Sharp well-posedness and spatial decaying for a generalized dispersive-dissipative Kuramoto-type equation and applications to related models

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September 27, 2023

Abstract

We introduce a fairly general dispersive-dissipative nonlinear equation, which is characterized by fractional Laplacian operators in both the dispersive and dissipative terms. This equation contains as particular cases some physically relevant models of the fluid dynamics, among them, the *dispersive Kuramoto-Velarde* equation, the *Kuramoto-Sivashinsky* equation and some nonlocal perturbations of the *KdV* and the *Benjamin-Ono* equations. We acutely study the effects of the fractional Laplacian operators in the qualitative study of solutions: on the one hand, we prove a sharp well-posedness result in the framework of the Sobolev spaces of negative order. On the other hand, we study pointwise decaying properties of solutions in the spatial variable, which are optimal in some cases. These last results are of particular interest for the contained physical models. Precisely, they agree with previous numerical works on the spatially decaying of a particular kind of solutions so-called the solitary waves.

Keywords: Dispersive-dissipative models in fluid dynamics; Sharp well-posedness; Spatially decaying of solutions; Solitary waves.

AMS Classification: 35A01, 35B30.

Contents

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1 Introduction and motivation of the model

In the context of physical phenomena, the dispersive Kuramoto-Velarde equation

$$(KV) \quad \partial_t u + \partial_x^2 u + \partial_x^3 u + \partial_x^4 u + \gamma_2\, \partial_x^2(u^2) + \gamma_3\, (\partial_x u)^2 = 0,$$

describes slow space-time variations of disturbances at interfaces, diffusion–reaction fronts and plasma instability fronts [5, 11, 12]. It also describes Benard–Marangoni cells that occur when there is large surface tension on the interface [14, 26, 27] in a microgravity environment. This situation arises in crystal growth experiments aboard an orbiting space station, although the free interface is metastable with respect to small perturbations. In particular, the nonlinearities, $\gamma_3\, (\partial_x u)^2$ and $\gamma_2\, \partial_x^2(u^2)$, model pressure destabilization effects striving to rupture the interface. Likewise, the equation (KV) is a variation of the Kuramoto–Sivashinsky equation,

$$(KS) \quad \partial_t u + \partial_x^2 u + \partial_x^3 u + \partial_x^4 u + \gamma_3\, (\partial_x u)^2 = 0,$$

which describes slow space-time variations of disturbances at interfaces, flame fronts, diffusion-reaction fronts, plasma instability fronts and the long waves on the interface between two viscous fluids [13]. In this equation, the linear terms describe a balance between long-wave instability and short-wave stability, while with the nonlinear term provides a mechanism for energy transfer between wave modes. Finally, remark that the (KS) equation agrees with the (KV) equation at $\gamma_2 = 0$.

Taking the periodic case into account, the equation (KS) is one of the simplest partial differential equations which is capable of exhibiting chaotic behavior. The long time behavior of the KS equation is characterized by the negative (therefore destabilizing) second-order diffusion, the positive (therefore stabilizing) fourth-order dissipation, and the nonlinear coupling term [21].

On the other hand, thinking about models describing the behavior of other types of fluids such as stratified fluids, relevant equations with nonlocal terms appear. A case of this type of equations is, on the one hand, the *Ostrovsky, Stepanyams and Tsimring* (OST) equation:

$$(OST) \quad \partial_t u + \partial_x^3 u + u\partial_x u + \eta\mathcal{H}(\partial_x u + \partial_x^3 u) = 0,$$

which is a nonlocal perturbation of the celebrated *Korteweg-de Vries* (KdV) equation and, on the other hand, a nonlocal perturbed version of the well-known *Benjamin-Ono* equation [4]:

$$(npBO) \quad \partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u + \eta\mathcal{H}(\partial_x u + \partial_x^3 u) = 0.$$

where $\mathcal{H}$ is the Hilbert transform (see (2) for a precise definition) and $\eta > 0$ is a physical parameter.

The *OST* equation describes the radiational instability of long non-linear waves in a stratified flow caused by internal wave radiation from a shear layer. The parameter $\eta > 0$ represents the importance of amplification and damping relative to dispersion. The fourth term in equation represents amplification, while the fifth term in equation denotes damping. For a more complete physical description we refer to [18, 19, 20], while the nonlocal perturbed *Benjamin-Ono* is a good approximate model for long-crested unidirectional waves at the interface of a two-layer system of incompressible inviscid fluids. Moreover, it gives an analogous model of the *OST* equation in deep stratified fluids [4].

One of the main objectives of this article is to introduce a new general *theoretical* equation that encompasses the aforementioned equations as well as some other *physically relevant* variants. We also want to understand or at least shed some light on the interaction between the dispersive term and the dissipative term that our equation presents, in relation to the mathematical questions on local and global well-posedness; and persistence properties of spatially decaying. This latter is of great interest for the particular physics models contained in our equation. Precisely, when compared with previous numerical studies on the spatially decaying of their solitary waves.

For the parameters $\alpha > \beta > 0$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, we shall consider the following *dispersive-dissipative*, nonlocal and nonlinear equation:

$$\begin{cases} \partial_t u + D(\partial_x u) + \Big(D^\alpha_x - D^\beta_x\Big) u + \gamma_1\, \partial_x(u^2) + \gamma_2\, \partial^2_x(u^2) + \gamma_3\, (\partial_x u)^2 = 0, \quad (t,x) \in (0,+\infty)\times\mathbb{R}, \\ u(0,\cdot) = u_0. \end{cases} \tag{1}$$

Here, the function $u : [0,+\infty)\times\mathbb{R} \to \mathbb{R}$ denotes the solution and the function $u_0 : \mathbb{R} \to \mathbb{R}$ is the initial datum.

The *dispersive* effects are characterized by the term $D(\partial_x u)$, where the operator D is given by $D = \partial^2_x$ or by $D = \mathcal{H}\partial_x$. In this last expression $\mathcal{H}$ denotes the Hilbert transform, which is a nonlocal operator defined in the Fourier variable as

$$\widehat{\mathcal{H}}(\varphi) = -i\,\text{sing}(\xi)\,\widehat{\varphi}(\xi), \tag{2}$$

where $\text{sing}(\xi)$ is the sing function and $\varphi \in \mathcal{S}(\mathbb{R})$. Therefore, in the Fourier variable we have

$$\widehat{D\varphi}(\xi) = m(\xi)\widehat{\varphi}(\xi), \quad \text{where} \quad m(\xi) = \begin{cases} -|\xi|^2, \text{ when } D = \partial^2_x, \\ |\xi|, \text{ when } D = \mathcal{H}\partial_x. \end{cases} \tag{3}$$

The whole term $D(\partial_x u)$ describes the linearized dispersion relation in the equation (1).

The *dissipative* action of the equation is given by the term $D^\alpha_x - D^\beta_x$. These two fractional derivative operators are easily defined in the Fourier variable by the expressions

$$\widehat{D^\alpha_x\varphi}(\xi) = c_\alpha|\xi|^\alpha\,\widehat{\varphi}(\xi), \qquad \widehat{D^\beta_x\varphi}(\xi) = c_\beta|\xi|^\beta\,\widehat{\varphi}(\xi). \tag{4}$$

Thus, the total dissipative action of the equation (1), in terms of Fourier variable, is essentially given is given by the symbol $|\xi|^\alpha - |\xi|^\beta$.

From a purely physical perspective, this model is not unreal since physical phenomena that are purely dissipative or purely dispersive are rarely found. This same fact makes interesting to study equation (1) from a mathematical point of view.

Finally, the nonlinear part of equation (1) is described by the term $\gamma_1\,\partial_x(u^2)$, which represents the classical transport term in fluid models, and by the terms $\gamma_2\,\partial^2_x(u^2)$, $\gamma_3\,(\partial_x u)^2$, taken from the (KV) model introduced above. In particular, these last terms allow the model to have a greater mathematical richness, on the one hand, a blow-up criterion in the well-posedness theory (see the Proposition 1 below) and, on the other hand, some optimal spatial decaying rates of solutions (see the Corollary 1 below).

One of the main interest of the equation (1) is based on the fact that it contains the following *physically relevant models* as a particular case. This is not an exhaustive list, but we shall mention the most representative ones. We shall divide them into two main groups according to *dispersive effects* of the term $D(\partial_x)$.

Nonlocal dispersive effects. We consider here $D = \mathcal{H}\partial_x$ and then $D(\partial_x) = \mathcal{H}\partial^2_x$. The nonlocal effects of this term are given by the Hilbert transform $\mathcal{H}$ (defined in (2)) and in this group we have the following models.

- By setting $\alpha = 3$, $\beta = 1$ and $\gamma_2 = \gamma_3 = 0$, the equation (1) agrees with the nonlocal perturbed *Benjamin-Ono* equation:

$$\partial_t u + \mathcal{H}\partial^2_x u + \mathcal{H}\Big(\partial^3_x u + \partial_x u\Big) + \partial_x(u^2) = 0. \tag{5}$$

  This equation is a good approximated model for long-crested unidirectional waves at the interface of a two-layer system of deep stratified incompressible inviscid fluids [4].

- When $\alpha = 4$, $\beta = 1$ and $\gamma_2 = \gamma_3 = 0$, the equation (1) writes down as another relevant physical model:

$$\partial_t u + \mathcal{H}\partial_x^2 u + \partial_x^4 u + \mathcal{H}\partial_x u + \partial_x(u^2) = 0. \tag{6}$$

  This equation provides a successful model in plasma theory [25].

- More generally, for $\alpha > \beta > 0$, and $\gamma_2 = \gamma_3 = 0$, the equation (1) becomes the following modified *Benjamin-Ono* equation:

$$\partial_t + \mathcal{H}\partial_x^2 u + \left(D_x^\alpha - D_x^\beta\right)u + \partial_x(u^2) = 0. \tag{7}$$

  This equation was introduced in [22] as a theoretical model to sharply study the well-posedness issues, which are driven by the parameters α and β.

Local dispersive effects. In this case, we consider $D = \partial_x^2$ (the classical Laplacian operator) and we obtain $D(\partial_x) = \partial_x^3$. Among the models containing this dispersive term, it is worth mentioning the following ones.

- For $\alpha > \beta > 0$ and $\gamma_2 = \gamma_3 = 0$, the equation (1) writes down as the following modified *KdV* equation:

$$\partial_t u + \partial_x^3 u + \left(D_x^\alpha - D_x^\beta\right)u + \partial_x(u^2) = 0. \tag{8}$$

  To the best of our knowledge, this equation has not been studied before; and it is a *KdV*-counterpart of the equation (7). Precisely, its main interest is the study of the dispersive effects of the term $\partial_x^3 u$, when compared with the effects of the dispersive term $\mathcal{H}\partial_x^2 u$ in the equation (7).

- When $\alpha = 3$, $\beta = 1$ and $\gamma_2 = \gamma_3 = 0$ we have the *OST* equation:

$$\partial_t u + \partial_x^3 u + \mathcal{H}\left(\partial_x^3 u + \partial_x u\right) + \partial_x(u^2) = 0, \tag{9}$$

  which describes the radiational instability of long non-linear waves in a stratified flow caused by internal wave radiation from a shear layer [18, 19, 20].

- Finally, when $\alpha = 4$, $\beta = 2$ and $\gamma_1 = 0$, the equation (1) becomes the dispersive *Kuramoto-Velarde* equation:

$$\partial_t u + \partial_x^2 u + \partial_x^3 u + \partial_x^4 u + \gamma_2\, \partial_x^2(u^2) + \gamma_3\, (\partial_x u)^2 = 0, \tag{10}$$

  and moreover, when we set $\gamma_2 = 0$ we obtain the 1D- *Kuramoto-Sivashinsky* equation:

$$\partial_t u + \partial_x^2 u + \partial_x^3 u + \partial_x^4 u + \gamma_3\, (\partial_x u)^2 = 0. \tag{11}$$

  The physics interest of both models was explained above.

As mentioned, the main objective of this paper is to focus on two relevant issues for equation (1): a well-posedness theory in the setting of the Sobolev spaces and the persistence problem of the spatially decaying of solutions. It is worth emphasizing these qualitative properties deeply depend on the parameters α, β in the dissipative term $\left(D_x^\alpha - D_x^\beta\right)u$, on the parameters γ_2, γ_3 in the nonlinear term $\gamma_2\partial_x^2(u^2) + \gamma_3(\partial_x u)^2$ as well as on the operator D in the dispersive term $D(\partial_x u)$.

2 The main results

2.1 Well-posedness

We recall that the equation (1) is locally well-posed in the space $H^s(\mathbb{R})$ (with $s \in \mathbb{R}$) if for any initial datum $u_0 \in H^s(\mathbb{R})$ there exists a time $0 < T = T(\|u_0\|_{H^s})$ and there exists a unique solution $u(t,x)$ to the equation (1) in a space $E_T \subset \mathcal{C}([0,T], H^s(\mathbb{R}))$, such that the flow-map data-solution:

$$S : H^s(\mathbb{R}) \to E_T \subset \mathcal{C}([0,T], H^s(\mathbb{R})), \quad u_0 \mapsto S(t)u_0 = u(t,\cdot), \tag{12}$$

is a locally continuous function from $H^s(\mathbb{R})$ to E_T.

As mentioned, the local well-posedness (LWP) of the equation (1) is driven by the parameter α in its dissipative term. Precisely, the constraint $\alpha > 7/2$ will allow us to handle the strong nonlinear effects of the terms $\gamma_2 \partial_x^2(u^2) + \gamma_3(\partial_x u)^2$ (see the Remark 1 below for more technical details on this fact) while in the case $\gamma_2 = \gamma_3 = 0$ this constraint is relaxed to $\alpha > 2$. Thus, our first result states as follows:

Theorem 1 (LWP) *Let $\alpha > \beta > 0$ and let $\gamma_1, \gamma_2, \gamma_3$ be the parameters in the equation (1).*

1. *Let $\gamma_2, \gamma_3 \neq 0$. We set $\alpha > 7/2$ and then the equation (1) is locally well-posed in the Sobolev space $H^s(\mathbb{R})$ with $s > 1-\alpha/2$. Moreover, we have $u \in \mathcal{C}^1(]0,T], \mathcal{C}^\infty(\mathbb{R}))$ and the flow-map function S defined in (12) is smooth.*

2. *Let $\gamma_2 = \gamma_3 = 0$. We set $\alpha > 2$ and then the equation is locally well-posed in $H^s(\mathbb{R})$ with $s > \max(3/2-\alpha, -\alpha/2)$. As above, we have $u \in \mathcal{C}^1(]0,T], \mathcal{C}^\infty(\mathbb{R}))$ and the flow-map function S is smooth.*

It is important to emphasize that one of the main interest of this theorem lies in the understanding of the relationship between the parameters α, γ_2 and γ_3 with the well-posedness theory for the equation (1).

This theorem also recovers some known results on the local well-posedness for the particular physical models introduced above, among them, the nonlocal perturbed *Benjamin-Ono* equation (5) studied in [10], the plasma model (6) and the modified *Benjamin-Ono* equation (7) studied in [22], the *OST* equation (9) investigated in [28], the dispersive *Kuramoto-Velarde* equation (10) investigated in [23] and the $1D-$*Kuramoto-Sivashinsky* (11) equation studied in [3].

Compared with these results, the novelty of this theorem is the fact that even for negative values of s, H^s-initial data yield classical solutions to the equation (1) since they also belong to the space $\mathcal{C}^1(]0,T], \mathcal{C}^\infty(\mathbb{R}))$. In particular, for the equations (10) and (11) this theorem improves the result obtained in [7], where the existence of classical solutions is proven for H^2-initial data verifying some additional smallness conditions.

Finally, the second point above provides us a new locally well-posedness result for the modified *KdV* equation (8), which (to our knowledge) has not been studied before.

On the other hand, we observe that the minimal regularity (measured by the parameter s) to prove the local well-posedness in the space $H^s(\mathbb{R})$ also depends on the parameter α through the conditions $s > 1-\alpha/2$ when $\gamma_2, \gamma_3 \neq 0$ and $s > \min(3/2-\alpha, -\alpha/2)$ when $\gamma_2 = \gamma_3 = 0$. In our second result, we prove that the quantities $1-\alpha/2$ and $-\alpha/2$ are sharp in the local well-posedness theory in the following sense:

Theorem 2 (Sharp LWP)

1. *Let $\alpha > \beta > 0$ with $\alpha > 7/2$ and $\gamma_2, \gamma_3 \neq 0$. Let $s < 1-\alpha/2$. If the equation (1) is locally well-posed in $H^s(\mathbb{R})$ then the flow-map function S in not a C^2- function at $u_0 = 0$.*

2. *Let $\alpha > \beta > 0$ with $\alpha > 2$ and $\gamma_2 = \gamma_3 = 0$. Let $s < -\alpha/2$. If the equation (1) is locally well-posed in $H^s(\mathbb{R})$ then the flow-map function S in not a C^2- function at $u_0 = 0$.*

Next, we are interested in studying the global well-posedness (GWP) of the equation (1). We recall that this equation is globally well-posed in $H^s(\mathbb{R})$ if the properties mentioned above hold true for any time $0 < T$. In our third result, we show that the GWP is driven by the parameters γ_2 and γ_3 in the nonlinear term $\gamma_2 \partial_x^2(u^2) + \gamma_3(\partial_x u)^2$.

Theorem 3 (GWP) *Within the framework of Theorem 1, the equation (1) is globally well-posed in the space $H^s(\mathbb{R})$ (with $s > 1 - \alpha/2$ or $s > \max(3/2 - \alpha, -\alpha/2)$) when $-2\gamma_2 + \gamma_3 = 0$.*

In the particular case when $\gamma_2 = \gamma_3 = 0$, we recover the GWP for the set of models (5), (6), (7), (9) (see the references mentioned above). Moreover, we give a new GWP result for the *modified KdV* equation (8).

Concerning the *Kuramoto-Sivashinsky* equation (11) (where $\gamma_2 = 0$) the constraint $-2\gamma_2 + \gamma_3 = 0$ implies that $\gamma_3 = 0$, and consequently, this result trivially holds true for the linear version of this equation. It is worth emphasizing this fact is coherent with [24], where it is shown that the nonlinear term $\gamma_3(\partial_x u)^2$ yields finite blow-up of solutions to the equation (11) associated with a large class of initial data.

Finally, for the *dipersive Kuramoto-Velarde* equation (10), we are able to ensure its GWP as long as $-2\gamma_2 + \gamma_3 = 0$, which was pointed out in [23]. The GWP or blow-up phenom in the case $-2\gamma_2 + \gamma_3 \neq 0$ remains an open question far from obvious and, in future research, we aim to give a deeper understanding of the effects of the nonlinear term $\gamma_2 \partial_x^2(u^2) + \gamma_3(\partial_x u)^2$ in the GWP theory. However, by performing some new energy estimates we are able to prove the following:

Proposition 1 (Blow-up criterion) *Within the framework of Theorem 1, assume that γ_2 and γ_3 are such that $-2\gamma_2 + \gamma_3 \neq 0$. Then, for a time $0 < T^* < +\infty$ we have:*

$$\lim_{t \to T^*} \|u(t,\cdot)\|_{H^s} = +\infty \qquad \textit{if and only if} \qquad \int_0^{T^*} \|\partial_x^2 u(t,\cdot)\|_{L^\infty} dt = +\infty.$$

This result gives us a new blow-up criterion for the *dispersive Kuramoto-Velarde* equation (10) and its related models containing the nonlinear term $\gamma_2 \partial_x^2(u^2) + \gamma_3(\partial_x u)^2$.

2.2 Spatially decaying

In this section we study another relevant qualitative property of the equation (1): the pointwise decaying of solutions $u(t,x)$ respect to the spatial variable x. This question gives us a good comprehension of the terms in this equation governing the spatially behavior of solutions; and it is also of physical interest when particularizing in the models introduced above. Specifically, when comparing with the spatially behavior of a relevant kind of particular solutions, the so-called *solitary waves.*

From the nonlinear differential equations point of view, the existence of the *solitary wave* describes a perfect balance between the nonlinearity and the dispersive character of its linear part. We refer to the book [17] for more details. Concerning the physics models introduced above, there exist previous *numerical works* on the spatially decaying of solitary waves. These works give some light on the spatially decaying of solutions to these equations. In this context, the main contribution of this work is to use the general framework of the equation (1) to *analytically* study the spatially decaying of solutions, which simultaneously holds true for the particular physics models contained in this equation.

As we shall observe, our main remark is that these decaying properties of solutions to the equation (1) are driven by both the dispersive term $D(\partial_x u)$ and the dissipative term $(D_x^\alpha - D_x^\beta)u$ in the equation (1). Precisely, by the operator D defined in (3) and the parameters α and β. We introduce here the parameter

$n \geq 2$, which depends on D, α and β, as follows:

$$n = \begin{cases} \min(3, [\beta]+1), & \text{when } D = \mathcal{H}\partial_x, \\ \text{any natural number}, & \text{when } D = -\partial_2^2 \text{ and } \alpha, \beta \text{ are both even numbers}, \\ [\alpha]+1, & \text{when } D = -\partial_x^2,\ \beta \text{ is an even number and not } \alpha, \\ [\beta]+1, & \text{when } D = -\partial_x^2 \text{ and } \beta \text{ is not an even number}, \end{cases} \tag{13}$$

where $[\alpha]$ and $[\beta]$ denote the integer part of α and β respectively. The parameter n gives us a detailed description on the *pointwise decaying rate* of solutions to the equation (1); and our next result reads as follows:

Theorem 4 (Spatially pointwise decaying) *Let $s > \frac{5}{2}$ and let $u_0 \in H^s(\mathbb{R})$ be an initial datum. Let $\alpha > \beta \geq 1$, with $\alpha > 7/2$ when $\gamma_2, \gamma_3 \neq 0$ and $\alpha > 2$ when $\gamma_2 = \gamma_3 = 0$. Moreover, let $u \in \mathcal{C}([0,T], H^s(\mathbb{R}))$ be the solution to the equation (1) associated to u_0, given by Theorem 1.*

Let $\kappa > 1$ and assume that the initial datum u_0 verifies

$$|u_0(x)| \leq \frac{c_0}{1+|x|^\kappa}, \quad x \in \mathbb{R}, \tag{14}$$

with a constant $c_0 > 0$. Then the solution $u(t,x)$ verifies the following pointwise estimate

$$|u(t,x)| \leq \frac{c_1(t,u)}{1+|x|^{\min(\kappa,n)}}, \quad 0 < t < T, \quad x \in \mathbb{R}, \tag{15}$$

with a constant $c_1(t,u) > 0$ depending on t, u and $c_0 > 0$; and where the parameter $n \geq 2$ defined in (13).

Let us make the following comments. The assumption of the initial data $u_0 \in H^s(\mathbb{R})$ with $s > 5/2$ ensures that the arising solution verifies $u(t,\cdot) \in H^s(\mathbb{R})$ for all $t \geq 0$ (see Theorem 1). In particular, the technical constraint $s > 5/2$ allows us to handle the nonlinear terms in the equation (1).

