$$\begin{aligned}
&\|\mathbb{P}\big((\vec{u}_\alpha \otimes \vec{u}_\alpha) - (\vec{u}_2 \otimes \vec{u}_2)\big)(\tau,\cdot)\|_{L^\infty}\\
=&\left\|\big(\vec{u}_\alpha(\tau,\cdot) - \vec{u}_2(\tau,\cdot)\big) \otimes \mathbb{P}(\vec{u}_\alpha + \vec{u}_2)(\tau,\cdot)\right\|_{L^\infty}\\
\leq&\ \|\vec{u}_\alpha(\tau,\cdot) - \vec{u}_2(\tau,\cdot)\|_{L^\infty}\left(\|\mathbb{P}(\vec{u}_\alpha)(\tau,\cdot)\|_{L^\infty} + \|\mathbb{P}(\vec{u}_2)(\tau,\cdot)\|_{L^\infty}\right)\\
\leq&\ \|\vec{u}_\alpha(\tau,\cdot) - \vec{u}_2(\tau,\cdot)\|_{L^\infty}\left(\|\vec{u}_\alpha(\tau,\cdot)\|_{H^s} + \|\vec{u}_2(\tau,\cdot)\|_{H^s}\right)\\
\leq&\ \mathbf{C}_2\,\|\vec{u}_\alpha(\tau,\cdot) - \vec{u}_2(\tau,\cdot)\|_{L^\infty}.
\end{aligned}$$

These last two estimations allow us to control (28) as follows

$$J_{\alpha,2} \leq \mathbf{C}_2 \sup_{0\leq t\leq T}\int_0^t (t-\tau)^{-1/2}\,\|\vec{u}_\alpha(\tau,\cdot) - \vec{u}_2(\tau,\cdot)\|_{L^\infty} d\tau \leq \mathbf{C}_2\,T^{1/2}\left(\sup_{0\leq t\leq T}\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty}\right). \quad (29)$$

Thus far, we have controlled the terms I_α, $J_{\alpha,1}$ and $J_{\alpha,2}$ in (23), (27), and (29), respectively. We set the constant $\mathbf{C} = \max(\mathbf{C}_1, \mathbf{C}_2)$, and we get back to (18) to write

$$\sup_{0\leq t\leq T}\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty} \leq I_\alpha + J_{\alpha,1} + J_{\alpha,2} \leq I_\alpha + J_{\alpha,1} + \mathbf{C}\,T^{1/2}\left(\sup_{0\leq t\leq T}\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty}\right).$$

In the above estimate, we set a time $0 < T_1 \leq T$ such that $\mathbf{C}\,T_1^{1/2} \leq \frac{1}{2}$. This way, we derive the following control:

$$\sup_{0\leq t\leq T_1}\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty} \leq I_\alpha + J_{\alpha,1} + \frac{1}{2}\left(\sup_{0\leq t\leq T_1}\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty}\right),$$

and we can write

$$\frac{1}{2}\sup_{0\leq t\leq T_1}\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty} \leq I_\alpha + J_{\alpha,1}.$$

Then, by (23) and (27) we obtain

$$\sup_{0\leq t\leq T_1}\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty} \leq \mathbf{C}(1 + T_1 + T_1^2)\,\max\left((2-\alpha)^\gamma, (2-\alpha)\right). \quad (30)$$

By iterative application of this argument up to time $T_0 > 0$, we have

$$\sup_{0\leq t\leq T_0}\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty} \leq \mathbf{C}(1 + T_0 + T_0^2)\,\max\left((2-\alpha)^\gamma, (2-\alpha)\right). \quad (31)$$

Remark 2 *In the case of global in time mild solutions, we can iterate (30) to obtain (13) for any time* $0 < T < +\infty$.

To finish the proof of Theorem 1.1, we shall prove that estimate (31) yields

$$\sup_{0\leq t\leq T_0}\|p_\alpha(t,\cdot) - p_2(t,\cdot)\|_{BMO} \leq \mathbf{C}(1 + T_0 + T_0^2)\,\max\left((2-\alpha)^\gamma, (2-\alpha)\right). \quad (32)$$

Indeed, by expression (5), by the estimate $\|\mathcal{R}_i f\|_{BMO} \leq C\|f\|_{L^\infty}$ (see for instance [10, Theorem 6.2]), and by the uniform estimate (26), for $0 < t \leq T_0$ we write

$$\begin{aligned}
\|p_\alpha(t,\cdot) - p_2(t,\cdot)\|_{BMO} &\leq C\|\vec{u}_\alpha \otimes \vec{u}_\alpha(t,\cdot) - \vec{u}_2 \otimes \vec{u}_2(t,\cdot)\|_{L^\infty}\\
&\leq C\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty}\big(\|\vec{u}_\alpha(t,\cdot)\|_{L^\infty} + \|\vec{u}_2(t,\cdot)\|_{L^\infty}\big)\\
&\leq \mathbf{C}\,\|\vec{u}_\alpha(t,\cdot) - \vec{u}_2(t,\cdot)\|_{L^\infty},
\end{aligned}$$

which yields (32). Theorem 1.1 is now proven. ■

4.2 Proof of Corollary 1.1

The proof of this result is straightforward. Remark that the family of initial data also belongs to the space $L^2(\mathbb{R}^3)$, and by well-known arguments, for $1+\varepsilon<\alpha<1$ we have

$$\|\vec{u}_\alpha(t,\cdot)\|_{L^2}^2 \leq \|\vec{u}_{0,\alpha}\|_{L^2}^2 \leq C\|\vec{u}_{0,\alpha}\|_{H^s}^2 \leq \mathbf{C}_2.$$