In the expressions (14) and (15), we may observe that the parameter n controls the decaying properties of solutions: the solution $u(t,x)$ fulfills the decaying given by the initial datum only if $\kappa \leq n$. But, for initial data decaying fast enough ($\kappa > n$) the corresponding solution does not mimic this decaying rate and it decays at infinity like $1/|x|^n$.

From now on, we shall assume initial data decaying fast enough: $\kappa > n$, and we shall discuss more in detail the decaying estimate verified by the solution $|u(t,x)| \lesssim 1/|x|^n$. To do this, recall that the parameter n ultimately depends on the operator D and the parameters α, β according to the expression (13).

- When $D = \mathcal{H}\partial_x$ the nonlocal effects of this operator have a strong influence in the spatially decaying properties of solutions. Precisely, in this case we have $n = \max(3, [\beta]+1)$, where the number 3 is due to the presence of the Hilbert transform $\mathcal{H}$. See the Proposition 4.1 below for more details. Consequently, the physical models containing the dispersive term $\mathcal{H}\partial_x^2 u$ verify the estimate

$$|u(t,x)| \lesssim \frac{1}{|x|^{\max(3,[\beta]+1)}}. \tag{16}$$

  In particular, solutions to the *plasma model* (6) and solutions to the nonlocal perturbed *Benjamin-Ono* equation (5) (in both cases we have $\beta = 1$) have the spatially decaying $|u(t,x)| \lesssim \frac{1}{|x|^2}$. For this last equation, this information is coherent with [2], where the authors numerically prove that *solitary waves* this equation behave at infinity as $1/|x|^2$. On the other hand, the spatially decaying properties of solutions to the modified *Benjamin-Ono* equation (7) have not been studied before; and they satisfy the spatially decaying (16). Here, we also realize the effects of the parameter β in the dissipative term, while the parameter α does not intervene.

- When $D = -\partial_x^2$ it is interesting to observe that the local effects of this operator do not influence the decaying properties of solutions, which are now driven by the parameters α and β. Here, we have the following cases.
  - When α and β are both even numbers, one can set any parameter $n \in \mathbb{N}$ (with $n \geq 2$); and for initial data verifying (14) with $\kappa > N$, solutions to the equation (1) verify the estimate
$$|u(t,x)| \lesssim \frac{1}{|x|^n}.$$
This *persistence problem* is verified for the *dispersive Kuramoto-Velarde* equation (10) and the *Kuramoto-Sivashinsky* equation (11). Moreover, this fact is in concordance to some numerical studies on the well-localized solitary waves to these equations. See for instance [5] and [6].
  - When β is a even number but not α, solutions to the equation (1) have a decaying rate
$$|u(t,x)| \lesssim \frac{1}{|x|^{[\alpha]+1}},$$
while β is not an even number it holds
$$|u(t,x)| \lesssim \frac{1}{|x|^{[\beta]+1}}.$$
These decaying rates are verified by the modified *KdV* equation (8) according to these cases of the parameters α and β. Moreover, it is interesting to get back to the modified *Benjamin-Ono* equation (7), which verifies the decaying rate (16), to highlight the stronger effects of the dissipative term $\mathcal{H}\partial_x^2 u$ compared with the dissipative term $-\partial_x^3 u$.
  - Finally, for $\beta = 1$, solutions to the *OST* equation (9) verify the decaying rate
$$|u(t,x)| \lesssim \frac{1}{|x|^2}.$$
This spatially decaying agrees with numerical studies performed on *solitary waves* in [1]; and it was also analytically proven in our previous work [8].

Now, we are interested in studying the optimality of the decaying rate (15) (with $\kappa > n$). To do this, first we shall find an asymptotic profile for the solution $u(t,x)$ to the equation (1). In order to state our next theorem, we need to introduce function $K_\alpha, \beta(t,x)$ which is obtained as the solution of the linear problem (when $\gamma_1 = \gamma_2 = \gamma_3 = 0$) of the equation (1):

$$\begin{cases} \partial_t K_{\alpha,\beta} + D(\partial_x K_{\alpha,\beta}) + \left(D_x^\alpha - D_x^\beta\right) K_{\alpha,\beta} = 0, & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ K_{\alpha,\beta}(0,\cdot) = \delta_0, \end{cases} \tag{17}$$

where δ_0 denotes the Dirac mass at the origin. It is thus interesting to observe that the asymptotic profile of the solution $u(t,x)$ to the equation (1) is essentially given by the function $K_{\alpha,\beta}(t,x)$. Precisely, we start by studying the pointwise decaying (in the spatial variable) of this function.

Proposition 2 *Let $n \geq 2$ be the parameter given in the expression (13). Moreover, let $\alpha > \beta \geq 1$, with $\alpha > 2$. For all $t > 0$ fixed, there exists a quantity $I(t)$, which verifies $|I(t)| \leq C_1 e^{\eta_1 t}$ with constants $C_1, \eta_1 > 0$ depending on α and β, such that the following identity holds:*

$$|K_{\alpha,\beta}(t,x)| = \frac{|I(t)|}{|x|^n}, \quad t > 0, \quad x \neq 0. \tag{18}$$

Then, our next theorem writes down as follows:

Theorem 5 (Asymptotic profile) *Let $u_0 \in H^s(\mathbb{R})$ (with $s > \frac{5}{2}$) be an initial datum verifying (14) with $\kappa > n$; and where the parameter $n \geq 2$ is defined in (13). Let $u \in \mathcal{C}([0,T], H^s(\mathbb{R}))$ (with $s > \frac{5}{2}$) be the associated solution to the equation (1) given by Theorem 1.*

For $0 < t \leq T$ fixed, this solution has the following asymptotic development in spatial variable

$$u(t,x) = K_{\alpha,\beta}(t,x)\left(\int_{\mathbb{R}} u_0(y)dy\right) + \gamma_3\, K_{\alpha,\beta}(t,x)\int_0^t \left(\int_{\mathbb{R}} (\partial_x u)^2(\tau,y)dy\right) d\tau + R(t,x), \quad |x| \to +\infty, \tag{19}$$

where

$$|R(t,x)| \leq \frac{c_2(t,u)}{|x|^{n+\varepsilon}}, \quad 0 < \varepsilon \leq 1, \tag{20}$$

with a constant $c_2(t,u) > 0$ depending on t and u.

From this asymptotic development we can deduce some optimally decaying properties of the solution $u(t,x)$. First, the expression $\gamma_3\, K_{\alpha,\beta}(t,x)\displaystyle\int_0^t \left(\int_{\mathbb{R}} (\partial_x u)^2(\tau,y)dt\right) d\tau$ highlights interesting effects of the nonlinear $\gamma_3(\partial_x u)^2$ in the spatially decaying of solutions. Precisely, when $\gamma_3 \neq 0$ this expression yields the following estimate from below:

Corollary 1 *Within the framework of Theorem 5, assume that $\gamma_3 \neq 0$. Then the solution $u(t,x)$ to the equation (1) verifies:*

$$\frac{c_3(u_0,\gamma_3,t,u)}{|x|^n} \leq |u(t,x)|, \quad |x| \to +\infty, \tag{21}$$

where the quantity $c_3(u_0,\gamma_3,t,u) > 0$ (given in (106)) depends on u_0, γ_3, t and u but it is independent of the variable x.

Consequently, the physical models containing the nonlinear term $\gamma_3(\partial_x u)^2$ have an *optimal decaying rate*:

$$|u(t,x)| \sim \frac{1}{|x|^n}, \qquad |x| \to +\infty. \tag{22}$$

In particular, this optimal decaying rate is verified by the *dispersive Kuramoto-Velarde* equation (10) and the *Kuramoto-Sivashinsky* equation (11).

We study now the case when $\gamma_3 = 0$. Here, solutions to the equation (1) have the asymptotic profile:

$$u(t,x) = K_{\alpha,\beta}(t,x)\left(\int_{\mathbb{R}} u_0(y)dy\right) + R(t,x), \quad |x| \to +\infty,$$

where the optimality properties are now driven by the term $\displaystyle\int_{\mathbb{R}} u_0(y)dy$.

Corollary 2 *Within the framework of Theorem 5, assume that $\gamma_3 = 0$. In this case, we have the following scenarios:*

1. *If the initial datum u_0 verifies $\displaystyle\int_{\mathbb{R}} u_0(y)dy \neq 0$, then the solution $u(t,x)$ to the equation (1) verifies the estimate from below:*

$$\frac{c_4(u_0,t)}{|x|^n} \leq |u(t,x)|, \quad |x| \to +\infty, \tag{23}$$

where the quantity $c_4(u_0,t) > 0$ (given in (107)) depends on u_0 and t but it is independent of x.

2. *Otherwise, if the initial data verifies* $\displaystyle\int_{\mathbb{R}} u_0(y)dy = 0$*, then the solution* $u(t,x)$ *to the equation (1) verifies the estimate from above:*

$$|u(t,x)| \leq \frac{c_2(t,u)}{|x|^{n+\varepsilon}}, \quad 0 < \varepsilon \leq 1, \quad |x| \to +\infty. \tag{24}$$

In the first point above, we obtain an *optimal decaying rate* of solutions (22) as long as $\int_{\mathbb{R}} u_0(y)dy \neq 0$. In particular, this property is verified by the physical models from the equation (5) to the equation (9), with the respective values of the parameter n detailed above.

On the other hand, the second point above shows us that this decaying rate can be improved to $|u(t,x)| \lesssim 1/|x|^{n+\varepsilon}$ (with $0 < \varepsilon \leq 1$) in the case of zero-mean initial data. To the best of our knowledge, the value $\varepsilon = 1$ seems to be the maximal one to improve the decaying rate. This is due to the fact that the solutions of equation (1) are written in an explicit *mild formulation* (27) involving the function $K_{\alpha,\beta}(t,x)$ defined above. So, the sharp spatially decaying properties of this function (given in Proposition 2 above) eventually block an improvement in the decaying of the solution for $\varepsilon > 1$.

Notation. In order to get rid of some unsubstantial constants, for $A, B > 0$ we will use the notation $A \lesssim B$ to mean that $A \leq cB$ with a constant $c > 0$ which does not depend on A nor B. Similarly, we shall write $A \sim B$ when $c_1 A \leq B \leq c_2 B$. On the other hand, the Fourier transform (in the spatial variable) of a function f is denoted by $\widehat{f}$ or $\mathcal{F}(f)$, while $\mathcal{F}^{-1}(f)$ stands for the inverse Fourier transform.

Organization of the paper. This paper is divided in two big sections: in Section 3 we give a proof of all the results stated in the well-posedness theory, while Section 4 is devoted to proving all the results stated in the spatially decaying theory.

3 The well-posedness theory

3.1 Kernel estimates I

Let $K_{\alpha,\beta}(t,x)$ be the solution of the linear problem (17). Then, by definition of the operators D, D^α_x and D^β_x, given in the formulas (3) and (4) respectively, for all $t > 0$ we have

$$\begin{aligned} K_{\alpha,\beta}(t,x) =& \mathcal{F}^{-1}\left(e^{-(i\,m(\xi)\xi + (|\xi|^\alpha - |\xi|^\beta))t}\right)(x) \\ =& \mathcal{F}^{-1}\left(e^{-f(\xi)t}\right)(x), \end{aligned} \qquad \text{with} \quad f(\xi) = i\,m(\xi)\xi + (|\xi|^\alpha - |\xi|^\beta). \tag{25}$$

In what follows we summarize some properties of the kernel $K_{\alpha,\beta}$, which will be useful in the sequel.

Lemma 3.1 *Let* $\sigma \geq 0$*. For all* $t > 0$ *we have* $\left\| \left| t^{\frac{1}{\alpha}}\,\xi \right|^{2\sigma} \widehat{K_{\alpha,\beta}}(t,\cdot) \right\|_{L^\infty} \lesssim t^{\frac{2\sigma}{\alpha}}\, e^t + t^{\frac{2\sigma}{\alpha}} + 1.$

Proof. By (25) for all $\xi \in \mathbb{R}$ we have $\left|\widehat{K_{\alpha,\beta}}(t,\xi)\right| = e^{-(|\xi|^\alpha - |\xi|^\beta)t}$. Then, we write

$$\begin{aligned} \left\| \left| t^{\frac{1}{\alpha}}\,\xi \right|^{2\sigma} \widehat{K_{\alpha,\beta}}(t,\cdot) \right\|_{L^\infty} \leq & \left\| \left| t^{\frac{1}{\alpha}}\,\xi \right|^{2\sigma} e^{-(|\xi|^\alpha - |\xi|^\beta)t} \right\|_{L^\infty\left(|\xi| \leq 2^{\frac{1}{\alpha-\beta}}\right)} + \left\| \left| t^{\frac{1}{\alpha}}\,\xi \right|^{2\sigma} e^{-(|\xi|^\alpha - |\xi|^\beta)t} \right\|_{L^\infty\left(|\xi| > 2^{\frac{1}{\alpha-\beta}}\right)} \\ =& I_1 + I_2. \end{aligned} \tag{26}$$

We observe that the term I_1 above can be split as:

$$I_1 \leq \left\| \left| t^{\frac{1}{\alpha}}\,\xi \right|^{2\sigma} e^{-(|\xi|^\alpha - |\xi|^\beta)t} \right\|_{L^\infty(|\xi| \leq 1)} + \left\| \left| t^{\frac{1}{\alpha}}\,\xi \right|^{2\sigma} e^{-(|\xi|^\alpha - |\xi|^\beta)t} \right\|_{L^\infty\left(1 < |\xi| \leq 2^{\frac{1}{\alpha-\beta}}\right)} = I_{1,a} + I_{1,b}.$$

Here, to estimate the term $I_{1,a}$, since $|\xi|\leq 1$ we write $-(|\xi|^\alpha-|\xi|^\beta)t=(|\xi|^\beta-|\xi|^\alpha)t\leq |\xi|^\beta t\leq t$. We thus get $I_{1,a}\lesssim t^{\frac{2\sigma}{\alpha}}e^t$. Similarly, to estimate the $I_{1,b}$, since $1<|\xi|\leq 2^{\frac{1}{\alpha-\beta}}$ we have $-(|\xi|^\alpha-|\xi|^\beta)<0$ and we obtain $I_{1,b}\lesssim t^{\frac{2\sigma}{\alpha}}$. On the other hand, to estimate the term I_2, since $|\xi|>2^{\frac{1}{\alpha-\beta}}$ we get $-(|\xi|^\alpha-|\xi|^\beta)\leq -\frac{1}{2}|\xi|^\alpha$ and we can write

$$I_2\lesssim \left\| \left| t^{\frac{1}{\alpha}}\xi\right|^{2\sigma} e^{-|\xi|^\alpha t}\right\|_{L^\infty\left(|\xi|>2^{\frac{1}{\alpha-\beta}}\right)} \lesssim \left\| \left| t^{\frac{1}{\alpha}}\xi\right|^{2\sigma} e^{-|t^{\frac{1}{\alpha}}\xi|^\alpha}\right\|_{L^\infty(\mathbb{R})}\lesssim 1.$$

By gathering the estimates on the terms $I_{1,a}$, $I_{1,b}$ and I_2 we obtain the wished result. ■

With this estimate, we are able to prove the following result.

Lemma 3.2 *Let $s\in\mathbb{R}$ and let $s_1\geq 0$. There exists a constant $\eta_0>0$, which depends on s and α, such that the following estimate holds $\|K_{\alpha,\beta}(t,\cdot)*\psi\|_{H^{s+s_1}}\lesssim \dfrac{e^{\eta_0 t}}{t^{\frac{s_1}{\alpha}}}\|\psi\|_{H^s}$.*

Proof. By the Hölder inequalities we write:

$$\|K_{\alpha,\beta}(t,\cdot)*\psi\|_{H^{s+s_1}}=\left\|(1+|\xi|^2)^{\frac{s+s_1}{2}}\widehat{K_{\alpha,\beta}}(t,\cdot)\,\widehat{u_0}\right\|_{L^2}\leq \left\|(1+|\xi|^2)^{\frac{s_1}{2}}\widehat{K_{\alpha,\beta}}(t,\cdot)\right\|_{L^\infty}\left\|(1+|\xi|^2)^{\frac{s}{2}}\widehat{\psi}\right\|_{L^2},$$

where we must estimate the quantity $\left\|(1+|\xi|)^{\frac{s_1}{2}}\widehat{K_{\alpha,\beta}}(t,\cdot)\right\|_{L^\infty}$. We thus have

$$\left\|(1+|\xi|^2)^{\frac{s_1}{2}}\widehat{K_{\alpha,\beta}}(t,\cdot)\right\|_{L^\infty}\lesssim \left\|\widehat{K_{\alpha,\beta}}(t,\cdot)\right\|_{L^\infty}+\left\||\xi|^{s_1}\widehat{K_{\alpha,\beta}}(t,\cdot)\right\|_{L^\infty}.$$

Now, in order to estimate the first term on the right-hand side, by Lemma 3.1 (with $\sigma=0$) we obtain

$$\left\|\widehat{K_{\alpha,\beta}}(t,\cdot)\right\|_{L^\infty}\lesssim e^t+1.$$

For the second term on the right-hand side, we use again the Lemma 3.1 (with $\sigma=\frac{s_1}{2}$) to get

$$\left\||\xi|^{s_1}\widehat{K_{\alpha,\beta}}(t,\cdot)\right\|_{L^\infty}=\frac{1}{t^{\frac{s_1}{\alpha}}}\left\|\left|t^{\frac{1}{\alpha}}\xi\right|^{s_1}\widehat{K_{\alpha,\beta}}(t,\cdot)\right\|_{L^\infty}\lesssim e^t+1+\frac{1}{t^{\frac{s_1}{\alpha}}}.$$

By gathering these estimates, and by setting a quantity $\eta_0=\eta_(s_1,\alpha)>0$ big enough, we finally obtain

$$\left\|(1+|\xi|^2)^{\frac{s_1}{2}}\widehat{K_{\alpha,\beta}}(t,\cdot)\right\|_{L^\infty}\lesssim e^t+1+\frac{1}{t^{\frac{s_1}{\alpha}}}=\frac{e^t\,t^{\frac{s_1}{\alpha}}+t^{\frac{s_1}{\alpha}}+1}{t^{\frac{s_1}{\alpha}}}\lesssim\frac{e^{\eta_0 t}}{t^{\frac{s_1}{\alpha}}}.$$

■

Finally, we state our last technical lemma. The proof essentially follows the same computations performed in [9, Lemma 4.1].

Lemma 3.3 *Let $s\in\mathbb{R}$, $\delta\geq 0$ and let $\varepsilon>0$. Then, these exists a constant $C=C(s,\varepsilon,\delta)>0$, such that for all $\varepsilon<t_1,t_2\leq T$ we have:*

$$\|K_{\alpha,\beta}(t_1,\cdot)*\psi-K_{\alpha,\beta}(t_2,\cdot)*\psi\|_{H^{s+\delta}}\leq C\,|t_1-t_2|^{1/2}\|\psi\|_{H^s}.$$

3.2 Sharp local well-posedness

Proof of Theorem 1

The case $\gamma_2, \gamma_3 \neq 0$ and $\alpha > 7/2$.

We divide the proof in three main steps, which we will prove in the technical theorems below. Precisely, in Theorem 3.1 we prove the local well-posedness of the equation (1) in a space $E^{s,\alpha}_T \subset \mathcal{C}([0,T], H^s(\mathbb{R}))$ for $1-\alpha/2 < s \leq 0$, while in Theorem 3.2 we prove the local-well posedness in a space $F^{s,\alpha}_T \subset \mathcal{C}([0,T], H^s(\mathbb{R}))$ for $0 < s$. Only for technical reasons, we shall divide or study in the cases $1-\alpha/2 < s \leq 0$ and $0 < s$. Finally, in Theorem 3.3 we study the regularity of solutions.

Local well-posedness. Solutions of the equation (1) are constructed as the solutions of the following (equivalent) problem:

$$u(t,x) = K_{\alpha,\beta}(t,\cdot) * u_0(x) - \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * \Big(\gamma_1 \partial_x(u^2) + \gamma_2\, \partial^2_x(u^2) + \gamma_3\, (\partial_x u)^2\Big)(\tau,x)\, d\tau, \tag{27}$$

where the kernel $K_{\alpha,\beta}(t,x)$ is defined in (25). The parameters $\gamma_1,\gamma_2,\gamma_3$ do not play any substantial role in local well-posedness theory, so for the sake of simplicity we shall set them as $\gamma_1 = \gamma_2 = \gamma_3 = 1$. On the other hand, we recall the following well-known estimate on the Beta function, which we shall fully use to study the nonlinear terms above. For $a > -1$ and $b > -1$ we have

$$\int_0^t (t-\tau)^a \tau^b d\tau \lesssim t^{a+b+1}. \tag{28}$$

The case $1-\alpha/2 < s \leq 0$. Let $7/2 < \alpha$ and let $0 < T$. We define the Banach space

$$E^{s,\alpha}_T = \{u \in \mathcal{C}([0,T], H^s(\mathbb{R})) : \|u\|_{s,\alpha} < +\infty\}, \tag{29}$$

with the norm

$$\|u\|_{s,\alpha} = \sup_{0\leq t \leq T} \|u(t,\cdot)\|_{H^s} + \sup_{0\leq t\leq T} t^{\frac{|s|}{\alpha}} \|u(t,\cdot)\|_{L^2} + \sup_{0\leq t\leq T} t^{\frac{1+|s|}{\alpha}} \|\partial_x u(t,\cdot)\|_{L^2}. \tag{30}$$

The second and the third term of this norm will be useful to handle the nonlinear terms in the equation (27) (see the Proposition 3.2 below). Now, for a time $0 < T \leq 1$ small enough, we shall construct a solution $u(t,x)$ to this equation in the space $E^{s,\alpha}_T$.

Theorem 3.1 *Let $\alpha > \beta > 0$ with $\alpha > 7/2$, let $1-\alpha/2 < s \leq 0$, and moreover, define the quantity*

$$\eta = \frac{s}{\alpha} - \frac{5}{2\alpha} + 1 > 0. \tag{31}$$

For any $u_0 \in H^s(\mathbb{R})$ there exists a time

$$T = T(\|u_0\|_{H^s}) < \min\left(\frac{1}{4^{1/\eta}\|u_0\|^{1/\eta}_{H^s}}, 1\right), \tag{32}$$

and a function $u \in E^{s,\alpha}_T$, which is the unique solution of the equation (27). Moreover, the flow-map function $S : H^s(\mathbb{R}) \to E^{s,\alpha}_T \subset \mathcal{C}([0,T], H^s(\mathbb{R}))$ defined in (12) is smooth.

Remark 1 *The quantity η appears in the estimates to handle the whole nonlinear term $\partial_x(u^2) + \partial^2_x(u^2) + (\partial_x u)^2$. See the Proposition 3.3 below. Thus, the constraints $7/2 < \alpha$ and $1-\alpha/2 < s$ ensure that $0 < \eta$. Indeed, by (31) the inequality $0 < \eta$ is equivalent to the inequality $\frac{5}{2} - \alpha < s$, but since $1-\alpha/2 < s$ and $7/2 < \alpha$ we can write $\frac{5}{2} - \alpha < 1 - \alpha/2 < s$.*

Proof. We start by studying the the linear term in the equation (27).

Proposition 3.1 *We have* $K_{\alpha,\beta}(t,\cdot)*u_0 \in E_T^{s,\alpha}$ *and* $\|K_{\alpha,\beta}(t,\cdot)*u_0\|_{s,\alpha} \lesssim \|u_0\|_{H^s}$.

Proof. We shall study separately each term in the norm $\|K_{\alpha,\beta}(t,\cdot)*u_0\|_{s,\alpha}$ defined in (30). For the first term, by Lemma 3.2 (with $s_1 = 0$) and since $0 < T \leq 1$ we get

$$\sup_{0\leq t\leq T} \|K_{\alpha,\beta}(t,\cdot)*u_0\|_{H^s} \lesssim \|u_0\|_{H^s}. \tag{33}$$

We also have $K_{\alpha,\beta}(t,\cdot)*u_0 \in \mathcal{C}([0,T], H^s(\mathbb{R}))$. Indeed, on the one hand, for $t=0$ by a standard convergence dominated argument we get $\lim_{t\to 0^+} \|K_{\alpha,\beta}(t,\cdot)*u_0 - u_0\|_{H^s} = 0$. On the other hand, by Lemma 3.3 (with $s_1 = 0$) we obtain $K_{\alpha,\beta}(t,\cdot)*u_0 \in \mathcal{C}((0,T], H^s(\mathbb{R}))$.

In order to estimate the second term, first we need to verify the following pointwise estimate:

$$t^{\frac{|s|}{\alpha}} \leq \frac{(1+\left|t^{\frac{1}{\alpha}}\xi\right|^2)^{\frac{|s|}{2}}}{(1+|\xi|^2)^{\frac{|s|}{2}}}. \tag{34}$$

Indeed, again by the fact that $0 \leq t \leq T \leq 1$, we just write

$$t^{\frac{|s|}{\alpha}}(1+|\xi|^2)^{\frac{|s|}{2}} = \left(t^{\frac{2}{\alpha}}\right)^{\frac{|s|}{2}}(1+|\xi|^2)^{\frac{|s|}{2}} = \left(t^{\frac{2}{\alpha}} + \left|t^{\frac{1}{\alpha}}\xi\right|^2\right)^{\frac{|s|}{2}} \leq \left(1+\left|t^{\frac{1}{\alpha}}\xi\right|^2\right)^{\frac{|s|}{2}}.$$

Once we have the estimate (34), we obtain

$$\begin{aligned} t^{\frac{|s|}{\alpha}} \|K_{\alpha,\beta}(t,\cdot)*u_0\|_{L^2} &\leq \left\| t^{\frac{|s|}{\alpha}} \widehat{K_{\alpha,\beta}}(t,\cdot)\,\widehat{u_0}\right\|_{L^2} \leq \left\| \frac{(1+\left|t^{\frac{1}{\alpha}}\xi\right|^2)^{\frac{|s|}{2}}}{(1+|\xi|^2)^{\frac{|s|}{2}}} \widehat{K_{\alpha,\beta}}(t,\cdot)\,\widehat{u_0}\right\|_{L^2} \\ &\leq \left\| \frac{(1+\left|t^{\frac{1}{\alpha}}\xi\right|^2)^{\frac{|s|}{2}}}{(1+|\xi|^2)^{\frac{|s|}{2}}(1+|\xi|^2)^{s/2}} \widehat{K_{\alpha,\beta}}(t,\cdot)\,(1+|\xi|^2)^{s/2}\,\widehat{u_0}\right\|_{L^2} \\ &= \left\| (1+\left|t^{\frac{1}{\alpha}}\xi\right|^2)^{\frac{|s|}{2}} \widehat{K_{\alpha,\beta}}(t,\cdot)\,(1+|\xi|^2)^{s/2}\,\widehat{u_0}\right\|_{L^2} = (A). \end{aligned}$$

The, by the Hölder inequalities and by Lemma 3.1 (by setting first $\sigma = 0$ and then $\sigma = \frac{|s|}{2}$) for all $0 \leq t \leq T \leq 1$ we get

$$\begin{aligned} (A) &\lesssim \left\| \widehat{K_{\alpha,\beta}}(t,\cdot)\,(1+|\xi|^2)^{s/2}\,\widehat{u_0}\right\|_{L^2} + \left\| (\left|t^{\frac{1}{\alpha}}\xi\right|^2)^{\frac{|s|}{2}} \widehat{K_{\alpha,\beta}}(t,\cdot)\,(1+|\xi|^2)^{s/2}\,\widehat{u_0}\right\|_{L^2} \\ &\lesssim \left\| \widehat{K_{\alpha,\beta}}(t,\cdot)\right\|_{L^\infty} \|u_0\|_{H^s} + \left\| (\left|t^{\frac{1}{\alpha}}\xi\right|^2)^{\frac{|s|}{2}} \widehat{K_{\alpha,\beta}}(t,\cdot)\right\|_{L^\infty} \|u_0\|_{H^s} \\ &\lesssim (e^t+1)\|u_0\|_{H^s} + (t^{\frac{|s|}{\alpha}}e^t + t^{\frac{|s|}{\alpha}} + 1)\|u_0\|_{H^s} \\ &\lesssim \|u_0\|_{H^s}. \end{aligned}$$

We thus obtain

$$\sup_{0\leq t\leq T} t^{\frac{|s|}{\alpha}} \|K_{\alpha,\beta}(t,\cdot)*u_0\|_{L^2} \lesssim \|u_0\|_{H^s}. \tag{35}$$

We study now the third term. First, we remark that by (34) we have

$$t^{\frac{1+|s|}{\alpha}} = t^{\frac{1}{\alpha}}\,t^{\frac{|s|}{\alpha}} \leq t^{\frac{1}{\alpha}} \frac{(1+\left|t^{\frac{1}{\alpha}}\xi\right|^2)^{\frac{|s|}{2}}}{(1+|\xi|^2)^{\frac{|s|}{2}}}.$$

Then, by following the same computations to prove the estimate (35), we write

$$
\begin{aligned}
t^{\frac{1+|s|}{\alpha}} \|\partial_x K_{\alpha,\beta}(t,\cdot) * u_0\|_{L^2} &\leq \left\| t^{\frac{1+|s|}{\alpha}} |\xi| \, \widehat{K_{\alpha,\beta}}(t,\cdot)\, \widehat{u_0} \right\|_{L^2} \\
&\leq \left\| \left| t^{\frac{1}{\alpha}}\xi \right| \frac{(1+\left|t^{\frac{1}{\alpha}}\xi\right|^2)^{\frac{|s|}{2}}}{(1+|\xi|^2)^{\frac{|s|}{2}}(1+|\xi|^2)^{\frac{s}{2}}} \widehat{K_{\alpha,\beta}}(t,\cdot)\,(1+|\xi|^2)^{\frac{s}{2}}\,\widehat{u_0} \right\|_{L^2} \\
&\lesssim \left\| \frac{(1+\left|t^{\frac{1}{\alpha}}\xi\right|^2)^{\frac{|s|+1}{2}}}{(1+|\xi|^2)^{\frac{|s|}{2}}(1+|\xi|^2)^{\frac{s}{2}}} \widehat{K_{\alpha,\beta}}(t,\cdot)\,(1+|\xi|^2)^{\frac{s}{2}}\,\widehat{u_0} \right\|_{L^2} \lesssim \|u_0\|_{H^s}.
\end{aligned}
$$

We thus obtain

$$
\sup_{0\leq t\leq T} t^{\frac{1+|s|}{\alpha}} \|\partial_x K_{\alpha,\beta}(t,\cdot) * u_0\|_{L^2} \lesssim \|u_0\|_{H^s}. \tag{36}
$$

The desired estimate follows from (33), (35) and (36). Proposition 3.1 is proven. ■

We study now the nonlinear terms in the equation (27). For this, we shall need the following useful technical estimates. Particularly, we shall observe the use of the second and the third terms in the norm $\|\cdot\|_{s,\alpha}$ given in (30).

Proposition 3.2 *Let $1-\alpha/2 < s \leq 0$ and let $0 \leq \sigma \leq 3$. For all $0 < t \leq T \leq 1$ the following estimates hold:*

1. $\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * (-\partial_x^2)^{\frac{\sigma}{2}}(u^2)(\tau,\cdot)\, d\tau \right\|_{H^s} \lesssim t^{\frac{s}{\alpha}-\frac{(\sigma+1/2)}{\alpha}+1} \left(\sup_{0\leq t\leq T} t^{\frac{|s|}{2}} \|u(t,\cdot)\|_{L^2} \right)^2.$

2. $\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * (-\partial_x^2)^{\frac{\sigma}{2}}(u^2)(\tau,\cdot)\, d\tau \right\|_{L^2} \lesssim t^{\frac{2s}{\alpha}-\frac{(\sigma+1/2)}{\alpha}+1} \left(\sup_{0\leq t\leq T} t^{\frac{|s|}{2}} \|u(t,\cdot)\|_{L^2} \right)^2.$

3. $\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * (-\partial_x^2)^{\frac{\sigma}{2}}((\partial_x u)^2)(\tau,\cdot)\, d\tau \right\|_{H^s} \lesssim t^{\frac{s}{\alpha}-\frac{(\sigma+5/2)}{\alpha}+1} \left(\sup_{0\leq t\leq T} t^{\frac{1+|s|}{\alpha}} \|\partial_x u(t,\cdot)\|_{L^2} \right)^2.$

4. $\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * (-\partial_x^2)^{\frac{\sigma}{2}}((\partial_x u)^2)(\tau,\cdot)\, d\tau \right\|_{L^2} \lesssim t^{\frac{2s}{\alpha}-\frac{(\sigma+5/2)}{\alpha}+1} \left(\sup_{0\leq t\leq T} t^{\frac{1+|s|}{\alpha}} \|\partial_x u(t,\cdot)\|_{L^2} \right)^2.$

Proof. Let us prove the first point. Since $s \leq 0$ we can write

$$
\begin{aligned}
&\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * (-\partial_x^2)^{\frac{\sigma}{2}}(u^2)(\tau,\cdot) d\tau \right\|_{H^s} \leq \int_0^t \left\| (1+|\xi|^2)^{s/2} |\xi|^\sigma \, \widehat{K_{\alpha,\beta}}(t-\tau,\cdot)(\widehat{u} * \widehat{u})(\tau,\cdot) \right\|_{L^2} d\tau \\
&\leq \int_0^t \left\| |\xi|^{s+\sigma} \widehat{K_{\alpha,\beta}}(t-\tau,\cdot)(\widehat{u} * \widehat{u})(\tau,\cdot) \right\|_{L^2} d\tau \leq \int_0^t \left\| |\xi|^{s+\sigma} \widehat{K_{\alpha,\beta}}(t-\tau,\cdot) \right\|_{L^2} \|(\widehat{u} * \widehat{u})(\tau,\cdot)\|_{L^\infty} d\tau.
\end{aligned} \tag{37}
$$

The first expression on the right-hand side can be estimated as follows:

$$
\left\| |\xi|^{s+\sigma} \widehat{K_{\alpha,\beta}}(t-\tau,\cdot) \right\|_{L^2} \lesssim (t-\tau)^{-\frac{s+\sigma}{\alpha}-\frac{1}{2\alpha}}. \tag{38}
$$

Indeed, we split $\left|\widehat{K_{\alpha,\beta}}(t-\tau,\xi)\right| = e^{(|\xi|^\alpha - |\xi|^\beta)(t-\tau)} = e^{-\frac{|\xi|^\alpha}{2}(t-\tau)}\, e^{-\left(\frac{|\xi|^\alpha}{2}-|\xi|^\beta\right)(t-\tau)}$; and for $\kappa = (t-\tau)^{\frac{1}{\alpha}}\xi$ we write

$$
\begin{aligned}
\left\| |\xi|^{s+\sigma} \widehat{K_{\alpha,\beta}}(t-\tau,\cdot) \right\|_{L^2} &\leq \left\| |\xi|^{s+\sigma} e^{-\frac{|\xi|^\alpha}{2}(t-\tau)} \right\|_{L^2} \left\| e^{-\left(\frac{|\xi|^\alpha}{2}-|\xi|^\beta\right)(t-\tau)} \right\|_{L^\infty} \\
&\leq (t-\tau)^{-\frac{s+\sigma}{\alpha}} \left\| \left|(t-\tau)^{\frac{1}{\alpha}}\xi\right|^{s+\sigma} e^{-\frac{\left|(t-\tau)^{\frac{1}{\alpha}}\xi\right|^\alpha}{2}} \right\|_{L^2} \left\| e^{-\left(\frac{|\xi|^\alpha}{2}-|\xi|^\beta\right)(t-\tau)} \right\|_{L^\infty} \\
&\leq (t-\tau)^{-\frac{s+\sigma}{\alpha}-\frac{1}{2\alpha}} \left\| |\kappa|^{s+\sigma} e^{-\frac{|\kappa|^\alpha}{2}} \right\|_{L^2} \left\| e^{-\left(\frac{|\xi|^\alpha}{2}-|\xi|^\beta\right)(t-\tau)} \right\|_{L^\infty} \\
&\lesssim (t-\tau)^{-\frac{s+\sigma}{\alpha}-\frac{1}{2\alpha}}.
\end{aligned}
$$

In order to estimate the second expression on the right-hand side, by the the Young inequalities (with $1+1/\infty = 1/2+1/2$), by the Plancherel's identity and by the second term in the norm $\|\cdot\|_{s,\alpha}$, we have:

$$
\|(\widehat{u}*\widehat{u})(\tau,\cdot)\|_{L^\infty} \lesssim \tau^{-\frac{2|s|}{\alpha}} \left(\sup_{0\leq\tau\leq T} \tau^{\frac{2|s|}{\alpha}} \|u(\tau,\cdot)\|^2_{L^2} \right). \tag{39}
$$

With the estimates (38) and (39) at hand, we get back to (37) and we write

$$
\begin{aligned}
&\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * (-\partial_x^2)^{\frac{\sigma}{2}}(u^2)(\tau,\cdot) d\tau \right\|_{H^s} \\
&\lesssim \left(\int_0^t (t-\tau)^{-\frac{s+\sigma}{\alpha}-\frac{1}{2\alpha}} \tau^{-\frac{2|s|}{\alpha}} d\tau \right) \left(\sup_{0\leq t\leq T} t^{\frac{2|s|}{\alpha}} \|u(t,\cdot)\|^2_{L^2} \right).
\end{aligned} \tag{40}
$$

To estimate the integral above, in the formula (28) we set $a = -\frac{s+\sigma}{\alpha} - \frac{1}{2\alpha}$ and $b = -\frac{2|s|}{\alpha}$.

Remark 2 *Since $0 \leq \sigma \leq 3$ and $s \leq 0$ we have $a = -\frac{s+\sigma}{\alpha} - \frac{1}{2\alpha} > -1$ as long as $\alpha > 7/2$. Moreover, since $s > 1-\alpha/2 > -\alpha/2$ we have $-\frac{2|s|}{\alpha} = \frac{2s}{\alpha} > -1$.*

Thus, a direct application of (28) yields $\displaystyle\int_0^t (t-\tau)^{-\frac{s+\sigma}{\alpha}-\frac{1}{2\alpha}} \tau^{\frac{2s}{\alpha}} d\tau \lesssim t^{s/\alpha-(\sigma+1/2)/\alpha+1}$. With this estimate the first point of this proposition follows.

The other points of this proposition follows similar estimates. Indeed, for the second point, we just remark that in the estimate (38) we set now $s=0$, which yields the integral $\displaystyle\int_0^t (t-\tau)^{-\frac{\sigma}{\alpha}-\frac{1}{2\alpha}} \tau^{\frac{2s}{\alpha}} d\tau \lesssim t^{2s/\alpha-(\sigma+1/2)/\alpha+1}$. For the third point, we follow the same computations in the estimates (37) and (38) to write

$$
\begin{aligned}
&\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * (-\partial_x^2)^{\frac{\sigma}{2}}((\partial_x u)^2)(\tau,\cdot)\, d\tau \right\|_{H^s} \lesssim \int_0^t (t-\tau)^{-\frac{s+\sigma}{\alpha}-\frac{1}{2\alpha}} \left\| (\widehat{\partial_x u} * \widehat{\partial_x u})(\tau,\cdot) \right\|_{L^\infty} d\tau \\
&\lesssim \left(\int_0^t (t-\tau)^{-\frac{s+\sigma}{\alpha}-\frac{1}{2\alpha}} \tau^{-\frac{2(1+|s|)}{\alpha}} d\tau \right) \left(\sup_{0\leq t\leq T} t^{\frac{2(1+|s|)}{\alpha}} \|\partial_x u(t,\cdot)\|^2_{L^2} \right).
\end{aligned} \tag{41}
$$

In the formula (28) we set now $b = -\frac{2(1+|s|)}{\alpha}$.

Remark 3 *We have $-\frac{2(1+|s|)}{\alpha} > -1$ as long as $1-\alpha/2 < s \leq 0$.*

Then we obtain $\displaystyle\int_0^t \tau^{-\frac{s+\sigma}{\alpha}-\frac{1}{2\alpha}} (t-\tau)^{-\frac{2(1+|s|)}{\alpha}} d\tau \lesssim t^{s/\alpha-(\sigma+5/2)/\alpha+1}$. Finally, for the fourth point, we also follow the same computations above, where we have the integral $\displaystyle\int_0^t (t-\tau)^{-\frac{\sigma}{\alpha}-\frac{1}{2\alpha}} \tau^{-\frac{2(1+|s|)}{\alpha}} d\tau \lesssim t^{2s/\alpha-(\sigma+5/2)/\alpha+1}$. Proposition 3.2 is proven. ■

With these estimates at our disposal, we directly obtain the following proposition.

Proposition 3.3 *Let $\eta > 0$ be the quantity defined in (31). The following estimates hold:*

1. $\displaystyle \left\| \int_0^t K_{\alpha,\beta}(\tau,\cdot) * \partial_x (u^2)(t-\tau,\cdot)\, d\tau \right\|_{s,\alpha} \lesssim T^\eta \, \|u\|^2_{s,\alpha}.$

2. $\displaystyle \left\| \int_0^t K_{\alpha,\beta}(\tau,\cdot) * \partial^2_x (u^2)(t-\tau,\cdot)\, d\tau \right\|_{s,\alpha} \lesssim T^\eta \, \|u\|^2_{s,\alpha}.$

3. $\displaystyle \left\| \int_0^t K_{\alpha,\beta}(\tau,\cdot) * (\partial_x u)^2(t-\tau,\cdot)\, d\tau \right\|_{s,\alpha} \lesssim T^\eta \, \|u\|^2_{s,\alpha}.$

Proof. The first estimate follows from the first point and the second point of Proposition 3.2 with $\sigma = 1$ and $\sigma = 2$. The second estimate also follows from these same points whit $\sigma = 2$ and $\sigma = 3$. Finally, the third estimate follows from the third point and the fourth point of Proposition 3.2 with $\sigma = 0$ and $\sigma = 1$. ■

Consequently, existence and uniqueness of a local in time solution $u \in E^{s,\alpha}_T \subset \mathcal{C}([0,T], H^s(\mathbb{R}))$ follow from standard arguments, provided that the condition (32) holds. Moreover, the smoothness of the flow-map function $S : H^s(\mathbb{R}) \to E^{s,\alpha}_T$ also follows from well-known arguments, see for instance [15]. Theorem 3.1 is proven. ■

<u>**The case $s > 0$**</u>**.** The key idea to prove the local-well posedness in this case is to use the estimates performed above. We thus start by proving the following useful lemma, which is a product law-type in the Sobolev spaces. Let us mention that for a parameter $z \in \mathbb{R}$ we shall denote the Bessel potential $(I_d - \partial^2_x)^{\frac{z}{2}} = \mathcal{J}^z$, which is defined in the Fourier level by the symbol $(1 + |\xi|^2)^{\frac{z}{2}}$.

Lemma 3.4 *Let $s_1 \leq 0 < s$. The following estimate holds: $\|f\, g\|_{H^s} \lesssim \|(\mathcal{J}^{s-s_1} f)\, g\|_{H^{s_1}} + \|f\, (\mathcal{J}^{s-s_1} g)\|_{H^{s_1}}$.*

Proof. The proof follows from the pointwise estimate:

$$(1+|\xi|^2)^{\frac{s}{2}} |(\widehat{f} * \widehat{g})(\xi)| \lesssim (1+|\xi|^2)^{\frac{s_1}{2}} \; \left(\left((1+|\xi|^2)^{\frac{s-s_1}{2}} |\widehat{f}| \right) * |\widehat{g}| \right)(\xi) + \left(|\widehat{f}| * \left((1+|\xi|^2)^{\frac{s-s_1}{2}} |\widehat{g}| \right) \right)(\xi). \qquad ■$$

With this lemma at our disposal, we are able to estimate the product $f\, g$ in the norm of the space $H^s(\mathbb{R})$ (with $s > 0$) in terms of the products $(\mathcal{J}^{s-s_1} f) g$ and $f(\mathcal{J}^{s-s_1} g)$ in the norm of the space $H^{s_1}(\mathbb{R})$ (with $s_1 \leq 0$). Consequently, we can use the estimates above as follows: for $\alpha > \frac{7}{2}$ we set $1 - \frac{\alpha}{2} < s_1 \leq 0$. Then, for $s > 0$ and for a time $0 \leq T \leq 1$ small enough, we define the Banach space

$$F^{s,\alpha}_T = \{ u \in \mathcal{C}([0,T], H^s(\mathbb{R})) : \|u\|_{s,\alpha,s_1} < +\infty \},$$

with the norm

$$\begin{aligned} \|u\|_{s,\alpha,s_1} = & \sup_{0\leq t\leq T} \|u(t,\cdot)\|_{H^s} + \sup_{0\leq t\leq T} t^{\frac{|s_1|}{\alpha}} \|u(t,\cdot)\|_{L^2} + \sup_{0\leq t\leq T} t^{\frac{1+|s_1|}{\alpha}} \|\partial_x u(t,\cdot)\|_{L^2} \\ & + \sup_{0\leq t\leq T} t^{\frac{|s_1|}{\alpha}} \|\mathcal{J}^{s-s_1}\, u(t,\cdot)\|_{L^2} + \sup_{0\leq t\leq T} t^{\frac{1+|s_1|}{\alpha}} \|\partial_x \, \mathcal{J}^{s-s_1}\, u(t,\cdot)\|_{L^2}. \end{aligned} \tag{42}$$

Let us briefly explain this norm. The second and the third terms are the same used in the norm $\|\cdot\|_{\sigma,\alpha}$ defined in (30). Moreover, as we consider here $s > 0$, the fourth and the fifth term will allow us to easily estimate the nonlinear terms of the equation (27) in the norm of the space $H^s(\mathbb{R})$. Thus, we can state our second technical theorem.