Estimate (14) follows from a standard interpolation argument (in Lebesgue spaces) between the estimate above and (12). Then, Corollary 1.1 is proven. ■

A Appendix

We prove here the estimate (17). We recall the expression

$$f^{'}_\xi(\alpha) = -t_1 e^{-t_1|\xi|^\alpha}|\xi|^\alpha \ln(|\xi|), \quad 1<\alpha<2+\delta, \quad 0<t_1\leq T.$$

Then, we write

$$\Big\| \|f^{'}_\xi\|_{L^\infty([1,2+\delta])} \Big\|_{L^\infty(\mathbb{R}^3)} \leq \Big\| \|f^{'}_\xi\|_{L^\infty([1,2+\delta])} \Big\|_{L^\infty(|\xi|\leq 1)} + \Big\| \|f^{'}_\xi\|_{L^\infty([1,2+\delta])} \Big\|_{L^\infty(|\xi|>1)} = A+B,$$

where we estimate the terms A and B separately. For the term A, as we have $|\xi|\leq 1$, $1<\alpha<2+\delta$, and moreover, as we have $\lim\limits_{|\xi|\to 0^+} |\xi|\ln(|\xi|)=0$, we deduce the following control:

$$A \leq T\left(\sup_{\xi\in\mathbb{R}^3} e^{-t_1|\xi|^{2+\delta}}|\xi|\ln(|\xi|)\right) \leq C\,T.$$

For the term B, since $|\xi|>1$ then we obtain

$$B \leq T\left(\sup_{\xi\in\mathbb{R}^3} e^{-t_1|\xi|}|\xi|^{2+\delta}\ln(|\xi|)\right) \leq C\,T.$$

B Appendix

We now prove the lower bound (8). By (7) we can set $0<\varepsilon\ll 1$ such that for all $1+\varepsilon<\alpha<2$ we have $\big|\|\vec{u}_{0,\alpha}\|_{H^s} - \|\vec{u}_{0,2}\|_{H^s}\big| \leq \frac{1}{2}\|\vec{u}_{0,2}\|_{H^s}$, hence we obtain $\|\vec{u}_{0,\alpha}\|_{H^s} \leq \frac{3}{2}\|\vec{u}_{0,2}\|_{H^s}$, and we can write

$$\frac{1}{2}\left(\frac{1-\frac{1}{\alpha}}{4C\|\vec{u}_{0,2}\|_{H^s}}\right)^{\frac{\alpha}{\alpha-1}} \leq T_\alpha, \quad 1+\varepsilon<\alpha<2.$$

Furthermore, the expression on the left-hand side is estimated from below by the quantity

$$T_0 = \frac{1}{2}\max\left[\left(\frac{1-\frac{1}{1+\varepsilon}}{4C\|\vec{u}_{0,2}\|_{H^s}}\right)^{\frac{2}{\varepsilon}}, \left(\frac{1-\frac{1}{1+\varepsilon}}{4C\|\vec{u}_{0,2}\|_{H^s}}\right)^{1+\varepsilon}\right].$$

Indeed, as we have $1+\varepsilon<\alpha<2$, then we get $1-\frac{1}{1+\varepsilon}<1-\frac{1}{\alpha}$, and we can write

$$\frac{1}{2}\left(\frac{1-\frac{1}{1+\varepsilon}}{4C\|\vec{u}_{0,2}\|_{H^s}}\right)^{\frac{\alpha}{\alpha-1}} \leq \frac{1}{2}\left(\frac{1-\frac{1}{\alpha}}{4C\|\vec{u}_{0,2}\|_{H^s}}\right)^{\frac{\alpha}{\alpha-1}}.$$

Thereafter, for the sake of simplicity, we denote $A = \dfrac{1 - \frac{1}{1+\varepsilon}}{4C\|\vec{u}_{0,2}\|_{H^s}}$, and we have $\dfrac{1}{2}A^{\frac{\alpha}{\alpha-1}} \leq \dfrac{1}{2}\left[\dfrac{1-\frac{1}{\alpha}}{4C\|\vec{u}_{0,2}\|_{H^s}}\right]^{\frac{\alpha}{\alpha-1}}$.

We now study the expression $\frac{\alpha}{\alpha-1}$. Since $1+\varepsilon < \alpha < 2$, then we get $1+\varepsilon < \frac{\alpha}{\alpha-1} < \frac{2}{\varepsilon}$. Thus, on the one hand, if the quantity A above verifies $A < 1$ then we have $\dfrac{1}{2}A^{\frac{2}{\varepsilon}} \leq \dfrac{1}{2}A^{\frac{\alpha}{\alpha-1}}$. On the other hand, if the quantity A verifies $1 \leq A$ then we have $\dfrac{1}{2}A^{1+\varepsilon} \leq \dfrac{1}{2}A^{\frac{\alpha}{\alpha-1}}$.

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Statements and Declarations

Data sharing does not apply to this article as no datasets were generated or analyzed during the current study. This work has not received any financial support. In addition, the authors declare that they have no conflicts of interest and all of them have equally contributed to this paper.