Theorem 3.2 *Let $\alpha > \beta > 0$ with $\alpha > 7/2$, let $s > 0$, $1 - \frac{\alpha}{2} < s_1 \leq 0$, and moreover, let $\eta > 0$ be the quantity given in (31).*

For any $u_0 \in H^s(\mathbb{R})$, there exists a time $T = T(\|u_0\|_{H^s})$ given in (32) and there exists $u \in F^{s,\alpha}_T$ a unique solution to the equation (27). Moreover, the flow-map function $S : H^s(\mathbb{R}) \to F^{s,\alpha,s_1}_T \subset \mathcal{C}([0,T], H^s(\mathbb{R}))$ defined in (12) is smooth.

Proof. As mentioned, the proof uses the estimates already proven in the previous case when $1-\alpha/2 < s \leq 0$; and it follows very similar ideas. So, it is enough to give a briefly proof. The linear term in the equation (27)) is easy to estimate and for $u_0 \in H^s(\mathbb{R})$ we have

$$\|K_{\alpha,\beta}(t,\cdot) * u_0\|_{s,\alpha,s_1} \lesssim \|u_0\|_{H^s}. \tag{43}$$

We study now the nonlinear terms. For the first term in the norm $\|\cdot\|_{s,\alpha,\sigma}$ (given in (42)), by Lemma 3.4, by the first point of Proposition 3.2 (with $\sigma = 1$), and moreover, by recalling that $\eta = \frac{s}{\alpha} - \frac{5}{2\alpha} + 1 < \frac{s}{\alpha} - \frac{3}{2\alpha} + 1$, for $0 < t \leq T \leq 1$ we write

$$\begin{aligned}
&\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * \partial_x(u^2)(\tau,\cdot) d\tau \right\|_{H^s} \\
&\lesssim \left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * \partial_x((\mathcal{J}^{s-s_1}u)u)(\tau,\cdot) d\tau \right\|_{H^{s_1}} + \left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * \partial_x(u\,(\mathcal{J}^{s-s_1}u))(\tau,\cdot) d\tau \right\|_{H^{s_1}} \\
&\lesssim t^{\frac{s}{\alpha}-\frac{3}{2\alpha}+1} \left(\sup_{0\leq t\leq T} t^{\frac{|\sigma|}{\alpha}} \|\mathcal{J}^{s-\sigma}u(t,\cdot)\|_{L^2} \right) \left(\sup_{0\leq t\leq T} t^{\frac{|s_1|}{\alpha}} \|u(t,\cdot)\|_{L^2} \right) \lesssim t^{\frac{s}{\alpha}-\frac{5}{2\alpha}+1} \|u\|^2_{s,\alpha,s_1} \leq T^\eta \|u\|^2_{s,\alpha,s_1}.
\end{aligned}$$

The other terms $\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * \partial_x^2(u^2)(\tau,\cdot) d\tau \right\|_{H^s}$ and $\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * (\partial_x u)^2(\tau,\cdot) d\tau \right\|_{H^s}$ are treated similarly, where we use again the Lemma 3.4 as well as the Proposition 3.2. Moreover, remark that the second to the fifth expressions in the norm $\|\cdot\|_{s,\alpha,\sigma}$ (see (42)) where already estimated in Proposition 3.2. Thus, the following estimate holds:

$$\left\| \int_0^t K_{\alpha,\beta}(\tau,\cdot) * \Big(\partial_x(u^2) + \partial_x^2(u^2) + (\partial_x u)^2 \Big)(t-\tau,\cdot)\, d\tau \right\|_{s,\alpha,s_1} \lesssim T^\eta \|u\|^2_{s,\alpha,s_1}. \tag{44}$$

Consequently, Theorem 3.2 follows from arguments already studied in the previous case when $1 - \frac{\alpha}{2} < s \leq 0$. ■

Regularity of solutions. In our last technical theorem, we study the regularity (in the spatial variable) of solutions constructed above. We recall the standard notation $H^\infty(\mathbb{R}) = \bigcap_{r\geq s} H^r(\mathbb{R})$.

Theorem 3.3 *Let $\alpha > \beta > 0$, with $\alpha > 7/2$. Let $u \in E_T^{s,\alpha}$ (when $1-\alpha/2 < s \leq 0$) or let $u \in F_T^{s,\alpha,\sigma}$ (when $0 < s$) be the solution of the integral equation (27) given by Theorems 3.1 and 3.2 respectively. Then we have $u \in \mathcal{C}((0,T], H^\infty(\mathbb{R}))$.*

Proof. We shall prove that each term on the right-hand side of the equation (27) belongs to the space $\mathcal{C}((0,T], H^\infty(\mathbb{R}))$. For the linear term $K_{\alpha,\beta}(t,\cdot) * u_0$ (with $u_0 \in H^s(\mathbb{R})$) by Lemmas 3.2 and 3.3 we directly have $K_{\alpha,\beta}(t,\cdot) * u_0 \in \mathcal{C}((0,T], H^\infty(\mathbb{R}))$.

We study now the nonlinear term in (27), where (for the sake of clearness) we shall consider the cases $1-\alpha/2s \leq 0$ and $0 < s$ separately.

The case $1-\alpha/2 < s \leq 0$. For the sake of simplicity, we shall write

$$B(u,u) = \partial_x(u^2) + \partial_x^2(u^2) + (\partial_x u)^2, \tag{45}$$

and for all $0 < t \leq T$ fixed, we will prove that there exists $0 < \delta$ small enough such that we have

$$\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * B(u,u)(\tau,\cdot) d\tau \right\|_{H^{s+\delta}} \lesssim t^{\frac{2s}{\alpha} - \frac{s+\delta+5/2}{\alpha}+1} \|u\|^2_{s,\alpha}, \quad \text{with} \quad \frac{2s}{\alpha} - \frac{s+\delta+5/2}{\alpha} + 1 > 0. \tag{46}$$

We consider here the following subcases: first when $s+\delta \leq 0$ and thereafter when $0 < s+\delta$.

In the case $s+\delta \leq 0$, by the first point of Proposition 3.2 (with $\sigma=1$ and $\sigma=2$ respectively), and moreover, by the third point of Proposition 3.2 (with $\sigma=0$) we directly have

$$\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * B(u,u)(\tau,\cdot)d\tau \right\|_{H^{s+\delta}} \lesssim \left(t^{\frac{s+\delta}{\alpha}-\frac{3}{2\alpha}+1} + t^{\frac{s+\delta}{\alpha}-\frac{5}{2\alpha}+1} \right) \|u\|^2_{s,\alpha}.$$

Moreover, since $1-\alpha/2 < s < s+\delta \leq 0$; and as we have $0 < t \leq T \leq 1$, the term on the right-hand side is estimated from above by $t^{\frac{s+\delta}{\alpha}-\frac{5}{2\alpha}+1}$. We thus get:

$$\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * B(u,u)(\tau,\cdot)d\tau \right\|_{H^{s+\delta}} \lesssim t^{\frac{s+\delta}{\alpha}-\frac{5}{2\alpha}+1} \|u\|^2_{s,\alpha}, \qquad \text{with} \quad \frac{s+\delta}{\alpha} - \frac{5}{2\alpha} + 1 > 0. \tag{47}$$

We consider now the case when $0 < s+\delta$. Here we write:

$$\begin{aligned} \left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * B(u,u)(\tau,\cdot)d\tau \right\|_{H^{s+\delta}} &= \left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * B(u,u)(\tau,\cdot)d\tau \right\|_{L^2} \\ &+ \left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * (-\partial_x^2)^{\frac{s+\delta}{2}} B(u,u)(\tau,\cdot)d\tau \right\|_{L^2}, \end{aligned} \tag{48}$$

where we must estimate each term on the right-hand side. For the first term, by the second point of Proposition 3.2 (with $\sigma=1$ and $\sigma=2$) and by the fourth term of Proposition 3.2 (with $\sigma=0$) we have

$$\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * B(u,u)(\tau,\cdot)d\tau \right\|_{L^2} \lesssim t^{\frac{2s}{\alpha}-\frac{5}{2\alpha}+1} \|u\|^2_{s,\alpha}, \qquad \text{with} \quad \frac{s}{\alpha} - \frac{5}{2\alpha} + 1 > 0. \tag{49}$$

For the second term, we use again the second point of Proposition 3.2 (with $\sigma = s+\delta+1$ and $\sigma = s+\delta+2$) and we use again the fourth point of Proposition 3.2 (with $\sigma = \frac{s+\delta}{2}$). Moreover, we set $0 < \delta < s - \frac{5}{2} + \alpha$ (since $\alpha > 7/2$ and $1-\alpha/2 < s$ we have $0 < s - \frac{5}{2} + \alpha$) to obtain that $\frac{2s}{\alpha} - \frac{s+\delta+5/2}{\alpha} + 1 > 0$. Then we have

$$\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * (-\partial_x^2)^{\frac{s+\delta}{2}} B(u,u)(\tau,\cdot)d\tau \right\|_{L^2} \lesssim t^{\frac{2s}{\alpha}-\frac{s+\delta+5/2}{\alpha}+1} \|u\|^2_{s,\alpha}. \tag{50}$$

Once we have the estimates (49) and (50), we remark that $\frac{2s}{\alpha} - \frac{5}{2\alpha} + 1 > \frac{2s}{\alpha} - \frac{s+\delta+5/2}{\alpha} + 1$, hence, since $0 < t \leq T \leq 1$ we get $t^{\frac{2s}{\alpha}-\frac{5}{2\alpha}+1} \leq t^{\frac{2s}{\alpha}-\frac{s+\delta+5/2}{\alpha}+1}$. We get back to (48) and we obtain

$$\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * B(u,u)(\tau,\cdot)d\tau \right\|_{H^{s+\delta}} \lesssim t^{\frac{2s}{\alpha}-\frac{s+\delta+5/2}{\alpha}+1} \|u\|^2_{s,\alpha}, \quad \text{with} \quad \frac{2s}{\alpha} - \frac{s+\delta+5/2}{\alpha} + 1 > 0. \tag{51}$$

Thus, for both cases when $s+\delta \leq 0$ and $0 < s+\delta$, the wished estimate (46) follows from (47) and (51) respectively. Moreover, remark that we have $t^{\frac{s+\delta}{\alpha}-\frac{5}{2\alpha}+1} \leq t^{\frac{2s}{\alpha}-\frac{s+\delta+5/2}{\alpha}+1}$.

We study now the continuity in the time variable. Let $\varepsilon > 0$ and let $\varepsilon < t_1, t_2 \leq T \leq 1$, without loss of generality we assume that $t_1 < t_2$. Then we write

$$\begin{aligned} &\left\| \int_0^{t_2} K_{\alpha,\beta}(t_2-\tau,\cdot) * B(u,u)(\tau,\cdot)d\tau - \int_0^{t_1} K_{\alpha,\beta}(t_1-\tau,\cdot) * B(u,u)(\tau,\cdot)d\tau \right\|_{H^{s+\delta}} \\ \leq &\left\| \int_{t_1}^{t_2} K_{\alpha,\beta}(t_2-\tau,\cdot) * B(u,u)(\tau,\cdot)d\tau \right\|_{H^{s+\delta}} \\ &+ \left\| \int_0^{t_1} (K_{\alpha,\beta}(t_2-\tau,\cdot) - K_{\alpha,\beta}(t_1-\tau,\cdot)) * B(u,u)(\tau,\cdot)d\tau \right\|_{H^{s+\delta}}. \end{aligned} \tag{52}$$

For the first term above, by following the same estimates performed in (46) we have

$$\left\| \int_{t_1}^{t_2} K_{\alpha,\beta}(t_2-\tau,\cdot) * B(u,u)(\tau,\cdot)d\tau \right\|_{H^{s+\delta}} \lesssim (t_2-t_1)^{\frac{2s}{\alpha}-\frac{s+\delta+5/2}{\alpha}+1} \|u\|^2_{s,\alpha}. \tag{53}$$

We estimate the second term above. Getting back to the expression (25), we have

$$\begin{aligned}&\left\|\int_0^{t_1}(K_{\alpha,\beta}(t_2-\tau,\cdot)-K_{\alpha,\beta}(t_1-\tau,\cdot))*B(u,u)(\tau,\cdot)d\tau\right\|_{H^{s+\delta}}\\ &\leq \int_0^{t_1}\left\|(1+|\xi|^2)^{\frac{s+\delta}{2}}\left|e^{-f(\xi)(t_2-\tau)}-e^{-f(\xi)(t_1-\tau)}\right|\widehat{B(u,u)}(\tau,\cdot)\right\|_{L^2}d\tau,\end{aligned}$$

where we study the expression $\left|e^{-f(\xi)(t_2-\tau)}-e^{-f(\xi)(t_1-\tau)}\right|$. First, we remark that the function $|f(\xi)|$ (given in (25)) is of polynomial growth and moreover for $i=1,2$ we have $|e^{-f(\xi)(t_i-\tau)}|=e^{-(|\xi|^\alpha-|\xi|^\beta)(t_i-\tau)}\lesssim 1$. Then, by the mean value theorem in the temporal variable there exists $t_0\in(0,t_2-t_1)$ such that

$$\begin{aligned}\left|e^{-f(\xi)(t_2-\tau)}-e^{-f(\xi)(t_1-\tau)}\right| &= |e^{-f(\xi)(t_1-\tau)}|\left|\left(e^{-f(\xi)(t_2-t_1)}-1\right)\right|\\ &\lesssim |e^{-f(\xi)(t_1-\tau)}|\,|f(\xi)|e^{-(|\xi|^\alpha-|\xi|^\beta)t_0}(t_2-t_1)\lesssim |f(\xi)|\,e^{-(|\xi|^\alpha-|\xi|^\beta)t_0}(t_2-t_1)\\ &\lesssim \left(|f(\xi)|\,e^{-(|\xi|^\alpha-|\xi|^\beta)\frac{t_0}{2}}\right)e^{-(|\xi|^\alpha-|\xi|^\beta)\frac{t_0}{2}}(t_2-t_1)\lesssim e^{-(|\xi|^\alpha-|\xi|^\beta)\frac{t_0}{2}}(t_2-t_1).\end{aligned}$$

We get back to previous estimate and we use the definition of $B(u,u)$ given in (45) to write

$$\begin{aligned}&\int_0^{t_1}\left\|(1+|\xi|^2)^{\frac{s+\delta}{2}}\left|e^{-f(\xi)(t_2-\tau)}-e^{-f(\xi)(t_1-\tau)}\right|\widehat{B(u,u)}(\tau,\cdot)\right\|_{L^2}d\tau\\ &\lesssim (t_2-t_1)\int_0^{t_1}\left\|(1+|\xi|^2)^{\frac{s+\delta}{2}}e^{-(|\xi|^\alpha-|\xi|^\beta)\frac{t_0}{2}}(|\xi|+|\xi|^2)(\widehat{u}*\widehat{u})(\tau,\cdot)\right\|_{L^2}d\tau\\ &\quad+(t_2-t_1)\int_0^{t_1}\left\|(1+|\xi|^2)^{\frac{s+\delta}{2}}e^{-(|\xi|^\alpha-|\xi|^\beta)\frac{t_0}{2}}(\widehat{\partial_x u}*\widehat{\partial_x u})(\tau,\cdot)\right\|_{L^2}d\tau\\ &\lesssim (t_2-t_1)\|u\|^2_{s,\alpha}.\end{aligned}$$

We thus have

$$\left\|\int_0^{t_1}(K_{\alpha,\beta}(t_2-\tau,\cdot)-K_{\alpha,\beta}(t_1-\tau,\cdot))*B(u,u)(\tau,\cdot)d\tau\right\|_{H^{s+\delta}}\lesssim\ (t_2-t_1)\|u\|^2_{s,\alpha}. \tag{54}$$

Finally, by the estimates (53) and (54) we obtain $\displaystyle\int_0^t K_{\alpha,\beta}(t-\tau,\cdot)*B(u,u)(\tau,\cdot)d\tau\in\mathcal{C}((0,T],H^{s+\delta}(\mathbb{R}))$, and consequently we have $u\in\mathcal{C}((0,T],H^{s+\delta}(\mathbb{R}))$ for $0<\delta<s-\frac{5}{2}+\alpha$. By bootstrapping this procedure (in order to obtain a gain of regularity for the nonlinear term) we conclude that $u\in\mathcal{C}((0,T],H^\infty(\mathbb{R}))$.

The case $0<s$. In this case we have very similar estimates to the previous ones: we essentially follow the ideas of the proof of Theorem 3.2, we use the Lemma 3.4 and the norm $\|u\|^2_{s,\alpha,s_1}$ instead of the norm $\|u\|^2_{s,\alpha}$. Theorem 3.3 is now proven. ■

Once we have proven the Theorems 3.1, 3.2 and 3.3, in order to conclude with the proof of the whole Theorem 1 we must verify that $u\in\mathcal{C}^1((0,T],\mathcal{C}^\infty(\mathbb{R}))$. Indeed, by Theorem 3.3 we have $u\in\mathcal{C}((0,T],H^\infty(\mathbb{R}))$ and then for $0<t\leq T$ the solution u of the integral equation (27) (constructed in Theorems 3.1 and 3.2) also solves the differential equation (1) in the classical sense. We thus write $\partial_t u=-D(\partial_x u)-\left(D^\alpha_x-D^\beta_x\right)u-\partial_x(u^2)-\partial_x^2(u^2)-(\partial_x u)^2$ to get that $\partial_t u\in\mathcal{C}(]0,T],H^\infty(\mathbb{R}))$. Thereafter, we can follow the same ideas at the end of the proof of [9, Proposition 4.2] to obtain $\partial_t u\in\mathcal{C}(]0,T],\mathcal{C}^\infty(\mathbb{R}))$ and therefore $u\in\mathcal{C}^1(]0,T],\mathcal{C}^\infty(\mathbb{R}))$.

The case $\gamma_2=\gamma_3=0$ and $\alpha>2$.

In this case, recall that that mild solutions of the equation (1) write down as

$$u(t,x)=K_{\alpha,\beta}(t,\cdot)*u_0(x)-\int_0^t K_{\alpha,\beta}(t-\tau,\cdot)*\partial_x(u^2)(\tau,x)d\tau. \tag{55}$$

As is the proof of Theorem 3.1, for $\alpha > \beta > 0$ with $\alpha > 2$ and for $-\frac{\alpha}{2} < s \leq 0$ this equation is locally well-posed in the space (with $0 < T < 1$):

$$\mathcal{E}^{s,\alpha}_T = \{u \in \mathcal{C}([0,T], H^s(\mathbb{R})) : \quad \|u\|_{s,\alpha,2} < +\infty\},$$

with the norm

$$\|u\|_{s,\alpha,2} = \sup_{0\leq t\leq T} \|u(t,\cdot)\|_{H^s} + \sup_{0<t\leq T} t^{\frac{|s|}{\alpha}} \|u(t,\cdot)\|_{L^2}.$$

Indeed, we shall detail the bilinear estimates. We get back to the estimate (40) (with $\sigma = 1$) to obtain

$$\left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * \partial_x(u^2)(\tau,\cdot)d\tau \right\|_{H^s} \lesssim \left(\int_0^t (t-\tau)^{-\frac{s+1}{\alpha}-\frac{1}{2\alpha}} \tau^{-\frac{2|s|}{\alpha}} d\tau \right) \left(\sup_{0\leq t\leq T} t^{\frac{2|s|}{\alpha}} \|u(t,\cdot)\|^2_{L^2} \right). \quad (56)$$

In order to study the integral above, observe that since $s \leq 0$ we have $-\frac{s+1}{\alpha} - \frac{1}{2\alpha} > -1$ as long as $\alpha > 2$, and moreover, we have $-\frac{2|s|}{\alpha} > -1$ as long as $-\frac{\alpha}{2} < s$. Then, we apply the estimate (28) to obtain this integral computes down as $t^{1+\frac{s}{\alpha}-\frac{3}{2\alpha}}$. Moreover, remark that $1 + \frac{s}{\alpha} - \frac{3}{2\alpha} > 0$ as long as $s > \frac{3}{2} - \alpha$. Consequently, we have

$$\sup_{0\leq t\leq T} \left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * \partial_x(u^2)(\tau,\cdot)d\tau \right\|_{H^s} \lesssim T^{1+\frac{s}{\alpha}-\frac{3}{2\alpha}} \|u\|^2_{s,\alpha,2}.$$

Then, by the second point of Proposition 3.2 (with $\sigma = 1$) we have

$$\sup_{0<t\leq T} t^{\frac{|s|}{\alpha}} \left\| \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * \partial_x(u^2)(\tau,\cdot)d\tau \right\|_{L^2} \lesssim T^{1+\frac{s}{\alpha}-\frac{3}{2\alpha}} \|u\|^2_{s,\alpha,2}.$$

By these estimates we have the locally well-posed in the space $\mathcal{E}^{s,\alpha}_T$, with $\max(3/2-\alpha, -\alpha/2) < s \leq 0$.

Thereafter, in the case $s > 0$, by following the same arguments in the proof of Theorem 3.2 we also have the locally well-posedness in the space

$$\mathcal{F}^{s,\alpha}_T = \{u \in \mathcal{C}([0,T], H^s(\mathbb{R})) : \ \|u\|_{s,\alpha,s_1,2} < +\infty\},$$

where for $\max(3/2-\alpha, -\alpha/2) < s_1 \leq 0$ we define

$$\|u\|_{s,\alpha,s_1,2} = \sup_{0\leq t\leq T} \|u(t,\cdot)\|_{H^s} + \sup_{0\leq t\leq T} t^{\frac{|s_1|}{\alpha}} \|u(t,\cdot)\|_{L^2} + \sup_{0\leq t\leq T} t^{\frac{|s_1|}{\alpha}} \|\mathcal{J}^{s-s_1} u(t,\cdot)\|_{L^2}.$$

Finally, with minor modifications, the statement of Theorem 3.3 (regularity of solutions) also holds true in this case. Theorem 1 is now proven. ■

Proof of Theorem 2

Let us briefly explain the strategy of the proof. We shall assume that the equation (1) is locally well-posed in the space $H^s(\mathbb{R})$ when $s < 1-\alpha/2$ (when $\gamma_2, \gamma_3 \neq 0$) and $s < -\alpha/2$ (when $\gamma_2 = \gamma_3 = 0$) respectively. Moreover, we shall assume that the flow-map function $S : H^s(\mathbb{R}) \to \mathcal{C}([0,T], H^s(\mathbb{R}))$ (defined in (12)) is a $\mathcal{C}^2-$ function at $u_0 = 0$. In particular, this implies that the second Fréchet derivative of $S(t)$ at $u_0 = 0$, defined as $D^2_0 S(t) : H^s(\mathbb{R}) \times H^s(\mathbb{R}) \to H^s(\mathbb{R})$, $(v_0, w_0) \mapsto D^2_0(v_0, w_0)$, is a linear and bounded operator. Our general strategy is to construct well-prepared initial data $(v_0, w_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ to contradict the boundness of the operator $D^2_0 S(t)$. The proof is divided in three steps: first, we shall explicitly compute the operator $D^2_0 S(t)$. Then, we shall construct the well-prepared initial data and in the last step we shall prove the unboundedness of the operator $D^2_0 S(t)$.

The operator $D^2_0 S(t)$. Our starting point is to explicitly compute this operator. In all the computations bellow, the limit is understood in the strong topology of the space $H^s(\mathbb{R})$. Let us start by computing the first Fréchet derivative of $S(t)$ at $u_0 \in H^s(\mathbb{R})$ in the direction $v_0 \in H^s(\mathbb{R})$. Recall that by (12) and (27), and moreover, for the bilinear form $B(\cdot,\cdot)$ defined in (45), we have

$$S(t)u_0 = u(t,\cdot) = K_{\alpha,\beta}(t,\cdot) * u_0 \underbrace{- \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * B(u,u)(\tau,\cdot)\, d\tau}_{\mathcal{B}(u,u)},$$

with

$$B(u,u)(\tau,\cdot) = \Big(\gamma_1\, \partial_x(u^2) + \gamma_2\, \partial_x^2(u^2) + \gamma_3(\partial_x u)^2\Big)(\tau,\cdot).$$

For the sake of simplicity, we shall write $S(t)u_0 = K_{\alpha,\beta}(t,\cdot) * u_0 + \mathcal{B}\Big(S(t)u_0, S(t)u_0\Big)$, where this bilinear form is symmetric. Then, we have

$$\begin{aligned} D_{u_0}S(t)v_0 &= \lim_{h\to 0} \frac{S(t)(u_0 + h\, v_0) - S(t)u_0}{h} \\ &= \lim_{h\to 0} \frac{K_{\alpha,\beta}(t,\cdot) * (u_0 + hv_0) - K_{\alpha,\beta}(t,\cdot) * u_0}{h} \\ &\quad + \lim_{h\to 0} \frac{\mathcal{B}\Big(S(t)(u_0+hv_0), S(t)(u_0+hv_0)\Big) - \mathcal{B}\Big(S(t)(u_0), S(t)(u_0)\Big)}{h} \\ &= K_{\alpha,\beta}(t,\cdot) * v_0 + 2\mathcal{B}\Big(S(t)u_0, S(t)v_0\Big). \end{aligned} \tag{57}$$

We compute now the second derivative $D^2_{u_0}S(t)$ at $u_0 = 0$. First, for $u_0, v_0 \in H^s(\mathbb{R})$ fixed, we define the function $x \in \mathbb{R} \mapsto D_{xu_0}S(t)v_0 \in H^s(\mathbb{R})$; and by following similar computations performed in (57) we have

$$\partial_x D_{xu_0}S(t)v_0 = 2\mathcal{B}\Big(D_{xu_0}S(t)u_0, D_{xu_0}S(t)v_0\Big) + 2\mathcal{B}\Big(S(t)(xu_0), D^2_{xu_0}S(t)(u_0,v_0)\Big).$$

We thus set $x = 0$, and moreover, by the identity $S(t)0 = 0$ and since by (57) we have $D_0S(t)v_0 = K_{\alpha,\beta}(t,\cdot) * v_0$, we obtain

$$\begin{aligned} D^2_0 S(t)(u_0,v_0) &= 2\mathcal{B}\Big(K_{\alpha,\beta}(t,\cdot) * u_0,\, K_{\alpha,\beta}(t,\cdot) * v_0\Big) \\ &= 2\int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * B\Big(K_{\alpha,\beta}(\tau,\cdot) * u_0,\, K_{\alpha,\beta}(\tau,\cdot) * v_0\Big)\, d\tau. \end{aligned} \tag{58}$$

Well-prepared initial data. Let $N \in \mathbb{N}^*$ fixed such that $N \gg 1$. Moreover, let $r \in \mathbb{R}$ fixed such that $r \sim 1$. We consider the disjoint intervals $[-N, -N+r]$ and $[N+r, N+2r]$. Then, we define the functions v_0 and w_0 as

$$v_0 = r^{-1/2}\, N^{-s}\, \mathcal{F}^{-1}\left(\mathbb{1}_{[-N,-N+r]}(\xi)\right), \quad v_0 = r^{-1/2}\, N^{-s}\, \mathcal{F}^{-1}\left(\mathbb{1}_{[N+r,N+2r]}(\xi)\right). \tag{59}$$

We will verify that $\|v_0\|_{H^s} \sim 1$ and $\|w_0\|_{H^s} \sim 1$. Indeed, for the function v_0 defined above we write

$$\|v_0\|^2_{H^s} = \int_{\mathbb{R}} (1+|\xi|^2)^s r^{-1}\, N^{-2s} \mathbb{1}_I(\xi)\, d\xi = r^{-1}N^{-2s} \int_{-N}^{-N+r} (1+|\xi|^2)^s d\xi.$$

Here, as $\xi \in [-N, -N+r]$, and moreover, as $N \gg 1$ and $r \sim 1$, we have $|\xi| \sim N$ and $1 + |\xi|^2 \sim N^2$. Consequently, $(1+|\xi|^2)^s \sim N^{2s}$. We thus obtain

$$r^{-1}N^{-2s} \int_{-N}^{-N+r} (1+|\xi|^2)^s d\xi \sim r^{-1}N^{-2s}\, N^{2s} \int_{-N}^{-N+r} d\xi = 1.$$

The function w_0 follows the same estimates and we also have $\|w_0\|_{H^s} \approx 1$.

The unboundedness of the operator $D_0^2S(t)$. With the particular initial data constructed above, we shall prove the following estimates from below: for $N\in\mathbb{N}$ such that $N\gg 1$

$$\begin{cases} N^{2(1-s-\alpha/2)}\lesssim \|D_0^2S(t)(u_0,v_0)\|_{H^s}, & \text{when}\;\; s<1-\alpha/2,\\ N^{2(-s-\alpha/2)}\lesssim \|D_0^2S(t)(u_0,v_0)\|_{H^s}, & \text{when}\;\; s<-\alpha/2.\end{cases} \tag{60}$$

Indeed, by the identity (58) we write

$$\|D_0^2S(t)(u_0,v_0)\|_{H^s}=2\left\|(1+|\xi|^2)^{\frac{s}{2}}\,\mathcal{F}\left(\int_0^t K_{\alpha,\beta}(t-\tau,\cdot)*B\Big(K_{\alpha,\beta}(\tau,\cdot)*u_0,\,K_{\alpha,\beta}(\tau,\cdot)*v_0\Big)\,d\tau\right)\right\|_{L^2}. \tag{61}$$

Then, for $t>0$ and $\xi\in\mathbb{R}$ we define the function

$$g(t,\xi)=\mathcal{F}\left(\int_0^t K_{\alpha,\beta}(t-\tau,\cdot)*B\Big(K_{\alpha,\beta}(\tau,\cdot)*u_0,\,K_{\alpha,\beta}(\tau,\cdot)*v_0\Big)\,d\tau\right)(\xi),$$

and recalling that $\widehat{K_{\alpha,\beta}}(t,\xi)=e^{-f(\xi)t}$ (with $f(\xi)=i\,m(\xi)\,\xi+(|\xi|^\alpha-|\xi|^\beta)$) we can prove the following identity which, for the reader's convenience, will be postponed to the Appendix 4.3:

$$g(t,\xi)=\int_{\mathbb{R}}\big(\gamma_1\,i\xi-\gamma_2\,\xi^2-\gamma_3(\xi-\eta)\eta\big)\,\widehat{u_0}(\xi-\eta)\widehat{v_0}(\eta)\,\frac{e^{-f(\eta)t-f(\xi-\eta)t}-e^{-f(\xi)t}}{f(\xi)-f(\eta)-f(\xi-\eta)}\,d\eta. \tag{62}$$

Now, we prove the following estimate from below.

Lemma 3.5 *The following estimates hold:*

1. *When $\gamma_2,\gamma_3\neq 0$, we have $N^{2-2s-\alpha}\lesssim|g(t,\xi)|$.*
2. *When $\gamma_2=\gamma_3=0$, we have $N^{-2s-\alpha}\lesssim|g(t,\xi)|$.*

Proof. We must study each term inside the integral (62).

1. Assume that $\gamma_2,\gamma_3\neq 0$. For the first and the second term we have
$$\big(\gamma_1\,i\xi-\gamma_2\,\xi^2-\gamma_3(\xi-\eta)\eta\big)\,\widehat{u_0}(\xi-\eta)\widehat{v_0}(\eta)\sim N^{2-2s}. \tag{63}$$
Indeed, recall that u_0 and v_0 are defined in (59) and we thus have $\widehat{u_0}(\xi-\eta)=r^{-1/2}N^{-s}\,\mathbb{1}_{[-N,-N+r]}(\xi-\eta)$. Here $\xi-\eta\in[-N,-N+r]$ is equivalent to $N-r+\xi\leq\eta\leq N+\xi$. Moreover, we also have $\widehat{v_0}(\eta)=r^{-1/2}\,N^{-s}\,\mathbb{1}_{[N+r,N+2r]}(\eta)$. Then, since $N\in\mathbb{N}$ such that $N\gg1$ and $r\sim 1$, we remark that the intervals $N-r+\xi\leq\eta\leq N+\xi$ and $N+r\leq\eta\leq N+2r$ are not disjoint, provided that $r<\xi<3r$. Hence, we obtain $\xi\sim r$. On the other hand, since $N+r\leq\eta\leq N+2r$ we are able to write $\eta\sim N$. Consequently, we obtain the estimates $\big(\gamma_1 i\xi-\gamma_2\xi^2-\gamma_3(\xi-\eta)\eta\big)\,\widehat{u_0}(\xi-\eta)\widehat{v_0}(\eta)\sim\big(r+N^2+r^2\big)\,r^{-1}N^{-2s}\sim N^2r^{-1}N^{-2s}\sim N^{2-2s}$.

2. Assume that $\gamma_2=\gamma_3=0$. By following the same arguments above we have
$$(\gamma_1 i\xi)\,\widehat{u_0}(\xi-\eta)\widehat{v_0}(\eta)\sim N^{-2s}. \tag{64}$$

On the other hand, for the third term we have the estimate

$$f(\xi)-f(\eta)-f(\xi-\eta)\sim N^\alpha. \tag{65}$$

Indeed, recall that $f(\xi)=i\,m(\xi)\,\xi+(|\xi|^\alpha-|\xi|^\beta)$, where the symbol $m(\xi)$ is defined in (3) and we have $i\,m(\xi)\,\xi\sim\xi^3$ when $D=\partial_x^2$; or $i\,m(\xi)\,\xi\sim\xi^2$ when $D=\mathcal{H}\partial_x$. Moreover, recall that $\xi\sim1$ and $\eta\sim N$. Then, since $\alpha>\beta$ and $\alpha>7/2$, for both cases $D=\partial_x^2$ and $D=\mathcal{H}\partial_x$ we have $f(\xi)-f(\eta)-f(\xi-\eta)\sim f(\eta)\sim N^\alpha$.

With these estimates (63), (65) and (65) at hand, we get back to identity (62) to obtain the wished estimates from below. ■

Finally, we get back to the identity (61), hence we get the desired estimate (60). In this estimate, we consider first the case $s < 1 - \alpha/2$, hence we have $1 - s - \alpha/2 > 0$. Moreover, since $\|u_0\| \sim \|v_0\| \sim 1$ we have $N^{2(1-s-\alpha/2)} \lesssim \|D_0^2 S(t)(u_0, v_0)\|_{H^s} \lesssim \|u_0\|_{H^s} \|v_0\|_{H^s} \lesssim 1$, which is a contradiction as long as $N \gg 1$. Consequently, the flow-map function $S : H^s(\mathbb{R}) \to \mathcal{C}([0,T], H^s(\mathbb{R}))$ (given in (12)) is not a $\mathcal{C}^2-$ function at $u_0 = 0$. The case $s < -\alpha/2$ follows the same ideas. Theorem 2 is now proven. ■

3.3 Global well-posedness

Proof of Theorem 3

As the proof of Theorem 1, we shall consider the two cases of the parameter s:

The case $1 - \alpha/2 < s \leq 0$ **(when** $\gamma_2, \gamma_3 \neq 0$**) or** $\max(3/2 - \alpha, -\alpha/2) < s \leq 0$ **(when** $\gamma_2 = \gamma_3 = 0$**).**

By Theorem 3.3 the solution $u \in E_T^{s,\alpha}$ (constructed in Theorem 3.1) is regular enough and then, by multiplying the equation (1) by $u(t,x)$ and after some integration by parts (in the spatial variable), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 = -\int_{\mathbb{R}} \left(D_x^\alpha - D_x^\beta\right) u\, u\, dx - (2\gamma_2 - \gamma_3) \int_{\mathbb{R}} (\partial_x u)^2\, u\, dx. \tag{66}$$

Here we assume that $-2\gamma_2 + \gamma_3 = 0$ to get

$$\frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 = -\int_{\mathbb{R}} \left(D_x^\alpha - D_x^\beta\right) u\, u\, dx.$$

We estimate now the term on the right-hand side. By the Parseval's identity and for $M = 2^{\frac{1}{\alpha-\beta}}$ (remark that for $|\xi| \geq M$ we have $|\xi|^\beta - |\xi|^\alpha \leq -|\xi|^\beta$) we write

$$\begin{aligned}
&-\int_{\mathbb{R}} \left(D_x^\alpha - D_x^\beta\right) u\, u\, dx = -\int_{\mathbb{R}} \left(|\xi|^\alpha - |\xi|^\beta\right) |\widehat{u}|^2\, d\xi \\
&= \int_{|\xi|\leq M} \left(|\xi|^\beta - |\xi|^\alpha\right) |\widehat{u}|^2\, d\xi + \int_{|\xi|\geq M} \left(|\xi|^\beta - |\xi|^\alpha\right) |\widehat{u}|^2\, d\xi \leq \int_{|\xi|\leq M} |\xi|^\beta |\widehat{u}|^2\, d\xi - \int_{|\xi|\geq M} |\xi|^\beta |\widehat{u}|^2\, d\xi \\
&\leq \int_{|\xi|\leq M} |\xi|^\beta |\widehat{u}|^2\, d\xi \leq M^\beta \|u\|_{L^2}^2.
\end{aligned} \tag{67}$$

With this estimate and by the Grönwall inequality, for all $\frac{T}{2} < t < T$ we obtain

$$\|u(t,\cdot)\|_{H^s}^2 \leq \|u(t,\cdot)\|_{L^2}^2 \leq \|u(T/2,\cdot)\|_{L^2}^2 e^{2M^\beta t}.$$

hence, the solution can be extended to the whole interval $[0, +\infty[$.

The case $0 < s$. We shall follow similar ideas of [9, Proposition 4.3]. Let $u_0 \in H^s(\mathbb{R})$ (with $s > 1 - \alpha/2$ or $s > \max(3/2 - \alpha, -\alpha/2)$ respectively) be an initial datum. We define the time T^* as follows:

$$T^* = \sup\left\{T > 0 : \text{ there exists a unique solution } u \in \mathcal{C}([0,T], H^s(\mathbb{R})) \text{ of (1) arising from } u_0\right\}.$$

We assume the relationship $-2\gamma_2 + \gamma_3 = 0$ and we will prove that $T^* = +\infty$. Our strategy is to assume that $T^* < +\infty$ to obtain a contradiction. Always by Theorem 3.3 and by following the same estimates above, we have the estimate

$$\|u(t,\cdot)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 e^{2M^\beta T^*}. \tag{68}$$

where we set the constant $M_0 = \|u_0\|_{L^2}^2 e^{2M^\beta T^*} > 0$.

On the other hand, recall that by Theorem 3.2 for any initial datum $v_0 \in H^s(\mathbb{R})$ there exists $v \in F^{s,\alpha}_T \subset \mathcal{C}([0,T], H^s(\mathbb{R}))$ an arising solution of the equation (1), where the time $T = T(v_0)$ is given by the expression (32). Precisely, we have the bound from above $T(v_0) < \frac{1}{4^{1/\eta}\|v_0\|^{1/\eta}_{H^s}}$ and since $\|v_0\|_{L^2} \leq \|v_0\|_{H^s}$, we obtain $T(v_0) < \frac{1}{4^{1/\eta}\|v_0\|^{1/\eta}_{L^2}}$. Consequently, the time $T(v_0)$ is a decreasing function of $\|v_0\|_{L^2}$. This decreasing property yields that we can find a time $0 < T_1 < T^*$ such that for all initial datum $v_0 \in H^s(\mathbb{R})$ verifying $\|v_0\|_{L^2} \leq M_0$ the associated solution $v \in \mathcal{C}([0,T[, H^s(\mathbb{R}))$ exists at least on the interval $[0,T_1]$; and it verifies $v \in \mathcal{C}([0,T_1], L^2(\mathbb{R}))$.

Now, for $0 < \varepsilon < T_1$ and for the solution $u(t,x)$ (arising from u_0) we the initial datum $v_0 = u(T^* - \varepsilon, \cdot) \in H^s(\mathbb{R})$, which by (68) verifies $\|v_0\|_{L^2} \leq M_0$. So, there exists a solution v arising from $v_0 = u(T^* - \varepsilon, \cdot)$ which is defined at least on $[0, T_1]$. Thus, by gathering the functions $u(t,x)$ and $v(t,x)$ we get a solution

$$\tilde{u}(t,\cdot) = \begin{cases} u(t,\cdot), & \text{when} \quad 0 \leq t \leq T^* - \varepsilon, \\ v(t,\cdot), & \text{when} \quad T^* - \varepsilon \leq t \leq T^* - \varepsilon + T_1, \end{cases}$$

which arises from the datum u_0 and which is defined on the interval $[0, T^* - \varepsilon + T_1]$. But, since $0 < \varepsilon < T_1$ we have $T^* - \varepsilon + T_1 > T^*$, which contradicts the definition of the time T^*. We thus have $T^* = +\infty$. Theorem 3 is now proven. ■

Proof of Proposition 1

Our starting point is the identity (66), where we must estimate the second term on the right-hand side: $\displaystyle\int_{\mathbb{R}} (\partial_x u)^2 \, u \, dx$. We write

$$\begin{aligned} &\int_{\mathbb{R}} (\partial_x u)^2 \, u \, dx = \int_{\mathbb{R}} \partial_x u \, u \, \partial_x u \, dx = \frac{1}{2} \int_{\mathbb{R}} \partial_x u \partial_x (u^2) \, dx = -\frac{1}{2} \int_{\mathbb{R}} \partial^2_x u \, u^2 \, dx \\ &\leq \|\partial^2_x u\|_{L^\infty} \, \|u^2\|_{L^1} \leq \|\partial^2_x u\|_{L^\infty} \, \|u\|^2_{L^2}. \end{aligned}$$

With this estimate at hand, we get back to (66), hence, together with the estimate (67) we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2} &\leq M^\beta \, \|u(t,\cdot)\|^2_{L^2} + \frac{2\gamma_2 - \gamma_3}{2} \, \|\partial^2_x u(t,\cdot)\|_{L^\infty} \, \|u(t,\cdot)\|^2_{L^2} \\ &\lesssim (1 + \|\partial^2_x u(t,\cdot)\|_{L^\infty}) \, \|u(t,\cdot)\|^2_{L^2}. \end{aligned} \tag{69}$$

Then, by the Grönwall inequality for all $t > T/2$ (where the time T is given by (32)) we have

$$\|u(t,\cdot)\|^2_{L^2} \lesssim \|u(T/2,\cdot)\|^2_{L^2} \, e^{t - T/2 + \int_{T/2}^t \|\partial_x u(s,\cdot)\|_{L^\infty} ds},$$

hence we obtain

$$\|u(t,\cdot)\|^2_{L^2} \lesssim \|u(T/2,\cdot)\|^2_{L^2} \, e^{t + \int_0^t \|\partial_x u(s,\cdot)\|_{L^\infty} ds}. \tag{70}$$

From this estimate the blow-up criterion stated in Proposition 1 is obtained as follows: first, let us assume that $\lim\limits_{t \to T^*} \|u(t,\cdot)\|_{H^s} = +\infty$. This fact yields $\int_0^{T^*} \|\partial^2_x u(t,\cdot)\|_{L^\infty} dt = +\infty$. Indeed, if we assume that $\int_0^{T^*} \|\partial^2_x u(t,\cdot)\|_{L^\infty} dt < +\infty$, then by (70) we get that the quantity $\|u(t,\cdot)\|^2_{L^2}$ can be extended beyond the time T^*; and by following the same arguments in the proof of Theorem 3, we get that the quantity $\|u(t,\cdot)\|_{H^s}$ extends beyond T^*, which contradicts the definition of T^*.

Now, let us assume that $\int_0^{T^*} \|\partial^2_x u(t,\cdot)\|_{L^\infty} dt = +\infty$, which yields $\lim_{t \to T^*} \|u(t,\cdot)\|_{H^s} = +\infty$. Indeed, if we assume that $\lim\limits_{t \to T^*} \|u(t,\cdot)\|_{H^s} < +\infty$, then by Theorem 3.3 we have $u \in \mathcal{C}(0, T^* + \varepsilon], H^\infty(\mathbb{R}))$ with $\varepsilon > 0$.

Consequently, for $\sigma > 1/2$ we have $\int_0^{T^*} \|u(t,\cdot)\|_{H^{2+\sigma}} dt < +\infty$. Then, by the continuous Sobolev embedding $L^\infty(\mathbb{R}) \subset H^\sigma(\mathbb{R})$, we write

$$\int_0^{T^*} \|\partial_x^2 u(t,\cdot)\|_{L^\infty} dt \leq \int_0^{T^*} \|\partial_x^2 u(t,\cdot)\|_{H^\sigma} dt \leq \int_0^{T^*} \|u(t,\cdot)\|_{H^{2+\sigma}} dt < +\infty,$$

which is a contradiction. We thus have $\lim\limits_{t\to T^*} \|u(t,\cdot)\|_{H^s} = +\infty$. Proposition 1 is proven. ■

4 Spatially decaying properties

4.1 Kernel estimates II

Proof of Proposition 2

Since the definition of this kernel involves the operator D defined in (3), we shall consider the following cases: when $D = \mathcal{H}\partial_x$, we shall refer as the nonlocal dispersive effects due presence of the Hilbert transform. On the other hand, when $D = \partial_x^2$ we shall refer as the local dispersive effects. Moreover, recall that the action of the operator D is given in the Fourier level by the symbol $m(\xi)$, which is also given in the expression (3). Then, for the sake of clearness, we shall the identity (18) in the following technical propositions.

Proposition 4.1 (The nonlocal dispersive effects) *Let $D = \mathcal{H}\partial_x$, where we have $m(\xi) = |\xi|$. Let $\alpha > \beta \geq 1$ with $\alpha > 2$. For $t > 0$ there exists a quantity $I(t)$, which verifies $|I(t)| \leq Ce^{\eta_1 t}$ with $C > 0$ and $\eta_1 > 0$ depending on α and β, such that for all $x \neq 0$ the following estimate holds:*

$$|K_{\alpha,\beta}(t,x)| = \frac{|I(t)|}{|x|^{\min(3,[\beta]+1)}},$$

where $[\beta]$ denotes the integer part of β.

Proof. We start by explaining the general idea of the proof. This idea was inspired from the previous works [8, 9]. By the expression (25), for $x \neq 0$ and $t > 0$ we write

$$K_{\alpha,\beta}(t,x) = \int_{-\infty}^{0} e^{2\pi i x\xi} e^{-f(\xi)t} d\xi + \int_0^{+\infty} e^{2\pi i x\xi} e^{-f(\xi)t} d\xi.$$

In each term on the right-hand side, we multiply and we divide by $2\pi i x$ to get:

$$K_{\alpha,\beta}(t,x) = \frac{1}{2\pi i x} \int_{-\infty}^{0} 2\pi i x\, e^{2\pi i x\xi} e^{-f(\xi)t} d\xi + \frac{1}{2\pi i x} \int_0^{+\infty} 2\pi i x\, e^{2\pi i x\xi} e^{-f(\xi)t} d\xi.$$

Then, since $2\pi i x\, e^{2\pi i x\xi} = \partial_\xi \left(e^{2\pi i x\xi}\right)$ we integrate by parts respect to the variable ξ to obtain

$$\begin{aligned} K_{\alpha,\beta}(t,x) = &\frac{1}{2\pi i x}\left(-1 \int_{-\infty}^{0} e^{2\pi i x\xi}\, \partial_\xi \left(e^{-f(\xi)t}\right) d\xi + \left(e^{2\pi i x\xi} \left(e^{-f(\xi)t}\right)\right)\Big|_{-\infty}^{0}\right) \\ &+ \frac{1}{2\pi i x}\left(-1 \int_0^{+\infty} e^{2\pi i x\xi}\, \partial_\xi \left(e^{-f(\xi)t}\right) d\xi + \left(e^{2\pi i x\xi} \left(e^{-f(\xi)t}\right)\right)\Big|_0^{+\infty}\right). \end{aligned}$$

By iterating this process n times, we formally obtain the following expression:

$$\begin{aligned} K_{\alpha,\beta}(t,x) = &\frac{1}{(2\pi i x)^n}\left((-1)^n \int_{-\infty}^{0} e^{2\pi i x\xi}\, \partial_\xi^n \left(e^{-f(\xi)t}\right) d\xi + \left(e^{2\pi i x\xi}\, \partial_\xi^{n-1} \left(e^{-f(\xi)t}\right)\right)\Big|_{-\infty}^{0}\right) \\ &+ \frac{1}{(2\pi i x)^n}\left((-1)^n \int_0^{+\infty} e^{2\pi i x\xi}\, \partial_\xi^n \left(e^{-f(\xi)t}\right) d\xi + \left(e^{2\pi i x\xi}\, \partial_\xi^{n-1} \left(e^{-f(\xi)t}\right)\right)\Big|_0^{+\infty}\right), \end{aligned} \tag{71}$$

and this iterative process continues until we have one of the following scenarios:

- On the one hand, this process stops at the step n when for the next step $n+1$ we have

$$\int_{-\infty}^{0} e^{2\pi i\, x\xi}\, \partial_\xi^{n+1}\left(e^{-f(\xi)t}\right) d\xi = +\infty, \qquad \int_{0}^{+\infty} e^{2\pi i\, x\xi}\, \partial_\xi^{n+1}\left(e^{-f(\xi)t}\right) d\xi = +\infty. \tag{72}$$

Precisely, when both integrals diverge at $\xi = 0$ depending on the behavior of the function $f^{(n+1)}(\xi)$ when $\xi \to 0^-$ and $\xi \to 0^+$.

- On the other hand, this process stops at the step n when we have

$$L_n := \left(e^{2\pi i\, x\xi}\, \partial_\xi^{n-1}\left(e^{-f(\xi)t}\right)\right)\Big|_{-\infty}^{0} + \left(e^{2\pi i\, x\xi}\, \partial_\xi^{n-1}\left(e^{-f(\xi)t}\right)\right)\Big|_{0}^{+\infty} \neq 0. \tag{73}$$

We thus obtain

$$K_{\alpha,\beta}(t,x) = \frac{L_n}{(2\pi\, ix)^n} + \frac{(-1)^n}{2\pi\, ix)^n}\int_{-\infty}^{+\infty} e^{2\pi i\, x\xi} \partial_\xi^n\left(e^{-f(\xi)t}\right) d\xi. \tag{74}$$

In both scenarios, we conclude the wished identity stated in this proposition: $|K_{\alpha,\beta}(t,x)| = \frac{|I(t)|}{|x|^n}$. The generic quantity $I(t)$ may change in the different cases that we shall consider below, but we always have the control $|I(t)| \leq Ce^{\eta_1 t}$.

Now, we are able to prove this proposition. For the sake of clearness, we shall consider separately the following cases of the parameter $\beta \geq 1$.

- The case $1 \leq \beta < 2$. By the expression (74) (with $n = 2$) we write

$$K_{\alpha,\beta}(t,x) = \frac{L_2}{(2\pi\, ix)^2} + \frac{1}{(2\pi ix)^2}\left(\int_{-\infty}^{+\infty} e^{2\pi ix\xi}\partial_\xi^2\left(e^{-f(\xi)t}\right) d\xi\right).$$

Recall that the term L_2 (given in (73)) involves the expression $f'(\xi)$; and by a simple computation we have:

$$f'(\xi) = \begin{cases} -2i\xi - \alpha(-\xi)^{\alpha-1} + \beta(-\xi)^{\beta-1}, & \xi < 0, \\ 2i\xi + \alpha\xi^{\alpha-1} - \beta\xi^{\beta-1}, & \xi > 0. \end{cases} \tag{75}$$

Thus, when $\beta = 1$ by this expression we obtain $L_2 = -2t \neq 0$, and thus we can write

$$|K_{\alpha,\beta}(t,x)| = \frac{1}{|2\pi\, ix|^2}\left|-2t + \int_{-\infty}^{+\infty} e^{2\pi ix\xi}\partial_\xi^2\left(e^{-f(\xi)t}\right) d\xi\right| = \frac{|I(t)|}{|x|^2} = \frac{|I(t)|}{|x|^{\min(3,[\beta]+1)}},$$

where $[\beta] = 1$. Moreover, by the good decaying properties of the function $e^{-f(\xi)t}$ and by following the same computations performed in [8, Lemma 3.1], we have $|I(t)| \leq C^{\eta_1 t}$.

On the other hand, when $1 < \beta < 2$ by the identity (75) we have $L_2 = 0$ and we obtain

$$K_{\alpha,\beta}(t,x) = \frac{1}{(2\pi ix)^2}\left(\int_{-\infty}^{+\infty} e^{2\pi ix\xi}\partial_\xi^2\left(e^{-f(\xi)t}\right) d\xi\right).$$

To study the integral above we need to compute $f''(\xi)$ and we have:

$$f''(\xi) = \begin{cases} -2i + \alpha(\alpha-1)(-\xi)^{\alpha-2} - \beta(\beta-1)(-\xi)^{\beta-2}, & \xi < 0, \\ 2i + \alpha(\alpha-1)\xi^{\alpha-2} - \beta(\beta-1)\xi^{\beta-2}, & \xi > 0. \end{cases} \tag{76}$$

In particular, we have $f''(\xi) \sim \xi^{\beta-2}$ when $\xi \to 0$ and since $1 < \beta < 2$ this integral converges.

Finally, we remark that for the next value $n=3$, by the expression (74) we formally have

$$K_{\alpha,\beta}(t,x)=\frac{L_3}{(2\pi\, ix)^3}+\frac{1}{(2\pi ix)^2}\left(\int_{-\infty}^{+\infty}e^{2\pi ix\xi}\partial^3_\xi\left(e^{-f(\xi)t}\right)d\xi\right),$$

but the last integral diverges. Indeed, to study this integral we need to compute $f'''(\xi)$ and we have

$$f^{'''}(\xi)=\begin{cases}-c_\alpha(-\xi)^{\alpha-3}+c_\beta(-\xi)^{\beta-3}, & \xi<0,\\ c_\alpha\,\xi^{\alpha-3}-c_\beta\,\xi^{\beta-3}, & \xi>0.\end{cases} \tag{77}$$

By the expression (77) we observe that $f^{(3)}(\xi)\sim\xi^{\beta-3}$ when $\xi\to 0$, and since $1<\beta<2$ this fact yields (72). We thus obtain $|K_{\alpha,\beta}(t,x)|=\frac{|I(t)|}{|x|^{\min(3,[\beta]+1)}}$.

- The case $2\leq\beta$. By (74) (with $n=3$) we can write:

$$K_{\alpha,\beta}(t,x)=\frac{L_3}{(2\pi\, ix)^3}+\frac{1}{(2\pi ix)^2}\left(\int_{-\infty}^{+\infty}e^{2\pi ix\xi}\partial^3_\xi\left(e^{-f(\xi)t}\right)d\xi\right).$$

where by the expressions (73) and (76) we always have $L_3=-4i\,t\neq 0$. Then, by the identity (74) (with $n=3$) we obtain

$$|K_{\alpha,\beta}(t,x)|=\left|\frac{-4i\,t}{(2\pi\, ix)^3}+\frac{-1}{2\pi\, ix)^3}\int_{-\infty}^{+\infty}e^{2\pi i\,x\xi}\partial^3_\xi\left(e^{-f(\xi)t}\right)\right|=\frac{|I(t)|}{|x|^3}=\frac{|I(t)|}{|x|^{\min(3,[\beta]+1)}}.$$

Proposition 4.1 is proven. ■

Proposition 4.2 (The local dispersive effects I) *Let $D=\partial^2_x$, where we have $m(\xi)=-|\xi|^2$. Let $\alpha>\beta\geq 1$ with $\alpha>2$. Moreover, we assume that α and β are both even numbers. Then we have $K_{\alpha,\beta}(t,\cdot)\in\mathcal{S}(\mathbb{R})$.*

Proof. Since α and β are both even numbers, and moreover, since $m(\xi)=-|\xi|^2$, the function $f(\xi)$ given in (25) verifies $f\in\mathcal{C}^\infty(\mathbb{R})$. Consequently, by the good decaying properties of the function $e^{-f(\xi)t}$ when $|\xi|\to+\infty$, we have $e^{-f(\xi)t}\in\mathcal{S}(\mathbb{R})$. Then, always by (25) we conclude that $K_{\alpha,\beta}(t,\cdot)\in\mathcal{S}(\mathbb{R})$. ■

Proposition 4.3 (The local dispersive effects II) *Let $D=\partial^2_x$, where we have $m(\xi)=-|\xi|^2$. Let $\alpha>\beta\geq 1$ with $\alpha>2$. Moreover, we assume that α and β are* not *both even numbers.*

For $t>0$ there exists a quantity $I(t)$ which verifies $|I(t)|\leq C_1e^{\eta_1\, t}$ with the constants $C,\eta_1>0$ depending on α and β, such that for $x\neq 0$ the following estimates hold:

1 If $\beta>0$ is not an even number, we have $|K_{\alpha,\beta}(t,x)|=\dfrac{|I(t)|}{|x|^{[\beta]+1}}$,

2 If $\beta>0$ is an even number, we have $|K_{\alpha,\beta}(t,x)|=\dfrac{|I(t)|}{|x|^{[\alpha]+1}}$,

where $[\beta]$ and $[\alpha]$ denote the integer part of α and β respectively.

Proof. The proof essentially follows the same ideas of the proof of Proposition 4.1. Our starting point is the expression (71) with the value $n=[\beta]+1$:

$$\begin{aligned}K_{\alpha,\beta}(t,x)=&\frac{1}{(2\pi ix)^{[\beta]+1}}\left((-1)^{[\beta]+1}\int_{-\infty}^{0}e^{2\pi i\,x\xi}\,\partial^{[\beta]+1}_\xi\left(e^{-f(\xi)t}\right)d\xi+\left(e^{2\pi i\,x\xi}\,\partial^{[\beta]}_\xi\left(e^{-f(\xi)t}\right)\right)\Big|_{-\infty}^{0}\right)\\&+\frac{1}{(2\pi ix)^{[\beta]+1}}\left((-1)^{[\beta]+1}\int_{0}^{+\infty}e^{2\pi i\,x\xi}\,\partial^{[\beta]+1}_\xi\left(e^{-f(\xi)t}\right)d\xi+\left(e^{2\pi i\,x\xi}\,\partial^{[\beta]}_\xi\left(e^{-f(\xi)t}\right)\right)\Big|_{0}^{+\infty}\right).\end{aligned} \tag{78}$$

In order to study the study the terms

$$\left(e^{2\pi i x\xi}\,\partial_\xi^{[\beta]}\left(e^{-f(\xi)t}\right)\right)\Big|_{-\infty}^{0}, \qquad \left(e^{2\pi i x\xi}\,\partial_\xi^{[\beta]}\left(e^{-f(\xi)t}\right)\right)\Big|_{0}^{+\infty},$$

we need to compute the expression $f^{([\beta])}$ and we have

$$f^{([\beta])}(\xi) = \begin{cases} i\,c_\beta\,\xi^{3-[\beta]} + (-1)^{[\beta]}c_\alpha(-\xi)^{\alpha-[\beta]} - (-1)^{[\beta]}c_\beta(-\xi)^{\beta-[\beta]}, & \xi < 0, \\ i\,c_\beta\,\xi^{3-[\beta]} + c_\alpha\,\xi^{\alpha-[\beta]} - c_\beta\,\xi^{\beta-[\beta]}, & \xi > 0, \end{cases} \qquad 1 \leq [\beta] \leq 3, \tag{79}$$

$$f^{[\beta]}(\xi) = \begin{cases} (-1)^{[\beta]}c_\alpha(-\xi)^{\alpha-[\beta]} - (-1)^{[\beta]}c_\beta(-\xi)^{\beta-[\beta]}, & \xi < 0, \\ c_\alpha\,\xi^{\alpha-[\beta]} - c_\beta\,\xi^{\beta-[\beta]}, & \xi > 0, \end{cases} \qquad 4 \leq [\beta]. \tag{80}$$

At this point, we shall consider the following cases of the parameter β.

1. When β in not an even number. Within this setting, we still need to consider the next sub-cases.

   1.1 When β is an integer number. We thus have $\beta = [\beta]$ where $[\beta]$ is not an even number. Then, by the expressions (79) and (80) we have

   $$\left(e^{2\pi i x\xi}\,\partial_\xi^{[\beta]}\left(e^{-f(\xi)t}\right)\right)\Big|_{-\infty}^{0} + \left(e^{2\pi i x\xi}\,\partial_\xi^{[\beta]}\left(e^{-f(\xi)t}\right)\right)\Big|_{0}^{+\infty} = -2c_\beta\,t \neq 0.$$

   Consequently, we obtain

   $$\begin{aligned} |K_{\alpha,\beta}(t,x)| &= \left| \frac{-2c_\beta\,t}{(2\pi\,ix)^{[\beta]+1}} + \frac{(-1)^{[\beta]+1}}{(2\pi\,ix)^{[\beta]+1}} \int_{-\infty}^{+\infty} e^{2\pi i x\xi}\,\partial_\xi^{[\beta]+1}\left(e^{-f(\xi)t}\right)d\xi \right| \\ &= \frac{|I(t)|}{|x|^{[\beta]+1}}, \qquad |I(t)| \leq Ce^{\eta\,t}. \end{aligned}$$

   1.2 When β is not an integer number. In this case we have $[\beta] < \beta$ and by the expressions (79) and (80) we obtain

   $$\left(e^{2\pi i x\xi}\,\partial_\xi^{[\beta]}\left(e^{-f(\xi)t}\right)\right)\Big|_{-\infty}^{0} + \left(e^{2\pi i x\xi}\,\partial_\xi^{[\beta]}\left(e^{-f(\xi)t}\right)\right)\Big|_{0}^{+\infty} = 0.$$

   Therefore, we can write

   $$|K_{\alpha,\beta}(t,x)| = \left| \frac{(-1)^{[\beta]+1}}{(2\pi\,ix)^{[\beta]+1}} \int_{-\infty}^{+\infty} e^{2\pi i x\xi}\,\partial_\xi^{[\beta]+1}\left(e^{-f(\xi)t}\right)d\xi \right| = \frac{|I(t)|}{|x|^{[\beta]+1}}, \quad |I(t)| \leq Ce^{\eta_1\,t}.$$

   Moreover, we remark that we cannot continue the iterative process described in (71) with the next step $n = [\beta]+2$: the resulting integrals involve the expression $\partial_\xi^{[\beta]+2}\left(e^{-f(\xi)t}\right)$, which contains the term $f^{([\beta]+2)}(\xi)$. But, by a simple calculation, from the expressions (79) and (80) for $[\beta]+2 = 3$ we have

   $$f^{([\beta]+2)}(\xi) = \begin{cases} i\,c_\beta\,\xi^{3-([\beta]+2)} + (-1)^{[\beta]}c_\alpha(-\xi)^{\alpha-[\beta]-2} - (-1)^{[\beta]+2}c_\beta(-\xi)^{\beta-[\beta]-2}, & \xi < 0, \\ i\,c_\beta\,\xi^{3-([\beta]+2)} + c_\alpha\,\xi^{\alpha-[\beta]-2} - c_\beta\,\xi^{\beta-[\beta]-2}, & \xi > 0, \end{cases}$$

   and for $4 \leq [\beta]+2$ we have

   $$f^{([\beta]+2)}(\xi) = \begin{cases} (-1)^{[\beta]+2}c_\alpha(-\xi)^{\alpha-[\beta]-2} - (-1)^{[\beta]+2}c_\beta(-\xi)^{\beta-[\beta]-2}, & \xi < 0, \\ c_\alpha\,\xi^{\alpha-[\beta]-2} - c_\beta\,\xi^{\beta-[\beta]-2}, & \xi > 0, \end{cases}$$

   In both cases we get that $f^{([\beta]+2)}(\xi) \sim \xi^{\beta-[\beta]-2}$ when $\xi \to 0$. Consequently, these integrals are not convergent.

2. When β is an even number. We get back to the expression (78) and since β is an even number the expressions (79) and (80) write down as:

$$f^{(\beta)}(\xi)=\begin{cases} i\,c_\beta\,\xi^{3-\beta}+c_\alpha(-\xi)^{\alpha-\beta}-c_\beta, & \xi<0,\\ i\,c_\beta\,\xi^{3-\beta}+c_\alpha\,\xi^{\alpha-\beta}-c_\beta, & \xi>0,\end{cases} \qquad 1\leq\beta\leq 3,$$

$$f^{(\beta)}(\xi)=\begin{cases} c_\alpha(-\xi)^{\alpha-\beta}-c_\beta, & \xi<0,\\ c_\alpha\,\xi^{\alpha-\beta}-c_\beta, & \xi>0,\end{cases} \qquad 4\leq\beta,$$

Therefore, we obtain

$$\left(e^{2\pi i\,x\xi}\,\partial^{[\beta]}_\xi\left(e^{-f(\xi)t}\right)\right)\Big|^{0}_{-\infty}+\left(e^{2\pi i\,x\xi}\,\partial^{[\beta]}_\xi\left(e^{-f(\xi)t}\right)\right)\Big|^{+\infty}_{0}=0,$$

and we can continue with the iterative process described in (71) until the step $n=[\alpha]+1$ to write

$$\begin{aligned}K_{\alpha,\beta}(t,x)=&\frac{1}{(2\pi ix)^{[\alpha]+1}}\left((-1)^{[\alpha]+1}\int_{-\infty}^{0}e^{2\pi i\,x\xi}\,\partial^{[\alpha]+1}_\xi\left(e^{-f(\xi)t}\right)d\xi+\left(e^{2\pi i\,x\xi}\,\partial^{[\alpha]}_\xi\left(e^{-f(\xi)t}\right)\right)\Big|^{0}_{-\infty}\right)\\ &+\frac{1}{(2\pi ix)^{[\alpha]+1}}\left((-1)^{[\alpha]+1}\int_{0}^{+\infty}e^{2\pi i\,x\xi}\,\partial^{[\alpha]+1}_\xi\left(e^{-f(\xi)t}\right)d\xi+\left(e^{2\pi i\,x\xi}\,\partial^{[\alpha]}_\xi\left(e^{-f(\xi)t}\right)\right)\Big|^{+\infty}_{0}\right).\end{aligned}$$

From this identity and by following the same arguments detailed at the points 1.1 and 1.2 above (with α instead of β) we finally obtain $|K_{\alpha,\beta}(t,x)|=\dfrac{|I(t)|}{|x|^{[\alpha]+1}}$, with $|I(t)|\leq Ce^{\eta_1 t}$.

Proposition 4.3 is proven. ■

Summarizing, by the pointwise identities proven in Propositions 4.1, 4.2 and 4.3, and by the parameter $n\geq 2$ defined in the expression (13) we obtain the unified identity (18). Proposition 2 is now proven. ■

As a corollary of this identity we can easily estimate the kernel $K_{\alpha,\beta}$ in the L^p- norms, for both the nonlocal dispersive (when $D=\mathcal{H}\partial_x$) and the local dispersive (when $D=-\partial^2_x$) cases.

Proposition 4.4 *Let $\alpha>\beta\geq 1$ with $\alpha>2$. For all $t>0$ fixed, and for $1\leq p\leq +\infty$ the following estimate hold true:*

$$\|K_{\alpha,\beta}(t,\cdot)\|_{L^p}\leq C\,\frac{e^{\eta_1 t}}{t^{\frac{1}{\alpha}}},$$

where the constants $C>0$ and $\eta_1>0$ depend on α,β and p.

Proof. For $t>0$ fixed, we start by estimating the quantity $\|K_{\alpha,\beta}(t,\cdot)\|_{L^\infty}$. We recall that $f(\xi)=i\,m(\xi)\xi+(|\xi|^\alpha-|\xi|^\beta)$ (with $m(\xi)=|\xi|$ or $m(\xi)=|\xi|^2$) and since $\alpha>\beta$ for $M>0$ big enough we can write

$$\begin{aligned}\|K_{\alpha,\beta}(t,\cdot)\|_{L^\infty}\leq& C\,\|e^{-f(\cdot)t}\|_{L^1}\leq C\int_{|\xi|\leq M}e^{-(|\xi|^\alpha-|\xi|^\beta)t}\,d\xi+C\int_{|\xi|>M}e^{-(|\xi|^\alpha-|\xi|^\beta)t}\,d\xi\\ \leq& C\int_{|\xi|\leq M}e^{|\xi|^\beta t}d\xi+C\int_{|\xi|>M}e^{-|\xi|^\alpha t}d\xi\leq C\,e^{\eta t}+\frac{C}{t^{\frac{1}{\alpha}}}\leq\frac{C(t^{\frac{1}{\alpha}}e^{\eta t}+1)}{t^{\frac{1}{\alpha}}}\leq C\,\frac{e^{\eta_1 t}}{t^{\frac{1}{\alpha}}}.\end{aligned}$$

We thus have

$$\|K_{\alpha,\beta}(t,\cdot)\|_{L^\infty}\leq C\,\frac{e^{\eta_1 t}}{t^{\frac{1}{\alpha}}}. \tag{81}$$

On the other hand, we estimate now the quantity $\|K_{\alpha,\beta}(t,\cdot)\|_{L^1}$:

$$\|K_{\alpha,\beta}(t,\cdot)\|_{L^1}=\int_{|x|\leq 2}|K_{\alpha,\beta}(t,x)|\,dx+\int_{|x|>2}|K_{\alpha,\beta}(t,x)|\,dx\leq C\,\|K_{\alpha,\beta}(t,\cdot)\|_{L^\infty}+\int_{|x|>2}|K_{\alpha,\beta}(t,x)|dx.$$

The first term on the right-hand was already estimated in (81). On the other hand, by (18) we have

$$\int_{|x|>2} |K_{\alpha,\beta}(t,x)|dx \leq C\, e^{\eta_0\, t} \int_{|x|>2} \frac{dx}{|x|^n} \leq C\, e^{\eta_1\, t}. \tag{82}$$

We thus obtain $\|K_{\alpha,\beta}(t,\cdot)\|_{L^1} \leq \frac{C}{t^{\frac{1}{\alpha}}} + C\, e^{\eta_1\, t} \leq C\, \frac{e^{\eta_1\, t}}{t^{\frac{1}{\alpha}}}$.

Finally, the quantity $\|K_{\alpha,\beta}(t,\cdot)\|_{L^p}$, with $1 < p < +\infty$, follows from the standard interpolation inequalities. Proposition 4.4 is proven. ■

To close this section, remark that by the identity (18) and by the estimate $\|K_{\alpha,\beta}(t,\cdot)\|_{L^\infty} \leq C\, \frac{e^{\eta_1\, t}}{t^{\frac{1}{\alpha}}}$ proven above, for $t > 0$ and $x \in \mathbb{R}$ we have the following pointwise estimate:

$$|K_{\alpha,\beta}(t,x)| \leq C\, \frac{e^{\eta_1\, t}}{t^{\frac{1}{\alpha}}}\, \frac{1}{1+|x|^n}, \quad \text{with} \quad \alpha > 2 \quad \text{and} \quad n \geq 2. \tag{83}$$

This estimate will be very useful in the following section.

4.2 Spatial pointwise decaying

Proof of Theorem 4

Given an initial datum $u_0 \in H^s(\mathbb{R})$, with $s > \frac{5}{2}$, we assume now that the it verifies $u_0 \in L^\infty((1+|\cdot|^\kappa)dx)$, with $\kappa > 1$. Then, for a time $0 < T < 1$ we will construct a solution $u(t,x)$ of the equation (27) in the following Banach space

$$E_T = \left\{ u \in \mathcal{C}([0,T], H^s(\mathbb{R})) : \ \|u\|_T < +\infty \right\}, \tag{84}$$

where the norm $\|\cdot\|_T$ depends on the parameter κ, the parameter α, the parameter n given in (13), and moreover, it also depends on the previous norm $\|\cdot\|_{s,\alpha,0}$ (defined in (42)) as follows:

$$\|u\|_T = \|u\|_{s,\alpha,0} + \sup_{0\leq t\leq T} t^{\frac{1}{\alpha}} \|(1+|\cdot|^{\min(\kappa,n)})u(t,\cdot)\|_{L^\infty} + \sup_{0\leq t\leq T} t^{\frac{2}{\alpha}} \|(1+|\cdot|^{\min(\kappa,n+1)})\partial_x u(t,\cdot)\|_{L^\infty}. \tag{85}$$

In this expression, the first term norm $\|\cdot\|_{\alpha,s,0}$ will allow us to control in the space $H^s(\mathbb{R})$ each term of the nonlinear part of the equation (27). The second term characterizes the spatially decaying properties of solutions, while the third and the fourth terms are meant to treat the (more delicate) nonlinear term $(\partial_x u)^2$. Finally, the weights in the temporal variable $t^{\frac{1}{\alpha}}$ and $t^{\frac{2}{\alpha}}$ are essentially technical (due to the kernel estimates (83)) and they will be useful to carry up all our estimates.

Let us start by studying the linear term in the mild formulation (27).

Proposition 4.5 *We have $K_{\alpha,\beta} * u_0 \in E_T$ and $\|K_{\alpha,\beta} * u_0\|_T \lesssim (\|\vec{u}_0\|_{H^s} + \|(1+|\cdot|^\kappa)\vec{u}_0\|_{L^\infty})$.*

Proof. We must estimate each term in the norm given in (30), but recall that the first term was already considered in (43), and consequently, we shall focus on the second and the third term.

For the second term, since $u_0 \in L^\infty((1+|\cdot|^\kappa)dx)$ and moreover, by the kernel estimate (83), for $0 \leq t \leq T$ and $x \in \mathbb{R}$ fixed we write

$$\begin{aligned} |K_{\alpha,\beta}(t,\cdot) * u_0(x)| &\leq \int_{\mathbb{R}} |K_{\alpha,\beta}(t,x-y)||u_0(y)|dy \leq \int_{\mathbb{R}} |K_{\alpha,\beta}(t,x-y)|\frac{1+|y|^\kappa}{1+|y|^\kappa}|u_0(y)|dy \\ &\leq \|(1+|\cdot|^\kappa)u_0\|_{L^\infty} \int_{\mathbb{R}} \frac{|K_{\alpha,\beta}(t,x-y)|}{1+|y|^\kappa}dy \lesssim \|(1+|\cdot|^\kappa)u_0\|_{L^\infty}\, \frac{e^{c_\eta\, t}}{t^{\frac{1}{\alpha}}} \int_{\mathbb{R}} \frac{dy}{(1+|x-y|^n)(1+|y|^\kappa)} \\ &\lesssim \|(1+|\cdot|^\kappa)u_0\|_{L^\infty}\, \frac{e^{\eta_0\, t}}{t^{\frac{1}{\alpha}}}\, \frac{1}{1+|x|^{\min(\kappa,n)}}, \end{aligned}$$

hence we have

$$\sup_{0\leq t\leq T} t^{\frac{1}{\alpha}} \left\|(1+|\cdot|^{\min(\kappa,n)}) K_{\alpha,\beta}(t,\cdot)*u_0\right\|_{L^\infty} \lesssim \|(1+|\cdot|^{\kappa})u_0\|_{L^\infty}. \tag{86}$$

For the third term we shall need the following technical lemma, which was essentially proven in [8, Lemma 4.2]:

Lemma 4.1 *Let $\alpha>\beta\geq 1$ with $\alpha>2$. Let $K_{\alpha,\beta}$ the kernel given in (25) with $m(\xi)=|\xi|$ or $m(\xi)=-|\xi|^2$. Moreover, let $n\geq 2$ be the parameter defined in (13). Then for $t>0$ we have $K_{\alpha,\beta}(t,\cdot)\in \mathcal{C}^1(\mathbb{R})$ and the following estimates hold:*

1. *For all $x\neq 0$, $|\partial_x K_{\alpha,\beta}(t,x)|\leq C\,\dfrac{e^{\eta_1 t}}{|x|^{n+1}}$.*

2. *For all $x\in\mathbb{R}$, $|\partial_x K_{\alpha,\beta}(t,x)|\leq C\,\dfrac{e^{\eta_1 t}}{t^{\frac{2}{\alpha}}}\dfrac{1}{1+|x|^{n+1}}$,*

for two constants $C,\eta_1>0$ which depend on α and β.

By the second point above and by following very similar estimates done to prove (86) we obtain

$$\sup_{0\leq t\leq T} t^{\frac{2}{\alpha}} \left\|(1+|\cdot|^{\min(\kappa,n+1)})\partial_x\left(K_{\alpha,\beta}(t,\cdot)*u_0\right)\right\|_{L^\infty} \lesssim \|(1+|\cdot|^{\kappa})u_0\|_{L^\infty}. \tag{87}$$

Thus, the wished estimate follows from (43), (86) and (87). Proposition 4.5 is now proven. ■

We study now the nonlinear term in the mild formulation (27). For the sake of simplicity, we shall only consider the case $\gamma_1=\gamma_2=\gamma_3=1$ with $\alpha>7/2$. The other case: $\gamma_1=1$, $\gamma_2=\gamma_3=0$ with $\alpha>2$, essentially follows the same estimates with the obvious minor modifications.

Proposition 4.6 *Lat $\alpha>\frac{7}{2}$, $s>\frac{5}{2}$ and let $\eta>0$ be quantity given in (31). Define $0<\eta_2<\min(\eta,1-\frac{3}{\alpha})$. Then the following estimate holds:*

$$\left\|\int_0^t K_{\alpha,\beta}(t-\tau,\cdot)*\left(\partial_x(u^2)+\partial_x^2(u^2)+(\partial_x u)^2\right)(\tau,\cdot)d\tau\right\|_T \leq C\,T^{\eta_2}\,\|u\|_T^2.$$

Proof. We get back to the definition of the norm $\|\cdot\|_T$ given in (85), where we must estimate each term in this expression. We recall that the first term $\|\cdot\|_{s,\alpha,0}$ was already estimated in (44); and for the quantity $\eta>0$ given in (31) we have

$$\left\|\int_0^t K_{\alpha,\beta}(t-\tau,\cdot)*\Big(\partial_x(u^2)+\partial_x^2(u^2)+(\partial_x u)^2\Big)(\tau,\cdot)\,d\tau\right\|_{s,\alpha,0} \lesssim T^{\eta}\,\|u\|_T^2. \tag{88}$$

For the second term, the following estimate holds:

$$\sup_{0\leq t\leq T} t^{\frac{1}{\alpha}}\left\|(1+|\cdot|^{\min(\kappa,n)})\int_0^t K_{\alpha,\beta}(t-\tau,\cdot)*(\partial_x(u^2)+\partial_x^2(u^2)+(\partial_x u)^2)(\tau,\cdot)d\tau\right\|_{L^\infty} \lesssim T^{1-\frac{2}{\alpha}}\,\|u\|_T^2. \tag{89}$$

Indeed, to estimate the expression $\partial_x(u^2)$, for $t>0$ and $x\in\mathbb{R}$ fixed, by the kernel estimate (83) and by the first and the second expressions in (85), and moreover, by recalling that $s>\frac{5}{2}$ and we have the continuous embedding $\|\partial_x u\|_{L^\infty} \lesssim \|u\|_{H^s}$, we write

$$\begin{aligned}
&\left|\int_0^t K_{\alpha,\beta}(t-\tau,\cdot)*\partial_x(u^2)(\tau,x)\right|\\
\leq & \int_0^t \int_{\mathbb{R}} |K_{\alpha,\beta}(t-\tau,x-y)||u(t,y)||\partial_y u(\tau,y)|dy\,d\tau\\
\lesssim & \|u\|_T\, e^{\eta\, t}\int_0^t \frac{1}{(t-\tau)^{\frac{1}{\alpha}}}\frac{1}{\tau^{\frac{1}{\alpha}}}\left(\int_{\mathbb{R}}\frac{1}{1+|x-y|^n}\frac{1}{1+|y|^{\min(\kappa,n)}}dy\right)\|\partial_y u(\tau,\cdot)\|_{L^\infty}d\tau\\
\lesssim & \|u\|^2_T\, e^{\eta\, t}\int_0^t \frac{1}{(t-\tau)^{\frac{1}{\alpha}}}\frac{1}{\tau^{\frac{1}{\alpha}}}\left(\int_{\mathbb{R}}\frac{1}{1+|x-y|^n}\frac{1}{1+|y|^{\min(\kappa,n)}}dy\right)d\tau\\
\lesssim & \|u\|^2_T\, e^{\eta\, t}t^{-\frac{2}{\alpha}+1}\,\frac{1}{1+|x|^{\min(\kappa,x)}},
\end{aligned}$$

which yields the estimate $\sup\limits_{0\leq t\leq T} t^{\frac{1}{\alpha}}\left\|(1+|\cdot|^{\min(k,n)})\int_0^t K_{\alpha,\beta}(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)d\tau\right\|_{L^\infty}\lesssim T^{1-\frac{1}{\alpha}}\,\|u\|^2_T$.

To estimate the expression $\partial_x^2(u^2)$ we remark that we can write $K_{\alpha,\beta}*(\partial^2_x(u^2)) = \partial_x K_{\alpha,\beta}*2\,(u\partial_x u)$. By the second point of Lemma 4.1, we obtain

$$\begin{aligned}
&\left|\int_0^t K_{\alpha,\beta}(t-\tau,\cdot)*\partial^2_x(u^2)(\tau,x)\right|\\
\lesssim & \int_0^t \int_{\mathbb{R}} |\partial_x K_{\alpha,\beta}(t-\tau,x-y)||u(t,y)||\partial_y u(\tau,y)|dy\,d\tau\\
\lesssim & \|u\|_T\, e^{\eta\, t}\int_0^t \frac{1}{(t-\tau)^{\frac{2}{\alpha}}}\frac{1}{\tau^{\frac{1}{\alpha}}}\left(\int_{\mathbb{R}}\frac{1}{1+|x-y|^{n+1}}\frac{1}{1+|y|^{\min(\kappa,n)}}dy\right)\|\partial_x u(\tau,\cdot)\|_{L^\infty}d\tau\\
\lesssim & \|u\|^2_T\, e^{\eta_1\, t}t^{-\frac{3}{\alpha}+1}\,\frac{1}{1+|x|^{\min(\kappa,x)}}.
\end{aligned}$$

We thus have $\sup\limits_{0\leq t\leq T} t^{\frac{1}{\alpha}}\left\|(1+|\cdot|^{\min(k,n)})\int_0^t K_{\alpha,\beta}(t-\tau,\cdot)*\partial^2_x(u^2)(\tau,\cdot)d\tau\right\|_{L^\infty}\lesssim T^{1-\frac{2}{\alpha}}\,\|u\|^2_T$.

Similarly, for the expression $K_{\alpha,\beta}*(\partial_x u)^2$ we just write $K_{\alpha,\beta}*(\partial_x u)^2 = K_{\alpha,\beta}*((\partial_x u)(\partial_x u))$ and we have the estimate $\sup\limits_{0\leq t\leq T} t^{\frac{1}{\alpha}}\left\|(1+|\cdot|^{\min(k,n)})\int_0^t K_{\alpha,\beta}(t-\tau,\cdot)*(\partial_x u)^2(\tau,\cdot)d\tau\right\|_{L^\infty}\lesssim T^{1-\frac{2}{\alpha}}\,\|u\|^2_T$.

Finally, we recall that since $0<T<1$ we have $T^{1-\frac{1}{\alpha}}<T^{1-\frac{2}{\alpha}}$; and we thus obtain the wished estimate stated in (89).

For the third term in the norm $\|\cdot\|_T$ (given in (85)) we essentially follow the same arguments exposed above to obtain the estimate

$$\begin{aligned}
&\sup_{0\leq t\leq T} t^{\frac{2}{\alpha}}\left\|(1+|\cdot|^{\min(\kappa,n+1)})\int_0^t \partial_x K_{\alpha,\beta}(t-\tau,\cdot)*(\partial_x(u^2)+\partial^2_x(u^2)+(\partial_x u)^2)(\tau,\cdot)d\tau\right\|_{L^\infty}\\
\lesssim & T^{1-\frac{3}{\alpha}}\,\|u\|^2_T.
\end{aligned} \tag{90}$$

Indeed, for the reader's convenience, we shall only mention that to treat the expression $\partial^2_x(u^2)$ we write $\partial_x K_{\alpha,\beta}*\partial^2_x(u^2) = \partial_x K_{\alpha,\beta}*(\partial_x u)^2+\partial_x K_{\alpha,\beta}*(u\,\partial^2_x u)$. Here, to control the last term $\partial^2_x u$ we use the continuous embedding $\|\partial^2_x u\|_{L^\infty}\lesssim\|u\|_{H^s}$, which is valid for $s>\frac{5}{2}$.

To finish the proof, we set $\eta_2=\min(\eta,1-\frac{3}{\alpha})$ from which we get the desired estimate. Proposition 4.6 is proven. ■

With Propositions 4.5 and 4.6 at our disposal, there exists a solution $u \in E_{T_0}$ to the equation (27), for a time $0 < T_0 < T \leq 1$ small enough. But, by the embedding $E_{T_0} \subset \mathcal{C}([0, T_0], H^s(\mathbb{R}))$ and since the equation (1) is locally well-posed in this space (in particular we have the uniqueness of the solution) this solution is the same to the one constructed in Theorem 1.

Until now we have proven the estimate (15) for all time $0 < t \leq T_0$. Thereafter, by following the same arguments of [8, Theorem 4.2] this estimate is extended to the time T. Theorem 4 is now proven. ■

4.3 Asymptotic profiles and optimality

Once the problem of the spatial pointwise decaying of our problem (1) is finished, our next objective is to show in which cases we could speak of optimally of this decaying. To do this, we will start by giving an asymptotic profile of the solution in spatial variable, of problem (1).

Proof of Theorem 5

Since the solution $u(t, x)$ writes down as in the integral formulation (27), we start by proving that the first term on the right-hand side in has the following asymptotic development:

$$K_{\alpha,\beta}(t, \cdot) * u_0(x) = K_{\alpha,\beta}(t, x) \left(\int_{\mathbb{R}} u_0(y) dy \right) + R_1(t, x), \quad |x| \to +\infty, \tag{91}$$

with $|R_1(t, x)| = o(t)\,(1/|x|^n)$. Indeed, for $t > 0$ and $x \in \mathbb{R}$ fix this term can be decomposed as follows:

$$\begin{aligned}\int_{\mathbb{R}} K_{\alpha,\beta}(t, x - y) u_0(y) dy =& K_{\alpha,\beta}(t, x) \left(\int_{\mathbb{R}} u_0(y) dy \right) + \int_{|y| < \frac{|x|}{2}} (K_{\alpha,\beta}(t, x - y) - K_{\alpha,\beta}(t, x)) u_0(y) dy \\ & + \int_{|y| > \frac{|x|}{2}} K_{\alpha,\beta}(t, x - y) u_0(y) dy - K_{\alpha,\beta}(t, x) \left(\int_{|y| > \frac{|x|}{2}} u_0(y) dy \right) \\ =& K_{\alpha,\beta}(t, x) \left(\int_{\mathbb{R}} u_0(y) dy \right) + I_1 + I_2 + I_3,\end{aligned} \tag{92}$$

hence, we define $R_1 = I_1 + I_2 + I_3$ and we will verify that the following statement holds:

$$|R_1| \leq \frac{c(u_0, t)}{|x|^{n+\varepsilon}}, \quad |x| \to +\infty, \qquad \varepsilon > 0. \tag{93}$$

To estimate the term I_1 we need the Lemma 4.1. Since $K_{\alpha,\beta}(t, \cdot) \in \mathcal{C}^1(\mathbb{R})$, we write $K_{\alpha,\beta}(t, x - y) - K_{\alpha,\beta}(t, x) = -y\ \partial_x K_{\alpha,\beta}(t, x - \theta y)$, for some $0 < \theta < 1$. Then, by this identity and using the first point of Lemma 4.1 we get

$$\begin{aligned} I_1 &\leq \int_{|y| < \frac{|x|}{2}} |(K_{\alpha,\beta}(t, x - y) - K_{\alpha,\beta}(t, x))| |u_0(y)| dy \leq \int_{|y| < \frac{|x|}{2}} |y| |\partial_x K_{\alpha,\beta}(t, x - \theta y)| |u_0(y)| dy \\ &\lesssim e^{c_{\eta_1} t} \int_{|y| < \frac{|x|}{2}} \frac{|y| |u_0(y)|}{|x - \theta y|^{n+1}} dy. \end{aligned}$$

We study now the expression $\dfrac{1}{|x - \theta y|^{n+1}}$. As we have $0 < \theta < 1$, and moreover, as we have $|y| < \frac{|x|}{2}$, then we can write $|x - \theta y| \geq |x| - \theta |y| \geq |x| - |y| \geq \frac{|x|}{2}$; and thus we get $\dfrac{1}{|x - \theta y|^{n+1}} \lesssim \dfrac{1}{|x|^{n+1}}$. With this inequality and recalling that the initial datum verifies $|u_0(y)| \leq \dfrac{c}{1 + |y|^\kappa}$ (with $\kappa > n$), we can write

$$e^{c_{\eta_1} t} \int_{|y| < \frac{|x|}{2}} \frac{|y| |u_0(y)|}{|x - \theta y|^{n+1}} dy \lesssim \frac{e^{c_{\eta_1} t}}{|x|^{n+1}} \int_{|y| < \frac{|x|}{2}} \frac{|y|}{1 + |y|^\kappa} dy \lesssim \frac{e^{c_{\eta_1} t}}{|x|^{n+1}},$$

hence we have

$$I_1 \lesssim \frac{e^{c\eta_1 t}}{|x|^{n+1}}, \quad |x| \to +\infty. \tag{94}$$

For the term I_2, as $|u_0(y)| \leq \dfrac{c_0}{|y|^\kappa}$ (for $|y|$ large enough) and moreover, as we have $|y| > \frac{|x|}{2}$, then we write

$$\begin{aligned} I_2 &\leq \int_{|y|>\frac{|x|}{2}} |K_{\alpha,\beta}(t,x-y)||u_0(y)|dy \lesssim \int_{|y|>\frac{|x|}{2}} \frac{|K_{\alpha,\beta}(t,x-y)|}{|y|^\kappa} dy \lesssim \frac{1}{|x|^\kappa} \int_{|y|>\frac{|x|}{2}} |K_{\alpha,\beta}(t,x-y)| \\ &\lesssim \frac{1}{|x|^\kappa} \|K_{\alpha,\beta}(t,\cdot)\|_{L^1}, \end{aligned}$$

but, by Proposition 4.4 we have $\|K_{\alpha,\beta}(t,\cdot)\|_{L^1} \lesssim \frac{e^{c\eta_1 t}}{t^{\frac{1}{\alpha}}}$, and for $\kappa > n$ we get

$$I_2 \lesssim \frac{e^{c\eta_1 t}}{t^{\frac{1}{\alpha}}} \frac{1}{|x|^\kappa}, \quad |x| \to +\infty. \tag{95}$$

Finally, in order to study the term I_3, recall first that by the estimate (83) for $|x|$ enough enough we have $|K_{\alpha,\beta}(t,x)| \lesssim \dfrac{e^{c\eta_1 t}}{t^{\frac{1}{\alpha}}} \dfrac{1}{|x|^n}$. Moreover, recall that the initial datum verifies $|u_0(y)| \lesssim \dfrac{1}{1+|y|^\kappa}$ (with $\kappa = n+\varepsilon$). Then we write

$$\begin{aligned} I_3 &\lesssim \frac{e^{c\eta_1 t}}{t^{\frac{1}{\alpha}}} \frac{1}{|x|^n} \int_{|y|>\frac{|x|}{2}} |u_0(y)|dy \lesssim \frac{e^{c\eta_1 t}}{t^{\frac{1}{\alpha}}} \frac{1}{|x|^n} \int_{|y|>\frac{|x|}{2}} \frac{1}{1+|y|^{n+\varepsilon}} dy \\ &\lesssim \frac{e^{c\eta_1 t}}{t^{\frac{1}{\alpha}}} \frac{1}{|x|^{n+\varepsilon}} \int_{|y|>\frac{|x|}{2}} \frac{1}{1+|y|^n} dy \lesssim \frac{e^{c\eta_1 t}}{t^{\frac{1}{\alpha}}} \frac{1}{|x|^{n+\varepsilon}} \int_{\mathbb{R}} \frac{1}{1+|y|^n} dy \lesssim \frac{e^{c\eta_1 t}}{t^{\frac{1}{\alpha}}} \frac{1}{|x|^{n+\varepsilon}}. \end{aligned} \tag{96}$$

Thus, the desired estimate (93) follows from (94), (95) and (96); and we have the wished profile given in (91).

Now, we focus on the nonlinear term on the right-hand side of the equation (27). The first and the second nonlinear terms: $\partial_x(u^2)$ and $\partial_x^2(u^2)$, can be estimated as follows. For $t > 0$ and $x \in \mathbb{R}$ fix we write

$$\gamma_1 \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * \partial_x(u^2)(\cdot,\tau)d\tau = \gamma_1 \int_0^t \int_{\mathbb{R}} \partial_y K_{\alpha,\beta}(t-\tau,x-y)\, u^2(\tau,y)dy\, d\tau = (a).$$

Then, by the second point of Lemma 4.1, and moreover, since by (15) we have $|u(\tau,y)|^2 \lesssim \dfrac{1}{\tau^{\frac{2}{\alpha}}(1+|y|^{2n})}$, then we get

$$\begin{aligned} (a) &\lesssim \gamma_1 \int_0^t \frac{e^{c\eta_1(t-\tau)}}{(t-\tau)^{\frac{2}{\alpha}} \tau^{\frac{2}{\alpha}}} \int_{\mathbb{R}} \frac{1}{1+|x-y|^{n+1}} \frac{1}{1+|y|^{2n}} dy\, d\tau \\ &\lesssim \gamma_1 e^{c\eta_1 t} \int_0^t \frac{1}{(t-\tau)^{\frac{2}{\alpha}} \tau^{\frac{2}{\alpha}}} d\tau \left(\int_{\mathbb{R}} \frac{1}{1+|x-y|^{n+1}} \frac{1}{1+|y|^{2n}} dy \right) \\ &\lesssim \gamma_1 e^{c\eta_1 t} \left(\int_0^t \frac{1}{(t-\tau)^{\frac{2}{\alpha}} \tau^{\frac{2}{\alpha}}} \right) \frac{1}{1+|x|^{n+1}}. \end{aligned}$$

As $\alpha > 2$, this integral computes down as $\displaystyle\int_0^t \frac{d\tau}{(t-\tau)^{\frac{2}{\alpha}} \tau^{\frac{2}{\alpha}}} \lesssim \frac{1}{t^{\frac{4}{\alpha}-1}}$.

On the other hand, we write

$$\begin{aligned}
\gamma_2 \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * \partial_x^2(u^2)(\cdot,\tau)d\tau &= \gamma_2 \int_0^t \int_{\mathbb{R}} \partial_y^2 K_{\alpha,\beta}(t-\tau,x-y)\, u^2(\tau,y)dyd\tau \\
&\lesssim \int_0^t \frac{e^{c_{\eta_1}(t-\tau)}}{(t-\tau)^{\frac{3}{\alpha}}\,\tau^{\frac{2}{\alpha}}} \int_{\mathbb{R}} \frac{1}{1+|x-y|^{n+2}} \frac{1}{1+|y|^{2n}} dy\, d\tau \\
&\lesssim e^{c_{\eta_1} t} \int_0^t \frac{1}{(t-\tau)^{\frac{3}{\alpha}}\,\tau^{\frac{2}{\alpha}}} d\tau \left(\int_{\mathbb{R}} \frac{1}{1+|x-y|^{n+2}} \frac{1}{1+|y|^{2n}} dy \right) \\
&\lesssim e^{c_{\eta_1} t} \left(\int_0^t \frac{1}{(t-\tau)^{\frac{3}{\alpha}}\,\tau^{\frac{2}{\alpha}}} \right) \frac{1}{1+|x|^{n+2}}.
\end{aligned}$$

As $\alpha > \frac{7}{2}$, then this integral computes down as $\displaystyle\int_0^t \frac{d\tau}{(t-\tau)^{3/\alpha}\,\tau^{2/\alpha}} \leq \frac{c}{t^{5/\alpha-1}}$.

Finally, we must study the third nonlinear term $(\partial_x u)^2$, which must be treated differently from the previous ones.

Remark 4 *When studying this nonlinear term in the same fashion as the previous ones we obtain the integral* $\displaystyle\int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{\alpha}}\tau^{\frac{4}{\alpha}}}$*, which converges as long as* $\alpha > 4$*. But this constraint excludes the physically relevant value* $\alpha = 4$*.*

Remark 5 *The more precise analysis on the term* $(\partial_x u)^2$ *which we shall perform will allows us to prove an interesting optimally criterion of the pointwise decaying of solutions.*

We shall prove the following identity:

$$\int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * \gamma_3 (\partial_x u)^2(\tau,\cdot)\, d\tau = \gamma_3\, K_{\alpha,\beta}(t,x) \int_0^t \|u(\tau,\cdot)\|_{\dot{H}^1}^2 d\tau + R_2(t,x), \quad |x| \to +\infty, \tag{97}$$

with $|R_2(t,x)| = o(t)\left(\dfrac{1}{|x|^n}\right)$. Indeed, we follow the same ideas of the identity (92) to write

$$\gamma_3 \int_0^t K_{\alpha,\beta}(t-\tau,x) * (\partial_x u)^2(\tau,x)\, d\tau = \gamma_3 \int_0^t K_{\alpha,\beta}(t-\tau,x) \|u(\tau,\cdot)\|_{\dot{H}^1}^2\, d\tau + J_1(t) + J_2(t) + J_3(t),$$

where

$$J_1(t) = -\gamma_3 \int_0^t \int_{|y|<\frac{|x|}{2}} (K_{\alpha,\beta}(t-\tau,x-y) - K_{\alpha,\beta}(t-\tau,x))(\partial_y u)^2(y,\tau)dy\; d\tau,$$

$$J_2(t) = -\gamma_3 \int_0^t \int_{|y|>\frac{|x|}{2}} K_{\alpha,\beta}(t-\tau,x-y)((\partial_y u)^2)(\tau,y)dy\; d\tau,$$

and

$$J_3(t) = \gamma_3 \int_0^t K_{\alpha,\beta}(t-\tau,x) \left(\int_{|y|>\frac{|x|}{2}} (\partial_y u)^2(\tau,y)dy \right) d\tau.$$

Moreover, we write

$$
\begin{aligned}
&\gamma_3 \int_0^t K_{\alpha,\beta}(t-\tau,\cdot) * (\partial_x u)^2(\tau,x)\, d\tau \\
&= \gamma_3\, K_{\alpha,\beta}(t,x) \int_0^t \|u(\tau,\cdot)\|^2_{\dot{H}^1} dt - \gamma_3\, K_{\alpha,\beta}(t,x) \int_0^t \|u(\tau,\cdot)\|^2_{\dot{H}^1} dt + \int_0^t K_{\alpha,\beta}(t-\tau,x) \|u(\tau,\cdot)\|^2_{\dot{H}^1}\, d\tau \\
&\quad + J_1(t) + J_2(t) + J_3(t), \\
&= J_1(t) + J_2(t) + J_3(t) - \gamma_3 \int_0^t (K_{\alpha,\beta}(t,x) - K_{\alpha,\beta}(t-\tau,x) \|u(\tau,\cdot)\|^2_{\dot{H}^1} d\tau \\
&= J_1(t) + J_2(t) + J_3(t) + J_4(t).
\end{aligned}
$$

As before, we we define $R_2 = J_1(t) + J_2(t) + J_3(t) + J_4(t)$ and we will verify that the following statement holds:

$$
|R_2| \leq \frac{c(u,t)}{|x|^{n+\varepsilon}}, \quad \varepsilon > 0, \quad |x| \to +\infty. \tag{98}
$$

By an analysis similar to the one done above, for the term $J_1(t)$ we have

$$
\begin{aligned}
J_1(t) &\lesssim \gamma_3 \int_0^t \int_{|y|<\frac{|x|}{2}} |(K_{\alpha,\beta}(t-\tau,x-y) - K_{\alpha,\beta}(t-\tau,x))||(\partial_y u)^2(t,y)| dy\ d\tau \\
&\lesssim \gamma_3 \int_0^t \int_{|y|<\frac{|x|}{2}} |y||\partial_x K_{\alpha,\beta}(t-\tau,x-\theta y)||(\partial_y u)^2(\tau,y)| dy\ d\tau \\
&\lesssim \gamma_3 \int_0^t \frac{e^{c\eta_1\,(t-\tau)}}{(t-\tau)^{\frac{2}{\alpha}}} \int_{|y|<\frac{|x|}{2}} \frac{|y||(\partial_y u)^2(\tau,y)|}{|x-\theta y|^{n+1}} dy\ d\tau \\
&\lesssim \frac{\gamma_3}{|x|^{n+1}} \int_0^t \frac{e^{c\eta_1\,(t-\tau)}}{(t-\tau)^{\frac{2}{\alpha}}} \int_{|y|<\frac{|x|}{2}} \frac{|y|}{1+|y|^{n+1}} |(\partial_y u)^2(\tau,y)| dy\ d\tau \\
&\lesssim \frac{\gamma_3}{|x|^{n+1}} \int_0^t \frac{e^{c\eta_1\,(t-\tau)}}{(t-\tau)^{\frac{2}{\alpha}}} \|u(\tau,\cdot)\|^2_{\dot{H}^1} d\tau.
\end{aligned}
$$

Hence we obtain

$$
J_1(t) \leq \frac{C(t,u)}{|x|^{n+1}}, \quad |x| \to +\infty. \tag{99}
$$

For the term $J_2(t)$, recall the estimate $|((\partial_y u))(\tau,y)| \leq \frac{c}{\tau^{\frac{2}{\alpha}}} \frac{1}{|y|^{n+1}}$ (for $|y|$ large enough) and moreover, since $|y| > \frac{|x|}{2}$, then we write

$$
\begin{aligned}
J_2(t) &\lesssim \gamma_3 \int_0^t \int_{|y|>\frac{|x|}{2}} |K_{\alpha,\beta}(t-\tau,x-y)|((\partial_x u)^2)(\tau,y)|| dy d\tau \\
&\lesssim \gamma_3 \int_0^t \int_{|y|>\frac{|x|}{2}} \frac{|K_{\alpha,\beta}(t-\tau,x-y)\ |(\partial_x u)(\tau,y)}{|y|^{n+1}} dy \\
&\lesssim \frac{\gamma_3}{|x|^{n+1}} \int_0^t \frac{1}{\tau^{\frac{2}{\alpha}}} \|K_{\alpha,\beta}(t-\tau,\cdot)\|_{L^2}\ \|u(\tau)\|_{\dot{H}^1}\, d\tau \\
&\lesssim \frac{\gamma_3}{|x|^{n+1}} \int_0^t \frac{1}{\tau^{\frac{2}{\alpha}}} \frac{1}{(t-\tau)^{\frac{1}{\alpha}}}\ \|u(\tau)\|_{\dot{H}^1}\, d\tau.
\end{aligned}
$$

We thus have

$$
J_2(t) \lesssim \frac{C(t,u)}{|x|^{n+1}}, \quad |x| \to +\infty. \tag{100}
$$

In order to study the term $J_3(t)$, recall first that always by (83) for $|x|$ enough enough we have the estimate $|K_{\alpha,\beta}(t,x)| \leq C \frac{e^{c_{\eta_1} t}}{t^{\frac{1}{\alpha}}} \frac{1}{|x|^n}$. Then we write

$$
\begin{aligned}
J_3(t) &\lesssim \frac{\gamma_3}{|x|^n} \int_0^t \frac{e^{c_{\eta_1}(t-\tau)}}{\tau^{\frac{2}{\alpha}}} \Big(\int_{|y|>\frac{|x|}{2}} \frac{1}{1+|y|^{n+1}} |(\partial_x u)(\tau,y)| dy \Big) d\tau \\
&\lesssim \frac{\gamma_3}{|x|^{n+1}} \int_0^t \frac{e^{c_{\eta_1}(t-\tau)}}{\tau^{\frac{2}{\alpha}}} \Big(\int_{|y|>\frac{|x|}{2}} \frac{1}{1+|y|^{n}} |((\partial_x u))(\tau,y)| dy \Big) d\tau \\
&\lesssim \frac{\gamma_3}{|x|^{n+1}} \int_0^t \frac{e^{c_{\eta_1}(t-\tau)}}{\tau^{\frac{2}{\alpha}}} \|u(\tau,\cdot)\|_{\dot{H}^1} d\tau.
\end{aligned}
$$

Hence,

$$
J_3(t) \lesssim \frac{C(t,u)}{|x|^{n+1}}, \quad |x| \to +\infty. \tag{101}
$$

Finally, we must estimate the term $J_4(t)$. We use the mean value theorem (in the time variable) and for a time $t-\tau \leq \tau_1 \leq t$ we write

$$
\begin{aligned}
J_4(t) &\leq \gamma_3 \int_0^t |K_{\alpha,\beta}(t,x) - K_{\alpha,\beta}(t-\tau,x)| \, \|u(\tau,\cdot)\|^2_{\dot{H}^1} d\tau \\
&\leq \gamma_3 \int_0^t |\partial_t K_{\alpha,\beta}(\tau_1,x)| \, |\tau| \, \|u(\tau,\cdot)\|^2_{\dot{H}^1} d\tau.
\end{aligned} \tag{102}
$$

At this point, we need to estimate the expression $|\partial_t K_{\alpha,\beta}(\tau_1,x)|$:

Lemma 4.2 *Let $\alpha > \beta \geq 1$ with $\alpha > 2$. Let $K_{\alpha,\beta}$ the kernel given in (25) with $m(\xi) = |\xi|$ or $m(\xi) = -|\xi|^2$. Moreover, let $n \geq 2$ be the parameter defined in (13). For $t > 0$ the following estimate hold:*

$$
|\partial_t K_{\alpha,\beta}(t,x)| \leq C \frac{e^{\eta_1 t}}{|x|^{n+1}}, \quad |x| \to +\infty, \tag{103}
$$

for two constants $C, \eta_1 > 0$ depending on α and β.

Proof. Recall that the kernel $K_{\alpha,\beta}(t,x)$ solves the equation (17) and for $t > 0$ fixed we can write

$$
|\partial_t K_{\alpha,\beta}(t,x)| \leq |D(\partial_x K_{\alpha,\beta}(t,x))| + |D^\alpha_x K_{\alpha,\beta}(t,x)| + |D^\beta_x K_{\alpha,\beta}(t,x)|.
$$

Each term on the right-hand side is essentially a derivative of the Kernel $K_{\alpha,\beta}(t,x)$ in the spatial variable. Consequently, by following the same ideas in the proof of [8, Lemma 4.2], for $x \neq 0$ we get the following estimates

$$
|D(\partial_x K_{\alpha,\beta}(t,x))| \leq \begin{cases} C \frac{e^{\eta_1 t}}{|x|^{n+2}}, & m(\xi) = |\xi|, \\ C \frac{e^{\eta_1 t}}{|x|^{n+3}}, & m(\xi) = -|\xi|^2, \end{cases}
$$

$$
|D^\alpha_x K_{\alpha,\beta}(t,x)| \leq C \frac{e^{\eta_1 t}}{|x|^{n+[\alpha]}},
$$

and

$$
|D^\alpha_x K_{\alpha,\beta}(t,x)| \leq C \frac{e^{\eta_1 t}}{|x|^{n+[\beta]}},
$$

where, as before $[\alpha]$ and $[\beta]$ denote the integer part of the parameters α and β. Thereafter, recall that $\alpha > 2$ and $\beta \geq 1$. Then, each expression above is controlled by the term $C \frac{e^{\eta_1 t}}{|x|^{n+1}}$ when $|x| \to +\infty$; and we thus obtain the wished estimate (103). ■

Once we have the estimate (103), we get back to the estimate (102) to finally obtain

$$J_4(t) \lesssim \frac{\gamma_3}{|x|^{n+1}} \int_0^t e^{\eta_1 \tau_1} |\tau| \, \|u(\tau,\cdot)\|^2_{\dot{H}^1} d\tau \lesssim \frac{\gamma_3 C(t,u)}{|x|^{n+1}}, \qquad |x| \to +\infty. \tag{104}$$

With the estimates (99), (100), (101) and (104) at our disposal, we obtain the wished identity (97). This identity together with the identity (92) yield yield the asymptotic profile (91). Theorem 5 is proven. ■

Proof of Corollary 1

By the asymptotic profile (19) and by the identity (18), for $t>0$ fixed and for $|x|$ large enough we write

$$\begin{aligned} |u(t,x)| &= \left| K_{\alpha,\beta}(t,x) \left[\int_{\mathbb{R}} u_0(y)dy + \gamma_3 \int_0^t \|u(\tau,\cdot)\|^2_{\dot{H}^1} d\tau \right] + R(t,x) \right| \\ &\geq |K_{\alpha,\beta}(t,x)| \left| \int_{\mathbb{R}} u_0(y)dy + \gamma_3 \int_0^t \|u(\tau,\cdot)\|^2_{\dot{H}^1} d\tau \right| - |R(t,x)| \\ &= \frac{|I(t)|}{|x|^n} \left| \int_{\mathbb{R}} u_0(y)dy + \gamma_3 \int_0^t \|u(\tau,\cdot)\|^2_{\dot{H}^1} d\tau \right| - |R(t,x)|. \end{aligned} \tag{105}$$

Recall that $|R(t,x)| \leq \frac{c_2(t,u)}{|x|^{n+\varepsilon}}$ with $0<\varepsilon\leq 1$ (hence we have $|R(t,x)| = o(1/|x|^n)$) and for the quantity $\frac{|I(t)|}{2} \left| \int_{\mathbb{R}} u_0(y)dy + \gamma_3 \int_0^t \|u(\tau,\cdot)\|^2_{\dot{H}^1} d\tau \right| > 0$ there exists $M>0$ such that for $|x|>M$ we have

$$|R(t,x)| \leq \frac{|I(t)|}{2} \left| \int_{\mathbb{R}} u_0(y)dy + \gamma_3 \int_0^t \|u(\tau,\cdot)\|^2_{\dot{H}^1} d\tau \right| \frac{1}{|x|^n}.$$

We get back to the previous estimate from below to obtain

$$\frac{|I(t)|}{2} \left| \int_{\mathbb{R}} u_0(y)dy + \gamma_3 \int_0^t \|u(\tau,\cdot)\|^2_{\dot{H}^1} d\tau \right| \frac{1}{|x|^n} \leq |u(t,x)|, \quad |x| \to +\infty,$$

hence we set

$$c_3(u_0,\gamma_3,t,u) = \frac{|I(t)|}{2} \left| \int_{\mathbb{R}} u_0(y)dy + \gamma_3 \int_0^t \|u(\tau,\cdot)\|^2_{\dot{H}^1} d\tau \right|. \tag{106}$$

Corollary 1 is proven. ■

Proof of Corollary 2

The proof follows very similar ideas of the previous proof. Indeed, in the case $\gamma_3 = 0$ and $\int_{\mathbb{R}} u_0(y)dy \neq 0$ by the estimate (105) and for $|x|$ large enough we have

$$\frac{|I(t)|}{2} \left| \int_{\mathbb{R}} u_0(y)dy \right| \frac{1}{|x|^n} \leq |u(t,x)|,$$

where we set the quantity

$$c_4(u_0,t) = \frac{|I(t)|}{2} \left| \int_{\mathbb{R}} u_0(y)dy \right|. \tag{107}$$

On the other hand, in the case $\gamma_3 = 0$ and $\int_{\mathbb{R}} u_0(y)dy = 0$ by the identity (19) we obtain the estimate (24). Corollary 2 is proven. ■

Appendix

A proof of the identity (62). First, we recall that the bilinear for $B(\cdot,\cdot)$ is given in (45). Moreover, for the sake of simplicity, we shall write $K_{\alpha,\beta}(t,\cdot)*u_0=\tilde{u_0}(t,\cdot)$ and $K_{\alpha,\beta}(t,\cdot)*v_0=\tilde{v_0}(t,\cdot)$. Then, we have

$$
\begin{aligned}
g(t,\xi) &= \int_0^t e^{-f(\xi)(t-\tau)}\,\mathcal{F}\left(B\big(\tilde{u_0},\tilde{v_0}\big)\right)(\tau,\xi)\,d\tau\\
&= \int_0^t e^{-f(\xi)(t-\tau)}\,\Big(\gamma_1\, i\xi(\widehat{\tilde{u_0}}*\widehat{\tilde{v_0}})-\gamma_2\,\xi^2(\widehat{\tilde{u_0}}*\widehat{\tilde{v_0}})+\gamma_3(\widehat{i\xi\tilde{u_0}}*\widehat{i\xi\tilde{v_0}})\Big)(\tau,\xi)d\tau\\
&= \int_0^t e^{-f(\xi)(t-\tau)}\,\Big((\gamma_1\, i\xi-\gamma_2\,\xi^2)\int_{\mathbb{R}} e^{f(\xi-\eta)\tau}\widehat{u_0}(\xi-\eta)\,e^{f(\eta)\tau}\widehat{v_0}(\eta)d\eta\\
&\quad -\gamma_3\int_{\mathbb{R}}(\xi-\eta)e^{f(\xi-\eta)\tau}\widehat{u_0}(\xi-\eta)\,\eta\,e^{f(\eta)\tau}\widehat{v_0}(\eta)d\eta\Big)\,d\tau\\
&= \int_0^t e^{-f(\xi)(t-\tau)}\int_{\mathbb{R}}\left[\gamma_1\, i\xi-\gamma_2\,\xi^2-\gamma_3(\xi-\eta)\eta\right]\,e^{-f(\xi-\eta)\tau}\,e^{-f(\eta)\tau}\widehat{u_0}(\xi-\eta)\,\widehat{v_0}(\eta)d\eta\,d\tau\\
&= \int_{\mathbb{R}}\left[\gamma_1 i\xi-\gamma_2\,\xi^2-\gamma_3(\xi-\eta)\eta\right]\widehat{u_0}(\xi-\eta)\,\widehat{v_0}(\eta)\left(\int_0^t e^{-f(\xi)(t-\tau)}\,e^{-f(\xi-\eta)\tau}\,e^{-f(\eta)\tau}d\tau\right)d\eta.
\end{aligned}
$$

where, the integral in the time variable computes down as

$$
\int_0^t e^{-f(\xi)(t-\tau)}\,e^{-f(\xi-\eta)\tau}\,e^{-f(\eta)\tau}d\tau=\frac{e^{-f(\eta)t-f(\xi-\eta)t}-e^{-f(\xi)t}}{f(\xi)-f(\eta)-f(\xi-\eta)}.
$$

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